Automatic Structures

Sasha Rubin

A THESIS SUBMITTED IN PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY,
THE UNIVERSITY OF AUCKLAND, 2004.

Abstract

This thesis investigates structures that are presentable by finite automata working synchronously on tuples of finite words. The emphasis is on understanding the expressiveness and limitations of automata in this setting. In particular, the thesis studies the classification of classes of automatic structures, the complexity of the isomorphism problem, and the relationship between definability and recognisability.

Acknowledgements

Foremost I thank my supervisor Bakhadyr Khoussainov for his instruction and encouragement. His trust, generosity of time and guidance have deeply informed my development. Through his association I have had the privilege to work with some remarkable people. The work in this thesis has been co-authored, discussed with or read by Hajime Ishihara, Bakhadyr Khoussainov, Andre Nies and Frank Stephan. I am indebted to all of them. In particular I am very pleased to have had the opportunity to work with Frank Stephan whose prolific ideas are a continual source of inspiration.

Bakhadyr helped organise my year long visit to Madison and Houston. I am grateful to Steffen Lempp for hosting me at the University of Wisconsin. I was fortunate to attend classes there and discuss my work with Steffen and his postdocs Charlie McCoy and Reed Solomon. I particularly thank Reed for his boundless enthusiasm for discussing automatic structures.

At Rice University I was hosted by Moshe Vardi who introduced me to formal verification. I have benefited immeasurably from his insight and patience.

I also spent three months at the University of Heidelberg. I thank Klaus Ambos-Spies in the Department of Mathematics and Computer Science for his hospitality.

Eamonn O'Brien and Marston Conder provided early financial support and frank advice. I am grateful to have received a scholarship from the Department of Mathematics and the Department of Computer Science at the University of Auckland.

Finally a small note to my friends and family. To my parents Alan and Deborah who encouraged me to pursue my interests, thank you. To my friends and colleagues who have become friends, for your interest in me and my work, I am happy to be able to acknowledge you here.

Contents

Figures

Chapter A

Introduction

The simplest complexity class that has some robustness in terms of the details of the machine model is the finite automaton – commonly thought of as a Turing machine that uses constant work space [Odifreddi - 1999, VIII.1]. That is, variations in the machine model – involving additional work tapes, or the possible direction of the heads, or the addition of non-determinism – do not alter the class of languages (unary relations) that are recognised. This thesis deals with finite automata that recognise relations of higher arity. As a result algebraic structures, such as groups, orders and algebras, may be described by collections of finite automata. The motivation for doing this comes from various more or less independent directions. However the unifying idea is that certain algorithmic problems in these structures – expressible in logical terms – are computable, and sometimes even feasibly computable.

An abstract definition of finite automaton was given by Kleene [1956], who suggested the problem of characterising the sets recognised by finite automata in terms of logical definability [see Church - 1963]. This problem was solved independently by Büchi [1960], Elgot [1961] and Trahtenbrot [1962]. Their work establishes an equivalence between monadic second order definability and automata. Particularly, the work of the first two authors establishes the equivalence between weak monadic second order (WMSO) definability in the structure $(\mathbb{N}, S)$ and finite automata operating synchronously on tuples of finite words. Here monadic second order means that there are variables for subsets of the domain $\mathbb{N}$, and weakness means that these set variables range over finite subsets of $\mathbb{N}$.[1]

A recasting of these results concerns the *first order* (FO) theory of the structure $(\mathbb{N}, +, |_2)$ where $x|_2\, y$ means that x is a power of 2 and x divides y. An important consequence in mathematical logic is that this theory is decidable – there is an effective procedure by which the truth of FO sentences in $(\mathbb{N}, +, |_2)$ can be decided. Here is a brief description of a proof of this fact. There is an effective procedure that transforms a given first order definable relation R, say of arity k, of the structure $(\mathbb{N}, +, |_2)$ into a synchronous finite automaton $\mathcal{A}_R$ operating on k-tuples of finite words. The automaton $\mathcal{A}_R$ recognises an encoding of R. In the present case this is achieved

[1]Some authors use WMSO$(\mathbb{N}, S)$ to mean that the set variables range over arbitrary subsets of the domain, while it is merely *quantification* of these variables that are restricted to finite subsets.

by first encoding the natural numbers into their base 2 representations, least significant digit first, so that there are three automata recognising, respectively, the encodings of the domain $\mathbb{N}$, the graph of $+$ and the relation $|_2$. Then proceed inductively on the complexity of the formula defining R. The automata for the atomic relations $+$ and $|_2$ and the domain $\mathbb{N}$ take care of the base case. The logical operations $\vee$, $\neg$ and $\exists$ are respectively taken care of using the important fact that this class of automata is effectively closed with respect to union, complementation and projection. Decidability now follows from observing that the truth of a sentence of the form $\exists x \Psi(x)$ reduces to the decidable problem of testing whether the corresponding automaton for Ψ accepts some word or not (the emptiness problem).

This setting motivated Hodgson [1976] to introduce the qualifier *automatic* for a relational structure whose domain and atomic relations are recognised by finite automata. For instance as in the preceding paragraph, the structure (N, G_+, d_2) where N is the set of natural numbers written in base 2 notation, G_+ is the graph of addition over N and d_2 is the relation corresponding to $|_2$ on N, is automatic. By the argument just presented, the first order theory of every automatic structure is decidable. This gives a uniform approach for proving the decidability of certain theories that were previously proven using methods such as quantifier elimination; for instance the standard models of Presburger Arithmetic $(\mathbb{N}, +)$ and the rational ordering $(\mathbb{Q}, \leq)$.

The time complexity of the decision procedure using this method of translating logical formulae to finite automata is, in general, non-elementary – that is, it exceeds any fixed number of iterations of the exponential function, namely $2^n, 2^{2^n} \cdots$, where n is the size of a sentence. This upper bound is due to the fact that every alternation of quantifiers in a sentence may incur an exponential increase in the size of the automaton being constructed. This is because projection may result in a non-deterministic automaton, and so determinisation, which in the worst case requires exponentially more states, may be needed to complement the result. In point of fact the complexity of every decision procedure for the particular structure $(\mathbb{N}, +, |_2)$ is non-elementary [Grädel - 1990], or equivalently for WMSO$(\mathbb{N}, S)$ [Meyer - 1975]. In later work they consider automatic structures in the general setting of investigating algorithmic properties of certain infinite structures [2002]. The structures considered are those that can be represented in a finite way – for example by a finite state machine or by logical formulae – and for which there is an effective semantics – for example the model checking problem is decidable.

The feasibility of computing with automatic structures is most clear in the related and much studied concept of automatic groups from computational group theory. Automatic groups were introduced by Cannon et al. [1992] to enable computations on certain finitely generated groups arising in 3-manifold theory. Roughly, an automatic group, say with k generators, in their sense is one for which its Cayley graph $(G, f_1, \cdots, f_k)$ is automatic (in our sense) under a certain natural encoding of the domain. They prove a number of structural and complexity-theoretic results. Illustratively, that every automatic group is finitely presented and that its word problem is solvable in quadratic time [Cannon et al. - 1992].

There are other models of computation, besides finite automata, that could be used to present infinite structures. The study of algorithmic properties of computable algebraic structures and their theories, part of an area known as computable model theory, led Nerode, Remmel and

Cenzer in the 1980's to develop a corresponding theory for polynomial-time structures [see Cenzer and Remmel - 1998]. The general idea in this area, called complexity-theoretic model theory, is to fix a complexity class $\mathcal{C}$ and ask questions about what structures can be represented by machines operating with complexity in $\mathcal{C}$, called $\mathcal{C}$ structures. A typical question is whether a given infinite structure is isomorphic (or $\mathcal{C}$-isomorphic) to a $\mathcal{C}$ structure. A fundamental result here is that every relational computable structure is computably isomorphic to a polynomial-time structure coded over a binary alphabet. Khoussainov and Nerode [1995] reintroduced automatic structures in line with the definitions and ideas of complexity-theoretic model theory. In particular they define a structure to be *automatically presentable* if it is isomorphic to some automatic structure. They formulated some basic questions and suggest that computable, algebraic, model theoretic and complexity theoretic properties of automatic structures are amenable to systematic investigation – in contrast with computable or polynomial time structures. For instance, they characterise the class of automatic structures via a generalisation of the Myhill-Nerode theorem for regular languages.

There are various model theoretic characterisations of the class of automatic structures. A common theme amongst them consists in identifying a structure that serves to define all the automatic structures in some logic. A well known illustration of this idea is that every computable set is first order definable via a quantifier free formula in the structure $(\mathbb{N}, +, \cdot)$. Comparatively there are various structures for which the definable unary relations coincide with the regular sets [Benedikt et al. - 2001, see]. The structure $(\mathbb{N}, S)$ with weak-monadic second order definability was the first to be discovered, Büchi [1960], Elgot [1961]. Again resetting this result, there is an effective transformation of finite automata to first order formulae of $(\mathbb{N}, +, |_2)$ [see Bruyère et al. - 1994]. The idea is to construct a formula $\Phi(\bar{x})$ expressing that there is a successful run of the given automaton on an encoding of $\bar{x}$. Consequently the structure $(\mathbb{N}, +, |_2)$ is complete for the regular relations, not just the regular sets, in the following sense. A relation of arbitrary arity is first order definable in $(\mathbb{N}, +, |_2)$ if and only if the encoding of it is recognised by a finite automaton. A restatement of this says that a structure is automatically presentable if and only if it is first order interpretable in $(\mathbb{N}, +, |_2)$, namely isomorphic to a structure that is first order definable in $(\mathbb{N}, +, |_2)$.

In order to avoid the encoding of natural numbers into strings, one can work directly with strings. There is a similar equivalence between automata and first order definable relations in the string structure $\mathcal{W}(\Sigma) = (\Sigma^\star, (\sigma_a)_{a\in\Sigma}, \preceq_p, el)$, where $\Sigma = \{0, \cdots, k-1\}$, the binary relation σ_a holds on pairs (w, wa), the binary relation $\preceq_p$ is the prefix relation, and the binary relation el holds on words of equal length. This seems to have been first noted by Shepherdson [unpublished, see Section 4, Thatcher - 1966]. Though Eilenberg et al. [1969] provide a proof by transforming formulae to automata inductively as above, while the reverse transformation is done by reducing the problem to showing that certain recognisable sets are definable. They also note that the first order theory of $\mathcal{W}(\Sigma)$ is decidable but do not develop this.

What these results mean is that the study of automatic structures is equivalent to the study of definability in, for instance, the first order fragment of arithmetic $(\mathbb{N}, +, |_2)$. However this logical view does not immediately tell us whether a given structure is automatically presentable or not. The primary motivation for this work is to develop techniques that identify whether a given structure is automatically presentable or not. For example, to show that a structure is automatically presentable one can exhibit appropriate automata or formulae. But how does one tell if a structure, say a particular linear order, has no automatic presentation – or equivalently that it is not interpretable in $(\mathbb{N}, +, |_2)$? In other words there is a need to characterise the isomorphism types of (classes of) automatic structures in relevant algebraic terms. For instance, what can be said about the Cantor normal form of an automatic ordinal ? Or the Cantor-Bendixson rank of an automatic tree ? This thesis presents partial answers to these types of questions. The idea is to extract algebraic information about a structure from the automata presenting it. Importantly there is no *a priori* knowledge of how the domain of a structure is coded. For example, if one fixes the base 2 encoding of $\mathbb{N}$, one may then ask for a classification of the automatic structures of the form $(\mathbb{N}, +, R)$ where $R \subset \mathbb{N}$, [compare Bruyère et al. - 1994]. But this is different from asking whether a given structure $(\mathbb{N}, +, R)$ is automatically presentable, namely if it is *isomorphic* to some automatic structure. In this case there is no predefined encoding of $\mathbb{N}$ and new techniques are required. It is also worth pointing out that Benedikt et al. [2001] prove that the structure $\mathcal{W}(\Sigma)$ does not have quantifier elimination without admitting binary functions to the signature. So in contrast to applying model theoretic means to the study of automatic structures, we also see that analysis of automata lends itself to understanding those structures that are first order definable in $\mathcal{W}(\Sigma)$.

The theory of finite automata supplies alternative types of automata that could be used in the definition of an automatic structure. Indeed ω-automata and various tree automata have been considered in the literature [Hodgson - 1983; Khoussainov and Nerode - 1995; Blumensath - 1999; Libkin and Neven - 2003]. In particular uncountable structures can also be treated. The present work focuses only on automata operating synchronously on tuples of finite words, and hence with countable structures. The reason for this restriction is that these automata are comparatively simple and the algebraic character of the corresponding automatic structures is not well understood. Furthermore binary relations are the primary focus of this work – arbitrary graphs, partial orders (linear orders, well orders and trees), equivalence relations and permutations.

A.1 Summary of results

Following is an outline of this thesis. The reader is referred to the relevant chapter for formal definitions and proofs. It is assumed that the reader is familiar with the basics of finite automata theory and of first order logic, though to fix notation some definitions and results will be repeated. Bold page numbers in the index refer to the page on which the definition of the entry can be found. Finally, since activity in the area has been recently renewed, there is some overlap of results with other researchers. If a result has been independently or subsequently proved by others then it will be cited as ‘also [citation]’.

Chapter B – Definitions and first results

The preliminary chapter starts by recalling some classical finite automata theory. These classical automata are then generalised to automata operating on n input tapes and hence adapted to recognise n-ary relations – which are called *finite automaton* (FA) *recognisable*, or simply *regular* relations (Definition B.1.2). The automata have the property that the reading heads on the n tapes move synchronously. As in the classical case, these automata can be effectively determinised. Furthermore, they are effectively closed under the operations of union, complementation and projection and the emptiness problem is decidable.

All structures are assumed to be countable and relational (that is one replaces every function f by its graph $\{(a, f(a)) \mid a \in \text{dom}(f)\}$). A structure is *automatic* over an alphabet Σ if the domain and the atomic relations are finite automaton recognisable over Σ (Definition B.1.12). A structure is called *automatically presentable* if it is isomorphic to some automatic structure over some alphabet (Definition B.1.18). These definitions are followed by some explanatory examples of automatic structures as well as some constructions of new automatic structures from old ones (Proposition B.1.22). The important decidability theorem is then extended to include the quantifiers $\exists^\infty$, due to Blumensath [1999], and $\exists^{(k,m)}$, respectively interpreted as 'there exists infinitely many' and 'there exists k modulo m many'.

Theorem B.1.26. *If $\mathcal{A}$ is automatic over Σ then there exists an algorithm that given a relation R which is first order definable (with parameters) in $\mathcal{A}$ with additional quantifiers $\exists^\infty$ and $\exists^{(k,m)}$ $(0 \leq k < m \in \mathbb{N})$ constructs an automaton recognising R. Hence the first order theory with these additional quantifiers of $\mathcal{A}$ is decidable.*

The chapter finishes with examples of automatic structures, most of which make an appearance later.

Chapter C – Characterisations of automatic structures

This chapter presents the known logical and structural characterisations of automatic structures. The first section contains a decomposition theorem for FA recognisable relations. The second section deals with those structures and their logics that are complete for the class of automatic structures (Definition C.2.1).

Denote by $\mathcal{W}_k$ the structure

$$(\Sigma^\star, (\sigma_a)_{a\in\Sigma}, \preceq_p, el),$$

where $\Sigma = \{0, \cdots, k-1\}$, $\sigma_a(w) = wa$, $\preceq_p$ is the prefix relation, and $el(w, v)$ holds if w and v have the same length. Denote by $\mathcal{N}_k$ the structure

$$(\mathbb{N}, +, |_k)$$

where $x|_k y$ holds if x is a power of k and x divides y. Denote by S the successor function $n \mapsto n+1$ on $\mathbb{N}$. For a logic $\mathcal{L}$, say first order or second order, a structure $\mathcal{A}$ is $\mathcal{L}$ interpretable in a structure $\mathcal{B}$ if $\mathcal{A}$ is isomorphic to a structure that is $\mathcal{L}$ definable in $\mathcal{B}$.

Theorem C.2.2. *Let $\mathcal{A}$ be a structure. Then the following are equivalent.*

1. *$\mathcal{A}$ is automatically presentable.*
2. *$\mathcal{A}$ is first order interpretable in $\mathcal{W}_k$ for some, equivalently all, $k \geq 2$.*
3. *$\mathcal{A}$ is first order interpretable in $\mathcal{N}_k$ for some, equivalently all, $k \geq 2$.*
4. *$\mathcal{A}$ is weak monadic second order interpretable in $(\mathbb{N}, S)$.*

This theorem allow one to interchange between working with finite automata and logical formulae. This is most useful when constructing automatic presentations of structures.

The couching in terms of interpretability rather than definability follows Blumensath and Grädel [2000]. The equivalence of $1.$ and $4.$ is due to Büchi [1960] and Elgot [1961] and known as the Büchi-Elgot Theorem. The proof proceeds by translating formula into automata and conversely. The equivalence of $1.$ and $2.$ first appeared in Eilenberg et al. [1969]. The equivalence of $1.$ and $3.$ is known as the Büchi-Bruyère theorem. Following the terminology in Bruyère et al. [1994], a relation $R \subset \mathbb{N}^n$ is defined as k-recognisable if the encoding of its elements in the least-significant-digit-first base k representation yields a finite automaton recognisable relation (over alphabet $\Sigma = \{0, \cdots, k-1\}$).

Theorem C.2.5. *A relation $R \subset \Sigma^{\star m}$ is k-recognisable if and only if it is first order definable in the structure $(\mathbb{N}, +, |_k)$.*

A Myhill-Nerode type result (Theorem C.3.2) due to Khoussainov and Nerode [1995], characterises the automatic structures in terms of congruences of finite index.

The final section presents a reduction of automatic structures to automatic graphs. This follows immediately from the existence of a complete structure under interpretations, and the mutual interpretation of a structure with a graph. Two structures are *mutually interpretable* if each is interpretable in the other (definitions in Section B.1).

Theorem C.4.1. *For every structure $\mathcal{A}$ there is a graph $\mathcal{G}(\mathcal{A})$ with the following properties.*

1. *$\mathcal{A}$ is automatic if and only if $\mathcal{G}(\mathcal{A})$ is automatic. And an automatic presentation of $\mathcal{G}(\mathcal{A})$ can be constructed in linear time in the size of an automatic presentation of $\mathcal{A}$.*
2. *$\mathcal{A}$ and $\mathcal{G}(\mathcal{A})$ are mutually interpretable.*
3. *The interpretation of $\mathcal{A}$ into $\mathcal{G}(\mathcal{A})$ preserves embeddings. That is $\mathcal{A}$ embeds in $\mathcal{B}$ if and only if $\mathcal{G}(\mathcal{A})$ embeds in $\mathcal{G}(\mathcal{B})$. Moreover the interpretation preserves FA recognisable embeddings.*

Chapter D – Unary vs. non-unary

In order to understand the isomorphism types of automatic structures, we first consider the simpler case of structures that can be presented by automata over a unary alphabet. These are called *unary automatic* structures. Some examples are then followed by a characterisation of unary automatic graphs. In a simple case, the characterisation essentially says that the graph can be partitioned into infinitely many finite isomorphic subgraphs B_i, such that the edge relation between the subgraph B_i and B_j mimic that between B_0 and B_1, for all $i, j \in \mathbb{N}$. Such a graph is called an unwinding (Definition D.1.7). Examples of unwindings are given, followed by a proof of the characterisation theorem.

Theorem D.1.1. [also Blumensath - 1999] *A graph is automatically presentable over a unary alphabet if and only if it is isomorphic to some unwinding.*

The section concludes with the classification of certain classes of unary automatic structures. The main tool restricts their connected components as follows.

Lemma D.1.13. [also Blumensath - 1999] *If $\mathcal{G}$ is an unwinding, then it contains a* finite *number of infinite connected components, and a* finite bound *on the sizes of the finite connected components.*

For example, using this lemma one can characterise the unary automatic equivalence structures (E, ρ) (here ρ is an equivalence relation on E) and the unary automatic injection structures (D, f) (here $f : D \to D$ is a one-to-one mapping).

Theorem D.1.14, D.1.16. [also Blumensath - 1999]

1. *An equivalence structure (E, ρ) is automatically presentable over a unary alphabet if and only if there are finitely many infinite ρ-classes and there exists $\kappa < \omega$ such that the size of every finite ρ-class is less than κ.*

2. *A injection structure (D, f) is automatically presentable over a unary alphabet if and only if there are finitely many infinite f-orbits and there exists $\kappa < \omega$ such that the size of every finite f-orbit is less than κ.*

The unary automatic linear orders can be built from orders of the following types: the positive integers (ω), the negative integers (ω^*), and the finite orders $\mathbf{n}$ for $n < \omega$.

Theorem D.1.19. [also Blumensath - 1999] *A linear order $(L, \leq)$ is automatically presentable over a unary alphabet if and only if it is a finite sum of linear orders amongst the set ω, ω^*, and $\mathbf{n}$, for $n < \omega$.*

In particular, the order type of the rationals η is not automatically presentable over a unary alphabet. Also the least ordinal that is not automatically presentable over a unary alphabet is ω^2.

The second section illustrates the complexities involved in finding similar algebraic classifications in the non-unary case. Automatic equivalence relations and automatic injections are investigated. We present examples that indicate that in comparison with the unary case a classification of these classes is non-trivial.

Theorem D.2.5. *For every function $f : \mathbb{N} \to \mathbb{N}$ that is either a polynomial with coefficients in $\mathbb{N}$, or an exponential k^{an+b} with $k \geq 2$ and $a, b \in \mathbb{N}$, there exists an automatic equivalence relation $\mathcal{E} = (E, \rho)$ such that there is exactly one ρ-class of size $f(n)$ for every $n \in \mathbb{N}$.*

The idea is that if $R \subset \Sigma^\star$ is a regular language, then one can build an automatic equivalence relation that has an equivalence class of size $|R \cap \Sigma^n|$ for every $n \in \mathbb{N}$. But for every polynomial or exponential function f as above, there is a regular language R_f such that $f(n) = |R_f \cap \Sigma^n|$ for every n.

The class of automatic equivalence relations is then shown to be closed under certain natural operations. More specifically, the *height function* f of an equivalence relation has domain $\mathbb{N}$ and $f(n)$ is defined as the number of equivalence classes of size n. It is shown that the height functions of automatic equivalence relations are closed under addition, Dirichlet convolution and Cauchy product.

One measure of the complexity of characterising a class $\mathcal{C}$ of automatic structures is the isomorphism problem; namely, given automatic presentations of two structures in $\mathcal{C}$, are the structures isomorphic or not ? It is unknown whether the isomorphism problem for automatic equivalence relations is decidable; it is clearly in Π^0_1.

The undecidability of the isomorphism problem for automatic structures first appeared in Blumensath and Grädel [2002].

Theorem D.2.11. *The isomorphism problem for automatic permutation structures is undecidable.*

Chapter E – Automatic linear orders and trees

The classical ranking of a linear order $\mathcal{L}$ is based on iteratively factoring $\mathcal{L}$ by the equivalence relation stating that x is equivalent to y when the number of elements between x and y is finite, [see Rosenstein - 1982]. The least number of times, in general an ordinal, before a fixed point is reached is called the FC–rank of $\mathcal{L}$ (finite condensation). For example, the FC–rank of the ordinal ω^α is α.

Delhommé [2001a] proved that the least ordinal without an automatic presentation is ω^ω. We generalise this result to ranks of linear orders and trees.

Theorem E.2.7. *The FC–rank of every automatic linear order is finite.*

The proof proceeds via an analysis of the state diagram of an automaton presenting the linear order.

Corollary E.3.3. *The isomorphism problem for automatic ordinals is decidable.*

This theorem is then applied to trees, viewed as partial orders $(T, \preceq)$. One measure of the complexity of a tree is its Cantor-Bendixson rank. The idea is that the Kleene-Brouwer ordering of an automatic tree is FA recognisable, since it is FO definable from the tree $\mathcal{T}$ and the length-lexicographical ordering on T which is FA recognisable. Applying the previous theorem results in the following necessary condition for the automaticity of trees.

Theorem E.5.9. *The CB–rank of every automatic tree $\mathcal{T} = (T, \preceq)$ is finite.*

This is then applied to an analysis of the complexity of infinite paths in automatic trees. The starting point is the observation of an automatic version of König's Lemma.

Theorem E.6.2. *If $\mathcal{T} = (T, \preceq)$ is an infinite finitely branching automatic tree then it has a regular infinite path. That is, there exists a regular set $P \subset T$ so that P is an infinite path of $\mathcal{T}$.*

This is due to the fact that the length-lexicographically least infinite path is definable from an automatic presentation of $\mathcal{T}$ and the FA recognisable length-lexicographical order on T.

Clearly if such an automatic tree has finitely many infinite paths, then each must be regular. This is then generalised to countably many infinite paths and with some more work the assumption of being finitely branching can be dropped.

Theorem E.6.6. *If an automatic tree has countably many infinite paths, then every infinite path in it is regular.*

Chapter F – Classifying automatic structures

To show that a given structure is not automatic, one needs some general understanding of the way that automata present structures. A relation $R \subset A^{k+l}$ is *locally finite* if for every $\overline{x}$ of length k there are at most a finite number of $\overline{y}$ of length l such that $(\overline{x}, \overline{y}) \in R$.

Proposition F.1.2. *Suppose that $R \subset A^{k+l}$ is a locally finite FA recognisable relation. There exists a constant p, that depends only on the automaton for R, such that*

$$\max\{|y| \mid y \in \overline{y}\} - \max\{|x| \mid x \in \overline{x}\} \leq p$$

for every $(\overline{x}, \overline{y}) \in R$.

With more reasoning the proposition is used in a proof that there is no automatic infinite field. Also the automatic Boolean algebras are classified.

Theorem F.1.12. *A Boolean algebra is automatically presentable if and only if is isomorphic to the interval algebra of α for some ordinal $\alpha < \omega^2$.*

Corollary F.1.13. *The isomorphism problem for automatic Boolean algebras is decidable.*

Then next section contains another technique that is used to prove the non-automaticity of some ubiquitous algebraic structures, in particular certain universal homogeneous structures.

Corollary F.2.10, F.2.14, F.2.16. *The following structures are not automatically presentable.*

1. *The random graph.*
2. *The universal, homogeneous partial order.*
3. *The C_p-free random graph for every $p > 2$ (C_p is the complete graph on p vertices).*

These results indicate that the class of automatic structures is quite limited. Recall that the isomorphism problem for a class $\mathcal{C}$ of automatic structures is: given automata presentations of two automatic structures from $\mathcal{C}$, are the structures isomorphic or not ?

It is not surprising that the isomorphism problem for the class of all automatic structures is undecidable. The reason for the undecidability is that the configuration space of a Turing machine considered as a graph is an automatic structure, and the reachability problem in the configuration space is undecidable. Thus one can reduce the reachability problem to the isomorphism problem for automatic structures. Similar methods yield the following.

A *locally finite directed graph* is one for which in the underlying undirected graph every vertex is incident with only finitely many vertices.

Proposition F.3.1. *The complexity of the isomorphism problem for automatic locally finite directed graphs is Π^0_3-complete.*

In particular the complexity of the isomorphism problem for configuration spaces of Turing machines is Π^0_3. Hence it was unexpected, to us at least, that the complexity of the isomorphism problem for the class of all automatic structures is Σ^1_1-complete. The Σ^1_1-completeness is proved by reducing the isomorphism problem for computable trees, known to be Σ^1_1-complete, see Goncharov and Knight [2002], to the isomorphism problem for automatic structures.

Theorem F.3.5. *The complexity of the isomorphism problem for automatic structures is Σ^1_1-complete.*

Chapter G – Intrinsic regularity

This chapter investigates the relationship between regularity and definability in automatic structures. A relation R on the domain of an automatically presentable structure $\mathcal{A}$ is called *intrinsically regular* in $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to an automatic structure $\mathcal{B}$ is finite automaton recognisable (Definition G.1.1). If a relation R is first order definable with additional quantifiers $\exists^\infty$ and $\exists^{(k,m)}$ in an automatic structure $\mathcal{A}$, then R is intrinsically regular in $\mathcal{A}$. This is because in a given automatic presentation of $\mathcal{A}$, one may extract an automaton for the image of R from the given definition (Theorem B.1.26). The following are converses to this. Denote by $\mathrm{IR}(\mathcal{A})$ the set of intrinsically regular relations in $\mathcal{A}$, and $\mathrm{FO}^{\infty,\mathrm{mod}}(\mathcal{A})$ the set of relations that are first order definable with quantifiers $\exists^{\infty}$ and $\exists^{(k,m)}$ in $\mathcal{A}$.

Proposition G.2.1, G.2.2, G.2.3.

1. $\mathrm{IR}(\mathbb{N},+,|_m) = \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N},+,|_m)$, *for* $m > 1$.
2. $\mathrm{IR}(\mathbb{N},+) = \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N},+)$.
3. $\mathrm{IR}(\mathbb{N},\leq) = \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N},\leq)$.

The first item follows from the completeness of $\mathrm{IR}(\mathbb{N},+,|_m)$ for the class of FA recognisable relations. The second item follows from the Cobham-Semenov Theorem that states that if the image of R in base k is FA recognisable and the image of R in base l is FA recognisable, and $k^m \neq l^n$ for every $m, n \in \mathbb{N} \setminus \{0\}$, then R is first order definable in $(\mathbb{N},+)$. The third item follows from the characterisation of structures whose domain consists of strings over a single letter alphabet, due to Nabebin [1976]; Blumensath [1999]. These are those that are first order definable in $(\mathbb{N},\leq,(\equiv_m)_{m\in\mathbb{N}})$, where $\equiv_m$ is the binary relation of congruence modulo m.

For the structure $(\mathbb{N},S)$ we have only proved that a unary relation $R \subset N$ is intrinsically regular in $(\mathbb{N},S)$ if and only if $R \in \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N},S)$ (Corollary G.2.5). This follows from the fact that for every $k \geq 2$, there is an automatic presentation of $(\mathbb{N},S)$ in which the image of the set $\{n \in \mathbb{N} \mid k \text{ divides } n\}$ is not regular (Theorem G.2.4). The proof of this theorem may be adapted to yield automatic presentations with pathological properties:

Corollary G.2.7. *There is an automatic presentation of a graph with exactly two components, each of which is not regular.*

A *cut* of the structure $(\mathbb{Z},S)$ is a set of the form $\{x \in \mathbb{Z} \mid x \geq n\}$ where $n \in \mathbb{Z}$ is fixed.

Corollary G.2.8. *There is an automatic presentation of* $(\mathbb{Z},S)$ *in which no cut is regular.*

Chapter H – Open questions

This chapter contains a sample of possible directions for future work.

A.2 A chronological bibliography

Here is a chronological bibliography and brief description of closely related work.

Büchi [1960] and Elgot [1961] prove the equivalence between weak monadic second order logic and automata synchronously operating on tuples finite words.

Elgot and Mezei [1965] study closure properties of subclasses of rational transductions. Their ‘finite automaton definable relations’ coincide with our FA recognisable relations.

Elgot and Rabin [1966] study decidable and undecidable extensions of WMSO$(\mathbb{N}, S)$ as well as its bi-interpretability with $\mathcal{W}_2 = (\{0,1\}^\star, \sigma_0, \sigma_1, \preceq_p, el)$.

Eilenberg, Elgot, and Shepherdson [1969] prove that a relation $R \subset \Sigma^{\star n}$ is definable in the structure $\mathcal{W}(\Sigma) = (\Sigma^\star, (\sigma_a)_{a\in\Sigma}, \preceq_p, el)$ if and only if it is regular.

Hodgson [1976] introduces the term ‘automatic structure’ and studies decidability of structures presented by automata on finite and infinite words. He also discusses closure properties under certain products.

Johnson [1986] studies equivalence relations represented by a variety of automata.

Cannon, Epstein, Holt, Levy, Paterson, and Thurston [1992] is the standard reference for automatic groups.

Bruyère, Hansel, Christian, and Villemaire [1994] is an excellent exposition of the Büchi-Bruyère Theorem and related results.

Khoussainov and Nerode [1995] reintroduce automatic structures as part of complexitytheoretic model theory and initiate a systematic development of their model-theoretic, logical and algebraic properties.

Blumensath [1999]; Blumensath and Grädel [2000] explore the theory of automatic structures from a logical point of view. They provide many of the fundamental theorems. The former is an excellent introduction to automatic structures over words, ω–words and trees.

Rubin [1999]; Khoussainov and Rubin [2001] study unary automatic structures.

Blumensath [2001]; Blumensath and Grädel [2002] study a generalisation of unary automatic structures. They present several equivalent characterisations for graphs, such as the prefix-recognisable graphs.

Delhommé [2001a,b] proves that neither ω^ω nor the random graph have automatic presentations.

Benedikt, Libkin, Schwentick, and Segoufin [2001] compare the model theoretic properties, such as quantifier elimination and VC dimension, and complexity of evaluating formulae in subsystems of $\mathcal{W}(\Sigma)$.

Ishihara, Khoussainov, and Rubin [2002] presents some constructions of automatic structures.

Klaedtke [2003] presents an optimal decision procedure for Presburger Arithmetic based on automata theoretic considerations.

Khoussainov, Rubin, and Stephan [2003a] presents the finite rank arguments for linear orders and trees, as well as automatic versions of König’s Lemma and some variations.

Khoussainov, Rubin, and Stephan [2003b] study intrinsic regularity in subsystems of $(\mathbb{N}, +, |_2)$.

Kuske [2003] studies two automatic presentations of $(Q, \leq)$ and discusses issues of automatic versions of results related to Cantor’s Theorem.

Delhommé, Goranko, and Knapik [2003] consider automatic linear orders.

Ly [2003] consider automatic graphs and DOL-sequences of finite graphs.

Lohrey [2003] considers automatic structures of bounded degree.

Khoussainov and Rubin [2003] present an overview of the area and outline possible future work.

Khoussainov, Nies, Rubin, and Stephan [2004] characterise the automatic Boolean algebras, show that some Fraïssé limits have no automatic presentations, and prove that the isomorphism problem for automatic structures is Σ_1^1-complete.

Chapter B

Definitions and first results

The purpose of this chapter is to define the concept of an automatic structure (Definition B.1.12) and establish the important decidability result (Theorem B.1.17) and its extension (Theorem B.1.26). The second section contains some examples of automatic structures, most of which will be encountered and used in later chapters. A passing familiarity with the basics of finite automata and first order logic is assumed.

B.1 Finite automata and automatic structures

Finite automata

For the sake of completeness and in order to fix notation for the rest of the thesis, the basic definitions and results from automata theory are briefly repeated here.

Write $\mathbb{N}$ for the natural numbers including 0, and 2^Q for the powerset of a set Q. Denote by Σ a finite set of symbols, called an *alphabet*. The set of finite strings, also called *words*, from Σ is written $\Sigma^\star$ and λ denotes the string with no symbols. The length of a string w is denoted by $|w|$. The concatenation of strings w and v is written $w \cdot v$, or also wv.

A $($*non-deterministic*$)$ *finite automaton* $\mathcal{A}$ over an alphabet Σ is a tuple (S, ι, Δ, F), where S is a finite set of *states*, $\iota \in S$ is the *initial state*, $\Delta \subset S \times \Sigma \times S$ is the *transition table* and $F \subset S$ is the set of *final states*. A *run* $($*or computation*$)$ of $\mathcal{A}$ on a word $\sigma_1\sigma_2\cdots\sigma_n$ ($\sigma_i \in \Sigma$) is a sequence of states, say $q_0, q_1, \cdots, q_n$, such that $q_0 = \iota$ and $(q_i, \sigma_{i+1}, q_{i+1}) \in \Delta$ for all $i \in \{0, 1, \ldots, n-1\}$. If $q_n \in F$, then the computation is *successful* and we say that automaton $\mathcal{A}$ *accepts* $($*or recognises*$)$ the word. The *language* accepted by the automaton $\mathcal{A}$ is the set of all words accepted by $\mathcal{A}$. In general, $D \subset \Sigma^\star$ is *finite automaton* $($*FA*$)$ *recognisable*, or *regular*, if D is equal to the language accepted by $\mathcal{A}$ for some finite automaton $\mathcal{A}$. Two automata are called *equivalent* if they accept the same language. An automaton is called *deterministic* if for every $s \in S$ and $\sigma \in \Sigma$, there is exactly one $t \in S$ with $\Delta(s, \sigma, t)$. Deterministic automata have the property that for every word $w \in \Sigma^\star$ there is a unique run of $\mathcal{A}$ on w. For a deterministic

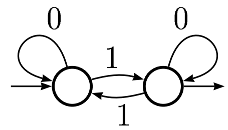

Figure B.1: An automaton accepting those strings over $\{0, 1\}$ that have an odd number of 1's.

automaton, the transition table may be written in functional notation, for instance $\Delta(s, \sigma) = t$. The natural extension of the transition function to domain $S \times \Sigma^\star$ is also denoted by Δ. Hence in this case we write $\Delta(s, w) = t$ for $w \in \Sigma^\star$.

We briefly mention Büchi automata as they will be used in a lemma in Chapter E. A non-deterministic Büchi automaton. (S, ι, Δ, F) over Σ accepts an infinite string $\alpha \in \Sigma^\omega$ if it has a run $(q_i)_{i \in \mathbb{N}}$ such that there is some state $f \in F$ with $f = q_j$ for infinitely many $j \in \mathbb{N}$. These automata were introduced in Büchi [1962]; for a more recent presentation see for instance Khoussainov and Nerode [2001].

Finite automata are usually pictured as directed labelled graphs. The states are represented as vertices, there is an arrow going into the initial state, arrows leaving the final states, and a transition (s, σ, t) is represented as a directed edge from vertex s to vertex t labelled with σ.

One classical method for proving that a language is not regular is the Pumping Lemma.

Pumping Lemma. *If $L \subset \Sigma^\star$ is regular then there exists a constant k such that for every word $x \in L$ and every factorisation $x = abc$ with $|b| > k$, there exists a factorisation uvw of b with $|v| > 0$ such that $auv^nwc \in L$ for all $n \in \mathbb{N}$.*

An important characterisation of regular languages is that they are exactly the languages of regular expressions:

Kleene's Theorem. *A language $L \subset \Sigma^\star$ is regular if and only if it can be obtained from the individual elements $\{\sigma\}$, for $\sigma \in \Sigma$, by applying a finite number of the rational operations, namely union, product and Kleene star.*

Here the product $L_1 \cdot L_2$ of languages is defined as $\{xy \mid x \in L_1 \text{ and } y \in L_2\}$. Define $L^0 = \{\lambda\}$ and L^{n+1} as $L \cdot L^n$. Then the Kleene star $L^\star$ is defined as $\cup_{n \in \omega} L^n$.

Classically finite automata recognise sets of words, namely unary relations. A finite automaton can be thought of as a one-way Turing machine with one input tape, that operates in constant space, see for instance Odifreddi [1999, VIII.1]. If one admits more than one such tape, say n many, then the language computed by such a machine is an n-ary relation. The resulting relations are called *rational relations* in the literature. In contrast to the case $n = 1$, the nondeterministic rational relations strictly contain the deterministic rational relations. This thesis deals with a particular class of deterministic rational relations, namely those recognisable by *synchronous n–tape automata* , that essentially first appears in the work of Büchi [1960] and Elgot [1961]. Here 'synchronous' refers to the movement of the n tapes. The following informal description follows Eilenberg et al. [1969]. A synchronous n–tape automaton can be thought of as a one-way Turing machine with n input tapes. Each tape is regarded as semi-infinite having written on it a word in the alphabet Σ followed by an infinite succession of blanks, $\perp$ symbols. The automaton starts in the initial state, reads simultaneously the first symbol of each tape, changes state, reads simultaneously the second symbol of each tape, changes state, etc., until it reads a blank on each tape. The automaton then stops and accepts the n–tuple of words if it is in a final state. The set of all n–tuples accepted by the automaton is the relation recognised by the automaton. Here is a formalisation.

Definition B.1.1 [cf. Büchi - 1960; Elgot - 1961] *Write $\Sigma_\perp$ for $\Sigma \cup \{\perp\}$ where $\perp$ is a symbol not in Σ. The* **convolution of a tuple** *$(w_1, \cdots, w_n) \in \Sigma^{\star n}$ is the string $\otimes(w_1, \cdots, w_n)$ of length $\max_i |w_i|$ over alphabet $(\Sigma_\perp)^n$ defined as follows. Its k-th symbol is $(\sigma_1, \ldots, \sigma_n)$ where σ_i is the k-th symbol of w_i if $k \leq |w_i|$ and $\perp$ otherwise.*

The **convolution of a relation** *$R \subset \Sigma^{\star n}$ is the relation $\otimes R \subset (\Sigma_\perp)^{n\star}$ formed as the set of convolutions of all the tuples in R. That is, $\otimes R = \{\otimes w \mid w \in R\}$.*

For example, for $\Sigma = \{1\}$, the convolution of $\{(1^n, 1^{n+1}) \mid n \in \mathbb{N}\}$ is $\{\binom{1}{1}^n \binom{\perp}{1} \mid n \in \mathbb{N}\}$. Note that we often write symbols of $(\Sigma_\perp)^n$ vertically. Also $\otimes(\lambda, \ldots, \lambda) = \lambda$.

Definition B.1.2 *An n–***tape automaton** *on Σ is a finite automaton over the alphabet $(\Sigma_\perp)^n$. An n–ary relation $R \subset \Sigma^{\star n}$ is* **finite automaton recognisable** *or* **regular** *if its convolution $\otimes R$ is recognisable by an n–tape automaton.*

1–tape automata coincide with the usual finite automata. In this case, $\otimes R = R \subset \Sigma^\star$. We postpone some examples of regular relations of higher arity until after the following useful constructions.

The next theorem says that n–tape automata can be transformed into deterministic automata.

Theorem B.1.3 *There is an effective procedure that given a non-deterministic n–tape automaton constructs an equivalent deterministic n–tape automaton.*

Proof Classically, 1–tape automata can be transformed into deterministic automata via the subset construction as follows. Given a non-deterministic automaton (S, ι, Δ, F) construct an equivalent deterministic automaton $(S', \iota', \Delta', F')$ where $S' = 2^S$ (subsets of S), $\iota' = \{\iota\}$, $\Delta'(Q, \sigma) = \bigcup_{q \in Q} \Delta(q, \sigma)$ and $F' = \{Q \in S' \mid Q \cap F \neq \emptyset\}$. Hence one can apply the subset construction to n–tape automata, since these are 1–tape automata over an alphabet of the form $(\Sigma_\perp)^n$. $\lhd$

The next definition introduces important operations on relations.

Definition B.1.4 *Let $R \subset \Sigma^{\star n}$.*

1. *The* **projection**, $\exists R$, *of the first co-ordinate of* R *is*

$$\{(x_2, \cdots, x_n) \mid (\exists x_1)\,(x_1, \cdots, x_n) \in R\}.$$

By convention if $n = 1$ *then* $\exists R$ *is empty if* R *is empty and* $\{\lambda\}$ *otherwise.*

2. *The* **instantiation** *of the first co-ordinate of* R *is*

$$\{(x_2, \cdots, x_n) \mid (a, x_2, \cdots, x_n) \in R\},$$

for a fixed $a \in \Sigma^\star$.

3. *The* **cylindrification** *of the first co-ordinate of* R *is*

$$\{(x_1, \cdots, x_{n+1}) \mid (x_2, \cdots, x_{n+1}) \in R, x_1 \in \Sigma^\star\}.$$

4. *The* **permutation of co-ordinates** *of* R *is*

$$\{(x_{\pi(1)}, \cdots, x_{\pi(n)}) \mid (x_1, \cdots, x_n) \in R\}$$

where π *is a fixed permutation of* $\{1, \cdots, n\}$.

Theorem B.1.5 *Let* $R_1, R_2 \subset \Sigma^{\star n}$ *be FA recognisable. Then the following relations are FA recognisable by automata that can be constructed given automata for* R_1 *and* R_2.

1. *The union* $R_1 \cup R_2$, *intersection* $R_1 \cap R_2$, *relative complementation* $R_1 \setminus R_2$,
2. *the projection of the first co-ordinate of* R_1,
3. *the instantiation of the first co-ordinate of* R_1,
4. *the cylindrification of the first co-ordinate of* R_1, *and*
5. *the permutation of co-ordinates of* R_1.

Proof Let $\mathcal{A}_i = (S_i, \iota_i, \Delta_i, F_i)$ be automata over Σ, recognising R_i, for $i = 1, 2$. Assume these automata are deterministic and that $S_1 \cap S_2 = \emptyset$.

1. Form the product automaton $\mathcal{A} = (S_1 \times S_2, \iota, \Delta, F_{\mathrm{OP}})$ over Σ, where $\iota = (\iota_1, \iota_2)$ and $\Delta((q, r), \sigma) = (\Delta_1(q, \sigma), \Delta_2(r, \sigma))$. For $\mathrm{OP} \in \{\cup, \cap, \setminus\}$, define F_{OP} as

$$(F_1 \times S_2)\mathrm{OP}(S_1 \times F_2).$$

Note that the product automaton is deterministic. Then for example with $\mathrm{OP} = \cup$, automaton $\mathcal{A}$ recognises $\otimes(R_1 \cup R_2)$ since $\mathcal{A}$ has a successful run on $\otimes w$ if and only if $\mathcal{A}_1$ has a successful run on $\otimes w$ or $\mathcal{A}_2$ has a successful run on $\otimes w$. Namely, $L(\mathcal{A}) = \otimes R_1 \cup \otimes R_2$ which is $\otimes(R_1 \cup R_2)$ from the definition of convolution. The cases of intersection $\cap$ and relative complement $\setminus$ are similar.

2. The required automaton is $\mathcal{A} = (S_1, \iota_1, \Delta, F)$ where $q \in \Delta(s, (\sigma_2, \cdots, \sigma_n))$ if and only if $\Delta_1(s, (\sigma_1, \cdots, \sigma_n)) = q$ for some $\sigma_1 \in \Sigma_\perp$. Define F as those states of $\mathcal{A}_1$ from which there is a path to a final state labelled $\otimes(x, \lambda, \cdots, \lambda)$ where $x \in \Sigma_\perp^\star$ and λ is the empty string. Note that this automaton is, in general, non-deterministic.

   Then $\mathcal{A}$ has a successful run on $\otimes(x_2, \cdots, x_n)$ if and only if there is a path labelled $\otimes(y_1, x_2, \cdots, x_n)$ in $\mathcal{A}_1$ from the initial state to some state q for some y_1 and a path labelled $\otimes(x_1, \lambda, \cdots, \lambda)$ in $\mathcal{A}_1$ from q to a final state for some x_1, if and only if there is a successful run in $\mathcal{A}_1$ on $\otimes(y_1 \cdot x_1, x_2, \cdots, x_n)$ for some $y_1 \cdot x_1$. So $L(\mathcal{A}) = \{\otimes\overline{x} \mid (\exists y) \otimes (y, \overline{x}) \in \otimes R_1\}$ which is $\otimes\{\overline{x} \mid (\exists y)\, (y, \overline{x}) \in R_1\}$, namely the convolution of the projection of R_1. Note that if $n = 1$ then $\mathcal{A}$ accepts no string if R_1 is empty and $\{\lambda\}$ if R_1 is not empty.

3. For a given $a \in \Sigma^\star$, transform automaton $\mathcal{A}_1$ into an equivalent automaton $\mathcal{B}$ with the property that every two paths of $\mathcal{B}$ starting from the initial state have only got the initial state in common amongst their first $|a|+1$ states. Write P for the set of paths of $\mathcal{B}$ starting in the initial state and containing $|a| + 1$ states. Construct an intermediate automaton $\mathcal{B}'$ by applying the following operations to $\mathcal{B}$. First remove from $\mathcal{B}$ all those edges on finite paths of P that satisfy the condition that the first component of the path is not labelled by a. Secondly if $s_1, \ldots, s_{|a|+1}$ is a path of P that is not removed in the previous step then delete every edge leaving $s_{|a|+1}$ where the first component of the edge's label is not $\perp$. Finally for every path of P not removed in the first step ensure that no state amongst its first $|a|$ states is an accept state, by removing it from the accepting set if necessary. Then $\mathcal{B}'$ has a successful run on $\otimes(x_1, x_2, \cdots, x_n)$ if and only if $x_1 = a$ and there is an accepting path in $\mathcal{B}$ (and hence $\mathcal{A}_1$) labelled by $\otimes(a, x_2, \cdots, x_n)$. Now project the first co-ordinate to get the required automaton.

4. Form an automaton $\mathcal{B} = (S_1, \iota_1, \Delta, F_1)$ where for every $\sigma \in \Sigma_\perp$,

   $$\Delta(s, (\sigma, \sigma_2, \cdots, \sigma_{n+1})) = \Delta_1(s, (\sigma_2, \cdots, \sigma_{n+1})).$$

   Now the required automaton can be formed by intersecting $\mathcal{B}$ and an automaton for $\otimes(\Sigma^{\star(n+1)})$, as in part 1.

5. If $\sigma = (\sigma_1, \cdots, \sigma_n) \in \Sigma_\perp^n$ then define $\pi(\sigma)$ as $(\sigma_{\pi(1)}, \cdots, \sigma_{\pi(n)})$. The required automaton is $\mathcal{A} = (S_1, \iota_1, \Delta, F_1)$ where $\Delta(s, \pi(\sigma)) = q$ if and only if $\Delta_1(s, \sigma) = q$.

   Then $\mathcal{A}$ has a successful run on $(x_{\pi(1)}, \cdots, x_{\pi(n)})$ if and only if $\mathcal{A}_1$ has a successful run on $(x_1, \cdots, x_n)$, since every transition is relabelled accordingly. $\lhd$

The emptiness problem asks whether given an automaton $\mathcal{A}$, there exists a word $w \in \Sigma^\star$ accepted by $\mathcal{A}$.

Theorem B.1.6 *The emptiness problem for n–tape automata is decidable.*

Proof $\mathcal{A}$ accepts some word if and only if there is a path from the initial state of $\mathcal{A}$ to a final state. This can be effectively tested in linear time using a breadth-first search. $\lhd$

Example B.1.7 *The universe $\Sigma^{\star n}$ is FA recognisable.*
Figure B.2 illustrates the case $\Sigma = \{0, 1\}$ and $n = 2$. $\lhd$

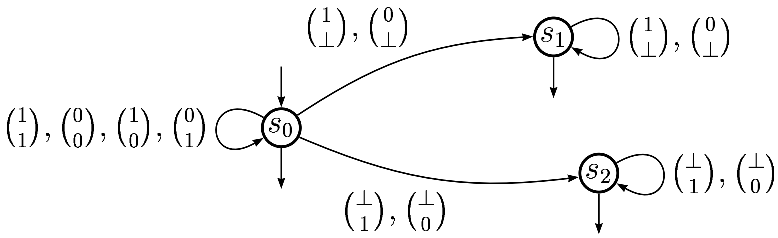

Figure B.2: An automaton recognising $\otimes(\Sigma^{\star 2})$.

This example is useful since we may build an automaton recognising some $R \subset (\Sigma_\perp^n)^\star$ so that $R \cap \otimes\Sigma^{\star n}$ is itself the convolution of some relation.

Example B.1.8 *For $k > 1$, consider strings over alphabet $\Sigma_k = \{0, 1, \cdots, k-1\}$ as least-significant-digit-first base-k representations of natural numbers. Then base-k addition is FA recognisable.*

Fix $k > 1$ and define a function $\mathrm{val}_k : \Sigma^\star \to \mathbb{N}$ as mapping the string $x_0 \cdots x_k$ to $\sum_{0 \le i \le k} x_i k^i$. Then the ternary relation $+(a, b, c) \subset \Sigma^{\star 3}$ satisfying those triples such that $\mathrm{val}_k(a) + \mathrm{val}_k(b) = \mathrm{val}_k(c)$ is FA recognisable over Σ_k. The idea is to implement the usual algorithm for addition with an automaton that uses its states to remember the carry. Formally, the automaton $\mathcal{A} = (S, \iota, \Delta, F)$ has $S = \{s_0, s_1, \cdots, s_{k-1}\}$, $\iota = s_0$, $F = \{s_0\}$, and for $(\sigma_a, \sigma_b, \sigma_c) \in \Sigma_\perp^3$,

$$\Delta(s_i, (\sigma_a, \sigma_b, \sigma_c)) = \{s_j \mid i + \sigma_a + \sigma_b = \sigma_c + kj\},$$

where $\perp$ is treated as 0 in the sum. Finally, to bring the automaton into the correct form (where no symbol from $\{0, 1\}$ follows a $\perp$), we intersect $\mathcal{A}$ with an automaton for $\otimes(\Sigma^{\star 3})$. $\lhd$

The following are important examples of FA recognisable orders.

Definition B.1.9 *Let Σ be a finite alphabet. For $x, y \in \Sigma^\star$ say that x is a* **prefix** *of y and write $x \preceq_p y$ if $y = xz$ for some $z \in \Sigma^\star$. If z is not the empty string λ then x is a* **proper prefix** *of y, written $x \prec_p y$. Then $\prec_p$ partially orders $\Sigma^\star$.*

Fix a linear ordering $<$ on Σ. Denote the **lexicographic ordering** *induced by $<$ as $<_{lex}$. Namely for $x, y \in \Sigma^\star$, $x <_{lex} y$ if either*

- *$x \prec_p y$, or*

- $x = a\sigma w$ *and* $y = a\delta z$ *for some* $a, w, z \in \Sigma^\star$, $\sigma, \delta \in \Sigma$ *with* $\sigma < \delta$.

Denote the **Kleene-Brouwer order** *induced by* $\prec_p$ *as* $<_{kb}$. *Namely* $x <_{kb} y$ *if either*

- $y \prec_p x$, *or*
- $x = a\sigma w$ *and* $y = a\delta z$ *for some* $a, w, z \in \Sigma^\star$, $\sigma, \delta \in \Sigma$ *with* $\sigma < \delta$.

Then $<_{kb}$ *linearly orders* $\Sigma^\star$.

Denote the **length lexicographic order** *induced by* $<_{lex}$ *as* $<_{llex}$. *Namely* $x <_{llex} y$ *if either*

- $|x| < |y|$, *or*
- $|x| = |y|$ *and* $x <_{lex} y$.

Then $<_{llex}$ *linearly orders* $\Sigma^\star$, *and the ordering is isomorphic to the usual ordering on* $\mathbb{N}$.

For example, if $\Sigma = \{0, 1\}$ with $0 < 1$ then $\lambda <_{llex} 0 <_{llex} 1 <_{llex} 00 <_{llex} 01 <_{llex} \ldots$.

Example B.1.10 *The orderings* $<_{lex}$, $<_{kb}$ *and* $<_{llex}$ *are FA recognisable.*
For example, suppose $\Sigma = \{0, 1\}$ with $0 < 1$. Then a regular expression for the prefix relation is

$$[\tbinom{1}{1} + \tbinom{0}{0}]^\star\,[\tbinom{\perp}{1} + \tbinom{\perp}{0}]^\star.$$

So a regular expression for the relation $\otimes(\leq_{lex}) \subset \Sigma^{\star 2}$ is

$$[\tbinom{1}{1} + \tbinom{0}{0}]^\star\,[\,[\tbinom{\perp}{1} + \tbinom{\perp}{0}]^\star + \tbinom{0}{1}(\otimes\Sigma^{\star 2})].$$

Furthermore, if D is a regular subset of $\Sigma^\star$ then $\leq_{lex}$ restricted to D is FA recognisable. Similarly, $<_{kb}$ and $<_{llex}$ are FA recognisable. $\triangleleft$

Example B.1.11 *The following relations are FA recognisable over* Σ.

1. The binary relations σ_a for $a \in \Sigma$ satisfying $(x, xa) \in \sigma_a$ for every $x \in \Sigma^\star$ have regular expressions
$$\bigcup_{\sigma \in \Sigma} [\tbinom{\sigma}{\sigma}^\star \tbinom{\perp}{a}].$$
2. The binary relation $el(x, y)$ for $|x| = |y|$ has regular expression
$$[\bigcup_{\sigma,\delta \in \Sigma} \tbinom{\sigma}{\delta}]^\star.$$
3. Let $\Sigma = \{0, 1\}$. In the following $\perp$ is treated as 0.
    (a) The ternary relations $and(x, y, z)$ where $z \in \{0, 1\}^\star$ is the bitwise *and* of x and y. Similarly $or(x, y, z)$ the bitwise *or*.
    (b) The binary relation $neg(x, z)$ where z is the bitwise *not* of x. $\triangleleft$

Structures

Mathematical objects are treated via the notion of structures in the formalism of logic, see for example Hodges [1993]. Basic familiarity with first order logic is assumed, though some definitions are summarised here to fix notation. A *structure* $\mathcal{A}$ consists of a set A called the *domain* and *atomic* relations and functions on A. Constants are regarded as functions of arity 0. The signature of $\mathcal{A}$ consists of the names and arities of the relations and functions of $\mathcal{A}$. There is always a symbol for equality, and so it is not explicitly mentioned in the signature. If $\mathcal{A}$ is a structure with signature S and $S' \subset S$, then the structure $\mathcal{A}'$ formed by removing the relations or functions named in $S \setminus S'$ from $\mathcal{A}$ is called a *reduct* of $\mathcal{A}$ and $\mathcal{A}$ is called an *expansion* of $\mathcal{A}'$. All structures considered are countable and signatures are finite unless otherwise specified.

A *relational structure* is one with no functions (or constants). We freely associate the structure $\mathcal{A}$ with the relational structure obtained by replacing the functions with their graphs. The *graph of* $f : A^n \to A$ is defined as $\{(\bar{x}, y) \in A^{n+1} \mid f(\bar{x}) = y\}$. Write $\mathcal{A} = (A, R_1^A, \ldots, R_k^A)$ where R_i^A is an n_i–ary relation on $\mathcal{A}$; if there is no confusion we drop the superscript A. An $\mathcal{A}$-*formula* $\Phi(x_1, \cdots, x_k)$ is a formula where the non-logical symbols of Φ are from the signature of $\mathcal{A}$. All formulae are first order unless specified otherwise. Note that the property of being an $\mathcal{A}$-formula depends solely on the signature of $\mathcal{A}$ and not on the particular structure $\mathcal{A}$. We often abbreviate $(x_1, \cdots, x_k)$ as $\overline{x}$ when no confusion could arise. Write Φ^A for the k-ary relation $\{\overline{x} \in A^k \mid \mathcal{A} \models \Phi(\overline{x})\}$.

The *first order* (FO) *theory* of a given structure $\mathcal{A}$ is the set of all first order sentences ($\mathcal{A}$-formulae without free variables) true in $\mathcal{A}$. This theory is *decidable* if there is an effective procedure that decides whether an arbitrary first order sentence is in the theory or not.

Two (relational) structures $\mathcal{A}, \mathcal{B}$ over the same signature are *isomorphic* if there exists a bijection $\nu : A \to B$ between their domains such that for every relation symbol R (of arity n say) and every tuple $(a_1, \cdots, a_n) \in A^n$,

$$(a_1, \cdots, a_n) \in R^A \text{ if and only if } (\nu(a_1), \cdots, \nu(a_n)) \in R^B.$$

We extend ν to A^k for every k and so write $\nu(R)$ for $\{(\nu(a_1), \cdots, \nu(a_n)) \mid (a_1, \cdots, a_n) \in R\}$. Similarly we extend ν to structures and so write $\nu(\mathcal{A}) = \mathcal{B}$. In case ν is as above but not necessarily onto, then it is called an *embedding*. If the identity mapping $\iota : A \to B$ is an embedding then $\mathcal{A}$ is called a *substructure* of $\mathcal{B}$. If $\nu : A \to B$ and $\mu : B \to A$ are functions, then $\nu\mu$ denotes the map $b \mapsto \nu(\mu(b))$.

The binary relation 'is isomorphic to' is an equivalence relation on the set of structures over a fixed signature. The *isomorphism type* of a structure $\mathcal{A}$ is the equivalence class of $\mathcal{A}$ with respect to this equivalence relation. If $\mathcal{A}$ is isomorphic to $\mathcal{B}$ then say that $\mathcal{A}$ is an *isomorphic copy* of $\mathcal{B}$.

A *relation R is first order definable in a structure $\mathcal{A}$* if there exists a first order formula $\Phi_R(\overline{x}, \overline{y})$ and elements $\overline{c}$ of $\mathcal{A}$ such that for all $\overline{x}$, $(\overline{x}, \overline{c}) \in R$ if and only if $\mathcal{A} \models \Phi_R(\overline{x}, \overline{c})$. The elements $\overline{c}$ are called *parameters* and may be empty, that is $\overline{c} = \emptyset$. The formula Φ_R is called a *first order definition in $\mathcal{A}$ of the relation R.*

A *congruence relation* on a structure $\mathcal{A}$ is an equivalence relation ϵ on A such that for every atomic relation R of $\mathcal{A}$ (say of arity r), and for every $\overline{x}, \overline{y}$ such that $(x_i, y_i) \in \epsilon$ for every $1 \leq i \leq r$ we have that $\overline{x} \in R^A$ if and only if $\overline{y} \in R^A$. In this case there is an induced *quotient structure* $\mathcal{A}/\epsilon$ over the same signature as $\mathcal{A}$. Its domain consists of the set of ϵ-classes $[a]$ of A and an atomic relation (of arity r say) $R^{A/\epsilon}$ satisfies those $([a_1], \cdots, [a_r])$ such that $(a_1, \cdots, a_r) \in R^A$.

Suppose we are given a $\mathcal{B}$-formula, possibly with parameters, $\Delta(\overline{x})$ and a collection of $\mathcal{B}$-formulae, possibly with parameters, $\Phi_{R_i}(\overline{x}_1, \cdots, \overline{x}_r)$ such that $\Phi_{R_i}^B$ is a relation on Δ^B. Then

$$(\Delta^B, \Phi_{R_1}^B, \cdots, \Phi_{R_n}^B)$$

is a structure. If further there is a $\mathcal{B}$-formula $\epsilon(\overline{x}, \overline{y})$ (possibly with parameters) such that ϵ^B is a congruence relation on this structure then the quotient structure

$$(\Delta^B, \Phi_{R_1}^B, \cdots, \Phi_{R_n}^B)/\epsilon,$$

is called *first order definable* in $\mathcal{B}$. In case there are infinitely many formulae Φ_{R_i} it is also required that the function sending i to Φ_{R_i} be computable. Here the tuples $\overline{x}_i, \overline{x}$ and $\overline{y}$ contain a fixed number of variables, say k, called the *dimension* of the definition.

If a structure $\mathcal{A}$ is isomorphic to this quotient structure, say via map $\nu : A \to B^k$, called the *co-ordinate map*, then $\mathcal{A}$ is called *first order interpretable* in $\mathcal{B}$, written $\mathcal{A} \leq_{\mathrm{FO}}^{\nu} \mathcal{B}$. The superscript is sometimes dropped when the particular co-ordinate map is not important. For every $\mathcal{A}$-formula $\Psi(x_1, \cdots, x_m)$ there exists a $\mathcal{B}$-formula $\Psi^\nu(\overline{y}_1, \cdots, \overline{y}_m)$ where each $\overline{y}_i$ is a tuple of k many variables, such that for every $(a_1, \cdots, a_m) \in A^m$,

$$\mathcal{A} \models \Psi(a_1, \cdots, a_m) \text{ if and only if } \mathcal{B} \models \Psi^\nu(\nu(a_1), \cdots, \nu(a_m)).$$

In fact Ψ^ν is defined inductively as follows: If $R(x_1, \cdots, x_r)$ is an atomic relation of $\mathcal{A}$ then define R^ν as $\Phi_R(\overline{x}_1, \cdots, \overline{x}_r)$; define $(\phi \wedge \psi)^\nu$ as $\phi^\nu \wedge \psi^\nu$ and $(\neg\phi)^\nu$ as $\neg(\phi^\nu)$ and $[(\forall x)\, \phi]^\nu$ as $(\forall \overline{x})\, [\Delta(\overline{x}) \to \phi^\nu]$.

Two structures $\mathcal{A}$ and $\mathcal{B}$ are *first order mutually interpretable* if $\mathcal{A} \leq_{\mathrm{FO}}^{\nu} \mathcal{B}$ and $\mathcal{B} \leq_{\mathrm{FO}}^{\mu} \mathcal{A}$ for some co-ordinate maps $\nu : A \to B^k$ and $\mu : B \to A^l$. If furthermore $\nu\mu : B \to B^{lk}$ is first order definable in $\mathcal{B}$ and $\mu\nu : A \to A^{lk}$ is first order definable in $\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are called *first order bi-interpretable*.

For ease of readability the qualifier 'first order' may be omitted from phrases such as 'first order definable' when no confusion will occur. All these definitions transfer to other logics, besides first order. There is one occasion, in Section C.2, where weak monadic second order logic (WMSO) is considered. However since this is an isolated case, we defer the definitions to that section.

Automatic Structures

The next definition is central since it describes what we mean by structures that are presented by a collection of automata. It originally appeared in Hodgson [1976] though more recently in Khoussainov and Nerode [1995].

Definition B.1.12 *A structure $\mathcal{A}$ is* **automatic over $\mathbf{\Sigma}$** *if its domain $A \subset \Sigma^\star$ and atomic relations $R_i^A \subset \Sigma^{\star n_i}$ are finite automaton recognisable over Σ.*

In case $\mathcal{A}$ has infinite signature $(R_i)_{i<\omega}$ one also requires that the function mapping i to the corresponding automaton be computable.

A structure $\mathcal{A}$ is **automatic** *if it is automatic over some alphabet Σ.*

So in order to show that a given structure is automatic, one may exhibit automata that recognise the domain and atomic relations. Here are some simple examples to illustrate the definition.

Example B.1.13 *The structure with domain $1^\star$ and unary function f sending 1^n to 1^{n+1} for $n \in \mathbb{N}$, is automatic over alphabet $\{1\}$.*
$\otimes 1^\star = 1^\star$ and the convolution of the graph of f is given by the regular expression $\binom{1}{1}^\star \binom{\perp}{1}$. ◁

Example B.1.14 *The structure with domain $\{0, 1, \cdots, k-1\}^\star$ and ternary relation satisfied by those triples of strings (a, b, c) for which $a + b = c$ $($base k-addition in the least-significant-digit-first representation$)$ is automatic.*
The domain and relations are regular as in Example B.1.8. ◁

For the next example let $\Sigma = \{0, \cdots, k-1\}$, $\sigma_a(x) = xa$ for $a \in \Sigma$, $\preceq_p$ denotes the prefix relation and $el(x, y)$ holds exactly when $|x| = |y|$.

Example B.1.15 *The structure $\mathcal{W}_k = (\Sigma^\star, (\sigma_a)_{a\in\Sigma}, \preceq_p, el)$ is automatic over Σ.*
The domain $\Sigma^\star$ is regular, and the convolution of each of the atomic relations is regular, see Examples B.1.11 and B.1.10. ◁

Here is an important theorem that motivates the study of automatic structures.

Theorem B.1.16 [cf. Khoussainov and Nerode - 1995; Hodgson - 1976] *If $\mathcal{A}$ is automatic over Σ then there exists an algorithm that given a first order definition (with parameters) in $\mathcal{A}$ of a relation R, constructs an automaton recognising $\otimes R$.*

Proof Proceed by induction on the complexity of a first order formula Φ defining a relation R. Assume first that Φ has no parameters. If Φ is atomic, then the automaton required is just the one that is supplied by the definition of the automaticity of $\mathcal{A}$. Suppose $\Phi(\bar{x}) = \Psi_1(\bar{x})\,\mathrm{OP}\,\Psi_2(\bar{x})$ where $\mathrm{OP} \in \{\cup, \cap, \setminus\}$. Then by induction and Theorem B.1.5 part 1. there is an automaton recognising Φ, and its construction is effective from the automata for Ψ_1 and Ψ_2. Finally, if $\Phi = \exists x_i \Psi(\bar{x})$ then by induction and Theorem B.1.5 part 2. and where necessary part 5., there is an automaton recognising Φ, and its construction is effective from the automaton for Ψ. Finally if $\Phi(\overline{x}, \overline{y})$ has parameters $\overline{c}$ then first construct the automaton for $\Phi(\overline{x}, \overline{y})$ without parameters as above, and then apply instantiation of $\overline{c}$ as in Theorem B.1.5 part 3. and where necessary part 5. $\lhd$

As a consequence one has the important decidability result.

Theorem B.1.17 *The first order theory of an automatic structure $\mathcal{A}$ is decidable.*

Proof Suppose $\mathcal{A}$ is automatic. Then given a first order *sentence* Φ, $\mathcal{A} \models \Phi$ if and only if the language of the automaton associated with Φ is non-empty. This condition can be tested effectively, see Theorem B.1.6. Hence the first order theory of $\mathcal{A}$ is decidable. $\lhd$

The domain of a given structure may or may not be given as a set (or relation) of strings over some alphabet. Though even if it is, it may be the case that it is not automatic while there is an automatic structure that is *isomorphic* to it. As an illustrative example, consider the structure with domain $B = \{0^n 1^n \mid n \in \mathbb{N}\}$ and unary function T mapping $0^n 1^n$ to $0^{n+1}1^{n+1}$ for every $n \in \mathbb{N}$. Then (B, T) is not automatic over alphabet $\{0, 1\}$ since B is not regular. However, (B, T) is isomorphic to the structure with domain $D = 1^\star$ and unary function N mapping 1^n to 1^{n+1}. This structure (N, D) is automatic over alphabet $\{0, 1\}$ and is isomorphic to the natural numbers with successor, $(\mathbb{N}, S)$. The point to note here is that although the structure $(\mathbb{N}, S)$ is not automatic for syntactic reasons (it may be given without mention of an alphabet coding the domain, say as the unique structure satisfying the relevant second order axioms) and the structure (B, T) is not automatic (over any alphabet), both can be *presented by*, namely are isomorphic to, an automatic structure, for instance (D, N). This leads us to the following definition from Khoussainov and Nerode [1995].

Definition B.1.18 *Suppose $\mathcal{A}$ is an automatic structure over Σ isomorphic to $\mathcal{B}$. This is expressed by saying that $\mathcal{A}$ is an* **automatic presentation (or an automatic copy)** *of $\mathcal{B}$; and that $\mathcal{B}$ is* **automatically presentable** $($*over* $\Sigma)$.

It will be convenient to overload the phrase 'automatic presentation' to refer to the automatic structure $\mathcal{A}$ or even the automata 'presenting' the structure $\mathcal{A}$.

Note that if a structure $\mathcal{B}$ is isomorphic to an automatic structure $\mathcal{A}$, then the first order theory of $\mathcal{B}$ equals that of $\mathcal{A}$, and hence is decidable.

Example B.1.19 *The natural numbers with the usual addition* $(\mathbb{N}, +)$ *is automatically presentable, and hence has decidable first order theory.*

Fix $k > 1$ and alphabet $\Sigma = \{0, 1, \cdots, k-1\}$. Recall that the structure $(\Sigma^\star, +)$ where $+$ represents base-k addition is automatic; see Example B.1.14. However each string in the set $w0^\star$ represents the same natural number. So define a function base_k that maps $n \in \mathbb{N}$ to its shortest least-significant-digit-first base-k representation. So $\mathrm{base}_k(0) = \lambda$ and for $n > 0$, $\mathrm{base}_k(n)$ ends in the symbol 1. Define an automatic structure $(A_k, +_k)$ where $A_k = \{\lambda\} \cup \{0, 1, \cdots, k-1\}^\star 1$ and $+_k$ is the restriction of $+$ to this set. Then $(\mathbb{N}, +)$ is isomorphic to $\mathcal{A}_k$ via the mapping base_k. So $\mathcal{A}_k$ is an automatic presentation of $(\mathbb{N}, +)$.

The first order theory of $(\mathbb{N}, +)$ is called Presburger Arithmetic (PA). The time complexity of the present algorithm for PA is non-elementary in the size n of the sentence, while the traditional method of exhibiting this decidability via quantifier elimination yields an algorithm of order $2^{2^{2^{kn}}}$ for some constant k, see Oppen [1978]. Also Fischer and Rabin [1974] give a double exponential non-deterministic lower bound on any decision procedure for PA. Recently Klaedtke [2003] has improved the time complexity of the automata theoretic decision procedure for PA. In fact he proves that there exists a constant $k \geq 0$ such that the minimal deterministic automaton for an arbitrary first order formula of $(\mathbb{N}, +)$ of length n has at most $2^{2^{2^{kn}}}$ states; and that this bound is tight. $\lhd$

Remark B.1.20 Let $\mathcal{A}$ be a structure with $A \subset \Sigma^\star$. Let Σ' be an alphabet such that $|\Sigma'| = |\Sigma|$ and $\Sigma \cap \Sigma' = \emptyset$. Let $\delta : \Sigma^\star \to \Sigma'^\star$ be the (monoid) morphism extending some fixed bijection from Σ to Σ'. Define $A' = \{\delta(w) \mid w \in A\}$. If R is an n-ary relation of $\mathcal{A}$ then define $R' = \{(\delta(x_1), \ldots, \delta(x_n)) \mid (x_1, \ldots, x_n) \in R\}$. The resulting structure $\mathcal{A}'$ is an *isomorphic copy* of $\mathcal{A}$. Note that if $\mathcal{A}$ is automatic over Σ then $\mathcal{A}'$ is automatic over Σ'.

Proposition B.1.21 [Blumensath - 1999] *Every automatically presentable structure has an automatic presentation over the alphabet* $\{0, 1\}$.

Proof Let $\mathcal{A}$ be automatic over Σ. If $|\Sigma| \leq 2$, then simply take an isomorphic copy over alphabet $\{0, 1\}$. Otherwise suppose $|\Sigma| > 2$. Let $k \in \mathbb{N}$ such that $2^k \geq |\Sigma|$. Let $\nu : \Sigma \to \{0, 1\}^k$ be some function that sends elements of Σ to binary strings of length k. Extend ν to $\Sigma^\star$ by mapping $\sigma_0 \cdots \sigma_n$ to $\nu(\sigma_0) \cdots \nu(\sigma_n)$. Then $\nu(\mathcal{A})$ is an isomorphic copy of $\mathcal{A}$. Moreover it is automatic over $\{0, 1\}$. Indeed form a regular expression over $\{0, 1\}$ for $\nu(R)$ from one for R over Σ, simply by replacing every symbol σ by $\nu(\sigma)$. $\lhd$

An important property of automatically presentable structures is that they are closed under first order interpretability.

Proposition B.1.22 [cf. Khoussainov and Nerode - 1995; Blumensath - 1999] *If* $\mathcal{A}$ *is automatic and* $\mathcal{B}$ *is first order definable in* $\mathcal{A}$ *then* $\mathcal{B}$ *is automatically presentable.*

Proof Suppose $\mathcal{A}$ is automatic over Σ and $\mathcal{B}$ is definable in $\mathcal{A}$, with dimension k. Then $\mathcal{B}$ is of the form $\mathcal{B}'/\epsilon$ where

$$\mathcal{B}' = (\Delta^A, \Phi^A_{R_1}, \cdots, \Phi^A_{R_n})$$

and $\Delta(x_1, \cdots, x_k)$, Φ_{R_i} and ϵ are $\mathcal{A}$-formulae. By Theorem B.1.16 each of Δ^A and $\Phi^A_{R_i}$ are finite automaton recognisable relations over Σ. Or alternatively $\mathcal{B}'$ is automatic over alphabet $\Sigma' = (\Sigma_\perp)^k$.

Now define the function F sending $\overline{a}$ to the length-lexicographically (with respect to Σ') least element in the ϵ-class $[\overline{a}]$. By Theorem B.1.16, F is finite automaton recognisable over Σ' since it is first order definable from the regular predicates ϵ and $<_{llex}$. So the structure $(\mathcal{B}', F)$ is automatic over Σ'. Define a substructure $\mathcal{C}$ of $\mathcal{B}'$ as follows. Its domain is the set

$$\{\overline{a} \in \mathcal{B}' \mid (\exists \overline{b})\, F(\overline{b}) = \overline{a}\}.$$

It has a relation R^C for every relation $R^{B'}$ of $\mathcal{B}'$ where for every $(\overline{a}_1, \cdots, \overline{a}_n)$,

$$(F(\overline{a}_1), \cdots, F(\overline{a}_n)) \in R^C \iff (\overline{a}_1, \cdots, \overline{a}_n) \in R^{B'}.$$

The domain C and relations R^C are first order definable from regular predicates, so using Theorem B.1.16, $\mathcal{C}$ is automatic over Σ'. Moreover $\mathcal{B}$ being isomorphic to $\mathcal{C}$, is automatically presentable over Σ'. $\triangleleft$

Corollary B.1.23 *If $\mathcal{A}$ and $\mathcal{C}$ are automatic then the following are automatically presentable.*

1. *a substructure of $\mathcal{A}$ with definable universe,*

2. *the factorisation of $\mathcal{A}$ by a definable congruence.*

3. *the direct product $\mathcal{A} \times \mathcal{C}$ (where $\mathcal{A}$ and $\mathcal{C}$ have the same signature).*

4. *$\mathcal{A}\omega$, the ω-fold disjoint union of $\mathcal{A}$.*

Proof For the first item note that if $D = \Phi(\overline{x})$ is a definable relation in $\mathcal{A}$, then the *relativised structure* $(D, R^D_1, \cdots, R^D_n)$ is definable in $\mathcal{A}$ since relation R^D_i, say of arity l, has first order definition

$$\bigwedge_{1 \le i \le l} \Phi(\overline{x}_i) \wedge R^A_i(\overline{x}_1, \cdots, \overline{x}_l).$$

The second item is an immediate corollary of the Proposition B.1.22. For the third item note that the direct product is definable in the disjoint union of $\mathcal{A}$ and $\mathcal{C}$, which is automatic. Indeed if R is an l-ary relation of the common signature, then $R^{A\times C}$ satisfies $((x_1, y_1), \cdots, (x_l, y_l))$ in the case that $R^A(x_1, \cdots, x_l) \wedge R^C(y_1, \cdots, y_l)$.

For the fourth item define domain $D = A \times \{1\}^\star$. Then define R^D as satisfying the tuple $((x_1, 1^{n_1}), \cdots, (x_l, 1^{n_l}))$ if and only if $(x_1, \cdots, x_l) \in R^A$ and $n_1 = \cdots = n_l$. $\triangleleft$

Remark B.1.24 Khoussainov and Nerode [1995] and Blumensath [1999] provide the following alternative definition of an automatic presentation. Let $\mathcal{A} = (A, R_1, \ldots, R_r)$ be a structure where relation R_j has arity n_j, $1 \leq j \leq r$. An *automatic presentation* over Σ of $\mathcal{A}$ consists of the following:

1. Finite automaton recognisable relations $L \subset \Sigma^\star$, $L_\epsilon \subset L^2$, and $L_j \subset L^{n_j}$ for $1 \leq j \leq r$.

2. A surjective mapping $\nu : L \to A$ with the property that

$$\begin{aligned} (x_1, x_2) \in L_\epsilon \quad &\iff \quad \nu(x_1) = \nu(x_2), \\ (x_1, \ldots, x_{n_j}) \in L_j \quad &\iff \quad (\nu(x_1), \ldots, \nu(x_{n_j})) \in R_j, \end{aligned}$$

   for every $x_i \in L$ and $1 \leq j \leq r$.

In this case the structure $(L, L_1, \ldots, L_r, L_\epsilon)$ is automatic over Σ as in Definition B.1.12. Hence by the proof of Proposition B.1.22, the structure $(L, L_1, \ldots, L_r)/L_\epsilon$, which is isomorphic to $\mathcal{A}$, is automatically presentable over $(\Sigma_\perp)^k$ as in Definition B.1.18. Conversely if $\mathcal{A}$ is automatically presentable over Σ as in Definition B.1.18 then $\mathcal{A}$ is automatically presentable here with ν being a bijection and $L_\epsilon = \{(x, x) \mid x \in L\}$. Hence the two definitions of being an automatic presentation coincide modulo the alphabets.

Definition B.1.25 *Write $\exists^\infty$ for the quantifier interpreted as 'there exists infinitely many'. For $k, m \in \mathbb{N}, 0 \leq k < m$ and $m > 1$, write $\exists^{(k,m)}$ for the quantifier interpreted as 'there exists k modulo m many'.*

For example, $\mathcal{A} \models (\exists^{(0,2)} x)\, \Phi(x)$ if and only if the cardinality of $\{x \in A \mid \mathcal{A} \models \Phi(x)\}$ is even. The following theorem says that one can extend Theorem B.1.16 to include these quantifiers. The case of $\exists^\infty$ is due to Blumensath [1999].

Theorem B.1.26 *If $\mathcal{A}$ is automatic over Σ then there exists an algorithm that given a relation R which is first order definable (with parameters) in $\mathcal{A}$ with additional quantifiers $\exists^\infty$ and $\exists^{(k,m)}$ constructs an automaton recognising R. Hence the first order theory with these additional quantifiers of $\mathcal{A}$ is decidable.*

Proof The proof is an extension of that of Theorem B.1.16 but includes cases for the additional quantifiers. Let $\Phi(\bar{x}) = (\exists^\infty x_i)\, \Psi(\bar{x})$. Expand the presentation of $\mathcal{A}$ to include the length lexicographical order on the domain A (see Example B.1.10). Then $\Phi(\bar{x})$ is equivalent (in $\mathcal{A}$) to the first order formula

$$(\forall y)(\exists x_i)\, [y <_{llex} x_i \wedge \Psi(\bar{x})].$$

Let $\Phi(y_1, \cdots, y_l) = (\exists^{(k,m)} x)\, \Psi(x, \bar{y})$ for fixed $k, m \in \mathbb{N}$. We construct an automaton over Σ that recognises Φ^A. If there are infinitely many x with $\mathcal{A} \models \Psi(x, \bar{y})$, a condition that can be tested using the $\exists^\infty$ quantifier, then the automaton accepts the empty set. Otherwise assuming that there are at most finitely many such x it proceeds as follows. Let $B = (Q, \iota, \rho, F)$ be the (deterministic) automaton recognising $\otimes\Psi^A$. Before defining the automaton we introduce some notation to make reading easier. For $q, q' \in Q$, and $\bar{y} \in \Sigma^{\star l}$ define $n(q, \otimes\bar{y}, q')$ to be number of strings $x \in \Sigma^\star$ with $|x| = |\otimes \bar{y}|$, such that there is a path in B labelled $\otimes(x, \bar{y})$ from state q to state q'. Then for $S \subset Q$ define $n(S, \otimes\bar{y}, q')$ as $\sum_{q\in S} n(q, \otimes\bar{y}, q')$.

The required automaton B' is of the form (Q', ι', Δ, F') where

1. The state set is $Q' = \prod_{1\leq i\leq m} 2^Q$,
2. The initial state is $\iota' = \{\iota\} \times \prod_{2\leq i\leq m}\{\emptyset\}$.
3. For $T = (T_1, \cdots, T_m)$ and $\sigma \in \otimes((\Sigma \cup \{\lambda\})^l)$, (recall that $\otimes(\lambda, \cdots, \lambda) = \lambda$), define $\Delta(T, \sigma) = (S_1, \cdots, S_m)$ as follows. For every i, let $q \in S_i$ if and only if
$$\sum_{1\leq j\leq m} j \times n(T_j, \sigma, q) \equiv i \pmod{m}.$$
4. Define $(S_1, \cdots, S_m) \in F'$ if and only if
$$\sum_{f\in F} \sum_{r<|Q|} \sum_{1\leq j\leq m} j \times n(S_j, (\{\perp\}^l)^r, f) \equiv k \pmod{m}.$$
Here $(\{\perp\}^l)^r$ is the concatenation of r copies of the tuple $(\perp, \ldots, \perp)$ of size l.

This completes the definition of the automaton. Note that it is deterministic. It remains to prove correctness.

Let $w = \otimes\bar{y}$ be the input to B' with $|w| \geq 1$ and let $(S_1, \cdots, S_m) = \Delta(\iota', w)$. Proceed by induction on $|w|$ to prove that for every i,

$$q \in S_i \iff n(\iota, w, q) \equiv i \pmod{m}. \tag{B.1}$$

For $|w| = 1$, the definition of Δ collapses to $q \in S_i$ if and only if $n(\iota, w, q) \equiv i \pmod{m}$ since $T_1 = \{\iota\}, T_j = \{\emptyset\}$ with $j \neq 1$. For $|w| > 1$ suppose $w = v \cdot \sigma$, $|\sigma| = 1$ and let $(T_1, \cdots, T_m) = \Delta(\iota', v)$. By the induction hypothesis, $q' \in T_j$ if and only if $n(\iota, v, q') \equiv j \pmod{m}$. So $T_1, \ldots, T_m$ partitions Q and for every $q \in Q$,

$$\begin{aligned} n(\iota, w, q) &= \sum_{1\leq j\leq m} \sum_{q'\in T_j} n(\iota, v, q') \times n(q', \sigma, q). \\ &\equiv \sum_{1\leq j\leq m} j \times \sum_{q'\in T_j} n(q', \sigma, q) \pmod{m} \\ &\equiv \sum_{1\leq j\leq m} j \times n(T_j, \sigma, q) \pmod{m}. \end{aligned}$$

Hence by definition of Δ, $q \in S_i$ if and only if $n(\iota, w, q) \equiv i \pmod{m}$, and this completes the induction.

Let $w \in \otimes(\Sigma^{\star l})$ and $(S_1, \ldots, S_m) = \Delta(\iota', w)$. It is sufficient to establish that the tuple $(S_1, \cdots, S_m) \in F'$ if and only if

$$\sum_{f \in F} \sum_{r \in \mathbb{N}} n(\iota, w \cdot (\{\perp\}^l)^r, f) \equiv k \pmod{m}. \tag{B.2}$$

First note that the range of the r's that give a non-zero value to the summand is finite, in fact $r < |Q|$, since it is assumed that the number of strings x with $\Psi(x, \bar{y})$ is finite (recall $w = \otimes\bar{y}$). Hence the sum can be restricted to $r < |Q|$.

case 1. If $|w| = 0$ then w is the empty string λ and $\Delta(\iota', w) = \iota'$. Also $n(\iota, w \cdot (\{\perp\}^l)^r, f)$ equals $n(\iota, (\{\perp\}^l)^r, f)$. Now since $S_1 = \{\iota\}$ and $S_j = \{\emptyset\}$ for $j \neq 1$, $n(S_j, (\{\perp\}^l)^r, f) = 0$ if $j \neq 1$. Applying these restrictions to the definition of F', we get that $\iota \in F'$ if and only if congruence B.2 holds.

case 2. Suppose $|w| > 0$. Now using Identity B.1 and the fact that $S_1, \ldots, S_m$ partitions Q one has that,

$$\begin{aligned} n(\iota, w \cdot (\{\perp\}^l)^r, f) &= \sum_{q' \in Q} n(\iota, w, q') \times n(q', (\{\perp\}^l)^r, f) \\ &\equiv \sum_{1 \leq j \leq m} \sum_{q' \in S_j} j \times n(q', (\{\perp\}^l)^r, f) \pmod{m} \\ &\equiv \sum_{1 \leq j \leq m} j \times n(S_j, (\{\perp\}^l)^r, f) \pmod{m}. \end{aligned}$$

Applying this congruence to the definition of F' we get that $(S_1, \cdots, S_m) \in F'$ if and only if congruence B.2 holds. $\lhd$

Here are two examples. Suppose (E, ρ) is automatic where ρ is an equivalence relation on E. Then the set of elements in infinite ρ-classes is definable as $(\exists^\infty z)\,[(x, z) \in \rho]$. Consider the structure $(\mathbb{N}, \leq)$, where $\leq$ is the usual ordering on $\mathbb{N}$. Then the set of even numbers is definable as $(\exists^{(0,2)} z)\,[z < x]$.

Finally we introduce two central structures. We present some definable relations that are used in the next theorem.

Example B.1.27 *Consider the structure*

$$\mathcal{W}_k = (\Sigma^\star, (\sigma_a)_{a \in \Sigma}, \preceq_p, el)$$

where

- $\Sigma = \{0, \cdots, k-1\}$,

- $\sigma_a(x) = xa$,
- $\preceq_p$ *is the prefix relation and*
- $el(x, y)$ *exactly when* $|x| = |y|$.

Here are examples of relations that are first order definable in $\mathcal{W}_k$.

The binary relation $|x| \le |y|$ is definable by $(\exists p)\,[p \preceq_p y \wedge el(x, p)]$.

The set consisting of the empty string $\{\lambda\}$ is definable as $(\forall y)\,[y \preceq_p x \to y = x]$; and consequently every singleton $\{a\}$ for $a \in \Sigma$ is definable as $x = \sigma_a(\lambda)$ which is really an abbreviation for $(\forall y)\,[y \in \{\lambda\} \to x = \sigma_a(y)]$.

Also, if $w = a_0 \cdots a_l \in \Sigma^\star$ then $\{y \cdot w \mid y \in \Sigma^\star\}$ is definable as $(\exists y)\,[y \preceq_p x \wedge \sigma_{a_l} \cdots \sigma_{a_0} y = x]$.

Fix $w \in \Sigma^\star$. Then $w^\star$ is definable as $w \in \{\lambda\}$ or

$$w \preceq_p x \wedge (\forall z)\,[z \cdot w \prec_p x \to z \cdot w \cdot w \preceq_p x].$$

Now for a fixed $n \in \mathbb{N}$, let w be a string of length n. Then the formula $(\exists y)\,[y \in w^\star \wedge el(y, x)]$ says that $|x|$ is a multiple of n, and is written $|x|_n$; and the formula $el(x, yw)$ defines the relation satisfying $|x| = |y| + n$.

The unary relations $\mathrm{first}_a(x)$, for $a \in \Sigma$, are defined by $(\exists y)\,[y \preceq_p x \wedge y = \sigma_a \lambda]$.

The unary relations $\mathrm{last}_a(x)$, for $a \in \Sigma$, defined by $(\exists y)\,[y \prec_p x \wedge \sigma_a y = x]$.

The function $\mathrm{maxl}(x_1, x_2, y)$ stating that $|y| = \max(|x_1|, |x_2|)$ is definable by

$$[|x_1| \le |x_2| \to el(y, x_2)] \wedge [|x_2| < |x_1| \to el(y, x_1)].$$

Similarly for $\mathrm{maxl}(x_1, \cdots, x_k, y)$ stating that $|z| = \max_i(|x_i|)$.

Let $co_a(x, p)$ mean that a is the symbol in the $|p| + 1$'st position of x, where the positions are counted from 1.

It is definable by the formula

$$\phi_a(x, p) : (\exists y)\,[y \prec_p x \wedge el(p, y) \wedge \sigma_a y \preceq_p x],$$

for $a \neq 0$ and in case $a = 0$ we use $\phi_a \vee |p| \ge |x|$.

The formula

$$\mathrm{last}_1(x) \wedge (\forall p)\,[p \preceq_p x \wedge \mathrm{last}_1(p) \to p = x],$$

defines the regular language $(\Sigma \setminus \{1\})^\star \cdot 1$. $\triangleleft$

Example B.1.28 *Consider the structure*

$$\mathcal{N}_k = (\mathbb{N}, +, |_k)$$

where

- $\mathbb{N}$ *denotes the natural numbers* (*that include* 0),
- $+$ *is the usual addition on* $\mathbb{N}$, *and*
- $x|_k y$ *means that* $x = k^n$ *for some* $n \in \mathbb{N}$ *and* $y = mx$ *for some* $m \in \mathbb{N}$.

Here are examples of relations that are first order definable in $\mathcal{N}_k$.
The singleton $\{0\}$ is definable by $(\forall m)\,[m + z = m]$.

The usual order $m < n$ is defined by $(\exists c)\,[c \neq 0 \to m + c = n]$.

Multiplication by a constant c is definable as $n + n + \cdots + n$, where there are c terms in the sum.

The unary relation expressing that n is a power of k, written $P_k(n)$, is definable by $n|_k n$.

For $0 \leq j < k$, the formula $\epsilon_j(m, p)$ defined by

$$(\exists a)(\exists b)\,[P_k(p) \wedge m = a + j.p + b \wedge a < p \wedge k.p|_k b]$$

which for a fixed m, satisfies $m = \sum_{i,j}\{j.k^i \mid \mathcal{N}_k \models \epsilon_j(m, k^i)\}$. In words, $\epsilon_j(m, p)$ holds if $p = k^i$ and the i-th digit in the shortest least-significant-digit-first base k representation of m is j. Here the least significant digit has position 0.

The formula

$$n|_k m \wedge (\forall c)\,[c|_k m \to c|_k n]$$

says that n is the highest power of k that divides m, written $V_k(m) = n$.

The formula $P_k(n) \wedge n \leq m < n \times k$ defines the function $\mathrm{size}_k(m) = n$ saying that n is a power of k and the number of symbols in the shortest base k expansion of m is $\log_k n$. $\lhd$

Theorem B.1.29 [cf. Grädel - 1990; Blumensath - 1999] *The structures* $\mathcal{N}_k$ *and* $\mathcal{W}_j$ *are FO mutually interpretable, for all* $j, k \geq 2$. *Moreover if* $k = j$ *then the structures are FO bi-interpretable.*

Proof We somewhat follow Blumensath [1999]. It suffices to establish for every $k \geq 2$, that $\mathcal{N}_k$ and $\mathcal{W}_k$ are bi-interpretable, and that $\mathcal{W}_2$ and $\mathcal{W}_k$ are mutually interpretable. We will make use of some definitions from the previous examples.

So fix $k \geq 2$ and $\Sigma = \{0, \cdots, k-1\}$. For $x = x_0 \cdots x_l \in \Sigma^\star$ define $\mathrm{val}_k(x) = \sum\{x_i k^i \mid 0 \leq i \leq l\}$ For $n \in \mathbb{N}$, define $\mathrm{base}_k : \mathbb{N} \to \Sigma^\star$ as sending n to the shortest string x such that $\mathrm{val}_k(x) = n$.

$\mathcal{N}_\mathbf{k} \leq_\mathbf{FO} \mathcal{W}_\mathbf{k}$: Define the co-ordinate mapping $\nu : n \mapsto \mathrm{base}_k(n)$. The image of $\mathcal{N}_k$ under ν is definable in $\mathcal{W}_k$ as follows. Define $\Delta(x)$ as $x = \lambda \vee x \in \Sigma^\star 1$. Note that $n|_k m$ holds if $\mathrm{base}_k(n) = 0^a 1$ and $\mathrm{base}_k(m) = 0^b y'$ for some $a, b \in \mathbb{N}$ with $a \leq b$ and $y' \in \Sigma^\star 1$. So define $\Phi_{|_k}(x, y)$ as

$$x \in 0^\star 1 \wedge (\exists p)\,[\sigma_1 p = x \wedge p \prec_p y].$$

Finally define $\Phi_+(x,y,z)$ as checking that there exists a string r encoding the sequence of carry bits in the base k sum $x+y=z$. Formally let

$$S = \{(a,b,c,d,e) \mid a+b+c = d+ke, \text{ for } a,b,c,d,e \in \{0,\cdots,k-1\}\}.$$

Then $\Phi_+(x,y,z)$ is defined as

$$\begin{aligned}
& (\exists m)\ \mathrm{maxl}(x,y,z,m) \\
\wedge\ & (\exists r)\ \mathrm{first}_0(r) \\
\wedge\ & (\forall p)\ [|p| \leq |m|+1 \rightarrow \\
& \bigvee_{(a,b,c,d,e)\in S} co_a(x,p) \wedge co_b(y,p) \wedge co_c(r,p) \wedge co_d(z,p) \wedge co_e(r,\sigma_1 p)]
\end{aligned}$$

The formula says that there is a string r encoding the carry so for every relevant position i, the tuple consisting of the ith digits of x,y,r,z and the $i+1$th digit of r is in S. Then $\mathcal{N}_k$ is isomorphic to

$$(\Delta^W, \Phi^W_{|_k}, \Phi^W_+).$$

$\mathcal{W}_\mathbf{k} \leq_\mathbf{FO} \mathcal{N}_\mathbf{k}$: Define the co-ordinate mapping $\mu : x \mapsto \mathrm{val}_k(x\cdot 1)$; namely

$$\mu(x_0\cdots x_l) = \mathrm{val}_k(x_0\cdots x_l) + k^{l+1}.$$

Recall that $\mathrm{size}_k(m) = n$ means that n is a power of k and the length of the shortest k-ary representation of m is $\log_k n$. So if $\mu(y) = m$ then $val_k(y) = m - k \times \mathrm{size}_k(m)$. The image of $\mathcal{W}_k$ under μ is definable in $\mathcal{N}_k$ as follows. Define $\mathcal{N}_k$-formulae

$$\begin{aligned}
\Delta(n) &\quad \text{as} \quad n \neq 0, \\
\Phi_{\sigma_a}(m,n) &\quad \text{as} \quad n = m - k \times \mathrm{size}_k(m) + a \times k \times \mathrm{size}_k(m) + k^2 \times \mathrm{size}_k(m), \\
\Phi_{\preceq_p}(m,n) &\quad \text{as} \quad m = 1 \vee [\mathrm{size}_k(m) < \mathrm{size}_k(n) \wedge \\
&\qquad\quad (\forall c)\ [c < \mathrm{size}_k(m) \rightarrow (\bigwedge_{a\in\Sigma} \epsilon_a(m,c) \longleftrightarrow \epsilon_a(n,c))]], \\
\Phi_{el}(m,n) &\quad \text{as} \quad \mathrm{size}_k(n) = \mathrm{size}_k(m).
\end{aligned}$$

Then $\mathcal{W}_k$ is isomorphic to

$$(\Delta^N, \Phi^N_{\sigma_0}, \Phi^N_{\sigma_1}, \Phi^N_{\preceq_p}, \Phi^N_{el}).$$

Hence $\mathcal{N}_k$ and $\mathcal{W}_k$ are mutually interpretable via 1-dimensional interpretations μ and ν and $\nu\mu : w \mapsto \mathrm{base}_k(\mathrm{val}_k(w1))$ and $\mu\nu : n \mapsto \mathrm{val}_k(\mathrm{base}_k(n)1)$. Then $\nu\mu(w)$ is definable in $\mathcal{W}_k$ as $\nu\mu(w) = \sigma_1(w)$, and $\mu\nu(n)$ is definable in $\mathcal{N}_k$ as

$$[n = 0 \rightarrow \mu\nu(n) = 1] \wedge [n > 0 \rightarrow \mu\nu(n) = n + k \times V_k(n)],$$

where $V_k(n)$ is highest power of k that divides n. Hence $\mathcal{N}_k$ and $\mathcal{W}_k$ are bi-interpretable, as required.

$\mathcal{W}_{\mathbf{k}} \leq_{\mathbf{FO}} \mathcal{W}_{\mathbf{2}}$: Let n be the least integer greater than or equal to $\log_2 k$. Define a mapping $c : \Sigma \to \{0,1\}^n$ such that $\mathrm{val}_2(c(j)) = n$. Define $\nu : \Sigma^\star \to \{0,1\}^\star$ by $\nu(x_0 \cdots x_l) = c(x_0) \cdots c(x_l)$. Then the image of $\mathcal{W}_k$ under ν is definable in $\mathcal{W}_2$ as follows. Define

$$\begin{array}{rcl} \Delta(x) & \text{as} & |x|_n \wedge (\forall y)\,[(y \prec_p x \wedge |y|_n) \to \bigvee_j y \cdot c(j) \preceq_p x], \\ \Phi_{\sigma_a}(x,y) & \text{as} & y = x \cdot c(a), \\ \Phi_{\preceq_p}(x,y) & \text{as} & x \preceq_p y, \\ \Phi_{el}(x,y) & \text{as} & el(x,y) \end{array}$$

Then $\mathcal{W}_k$ is isomorphic to

$$(\Delta^{\mathcal{W}_2}, (\Phi_{\sigma_a})^{\mathcal{W}_2}_{a\in\Sigma}, \Phi^{\mathcal{W}_2}_{\preceq_p}, \Phi^{\mathcal{W}_2}_{el}).$$

$\mathcal{W}_{\mathbf{2}} \leq_{\mathbf{FO}} \mathcal{W}_{\mathbf{k}}$: Define the co-ordinate mapping $\iota : w \mapsto w$, for $w \in \{0,1\}^\star$. The image of $\mathcal{W}_2$ under ι is definable in $\mathcal{W}_k$ as the substructure on domain

$$\Delta(x) = (\forall p \preceq_p x)\,[co_0(x,p) \vee co_1(x,p)].$$

This completes the proof. $\lhd$

B.2 Examples

We use this section to give examples of automatically presentable structures; as well as to fix structures and their signatures that will be encountered later. In some instances we give the finite automata explicitly. In others we show how the structure is definable from another automatic structure, usually $\mathcal{W}_2$.

Example B.2.1 *Every finite structure is automatically presentable.*
If a structure has a finite domain, then every relation is finite, and hence automatically presentable (over a unary alphabet). $\lhd$

An *injection structure* (D,f) consists of an injective map $f : D \to D$. Write f^i for the i-fold iteration of f. An orbit of f is a set of the form $\{b \in A \mid (\exists i \in \mathbb{N}),\ [b = f^i(a) \vee a = f^i(b)]\}$ for some $a \in A$. A finite orbit of size n is isomorphic to the injection structure $(\{1,\cdots,n\}, g)$ such that $g(i) = i+1$ for $1 \leq i < n$ and $g(n) = 1$. An infinite orbit isomorphic to one of two types: either the injection structure $(\mathbb{N}, S)$, called an $\mathbb{N}$-orbit, or the injection structure $(\mathbb{Z}, S)$, called a $\mathbb{Z}$-orbit, where $S : n \mapsto n+1$.

A *permutation structure* (D,f) is an injection structure where f is also surjective. Note that the only infinite orbits of a permutation structure are $\mathbb{Z}$-orbits.

Example B.2.2 *The permutation structure consisting of infinitely many infinite orbits is automatically presentable.*
A single $\mathbb{Z}$-orbit is isomorphic to the automatic structure $(1^\star, f)$ where $f(w) = w11$ if $|w|$ is even, $f(w11) = w$ if $|w|$ is odd, and $f(1) = \lambda$. Now take the ω-fold disjoint union. $\lhd$

Example B.2.3 *The injection structure consisting of infinitely many infinite orbits of both types and infinitely many finite orbits of every size is automatically presentable.*
For the finite orbits, let $R \subset \{0,1\}^\star$ be the regular language $00^\star 1^\star$. Then for every $n \in \mathbb{N}$, R has exactly n strings of length n. Define a function $f : R \to R$ that maps a string $x \in R$ of length n to the length-lexicographically next string $y \in R$ if $|y| = n$ and otherwise to the lexicographically least string z of length n. Note that f is regular since it is first order definable from the regular predicates $<_{llex}$ (length-lexicographical order) and el (equal length). Then (R, f) is an injection consisting of exactly one orbit of every finite size. Now take the union of (R, f) with an automatic $\mathbb{N}$-orbit and a $\mathbb{Z}$-orbit, and then take the ω-fold disjoint union. $\lhd$

A *partially ordered structure (partial order)* $(D, \preceq)$ consists of a binary relation $\preceq$ on a domain D that is reflexive $(\forall x \in D)\,[x \preceq x]$, symmetric $(\forall x, y \in D)\,[x \preceq y \wedge y \preceq x \to x = y]$ and transitive $(\forall x, y, z \in D)\,[x \preceq y \wedge y \preceq z \to x \preceq z]$.

A *linearly ordered structure (linear order)* $(L, \leq)$ is a partial order in which every two elements are comparable; that is, for all $x, y \in D$ either $x \leq y$ or $y \leq x$. Write $x < y$ for $x \leq y \wedge x \neq y$. Refer to Rosenstein [1982] for background on linear orders.

Example B.2.4 *The linear orders* $(\mathbb{N}, <)$ *and* $(\mathbb{Z}, <)$ *are automatically presentable.*
Fix Σ and recall that if D is a regular subset of Σ then the linearly ordered structure with domain D and binary relation $<_{lex}$ restricted to D, written $(D, <_{lex})$, is automatic over Σ.

Let $\Sigma = \{0,1\}$. Then $(1^\star, <_{lex})$ is an automatic copy of $(\mathbb{N}, <)$. And $(0^\star 1 \cup 1^\star 1, <_{lex})$ is an automatic copy of $(\mathbb{Z}, <)$. $\lhd$

Example B.2.5 *The ordering of the rationals* $(\mathbb{Q}, \leq)$ *is automatically presentable.*
Here are two automatic presentations of $(\mathbb{Q}, \leq)$. In each case we define an automatic structure over the alphabet $\Sigma = \{0,1\}$. We check that it is a (countable) dense linear order with no endpoints.

The structure $(\Sigma^\star 1, \leq_{lex})$ is an automatic presentation of $(\mathbb{Q}, \leq)$. Indeed, $\leq_{lex}$ is a linear ordering. Let $v, w \in \Sigma^\star$. Then $w01 <_{lex} w1 <_{lex} w11$ so there are no endpoints. Suppose $w1 <_{lex} v1$. Then $w1 <_{lex} v01 <_{lex} v1$, except when $w = v0$ in which case $w1 <_{lex} w11 <_{lex} v1$, so the order is dense.

For the second structure, define an ordering $\sqsubset$ on $\Sigma^\star$ as follows. For $u, v \in \Sigma^\star$ define $u \sqsubset v$ if either

1. $\exists x, y, z \in \Sigma^\star$ with $u = x0y$ and $v = x1z$, or

2. $\exists x \in \Sigma^\star$ with $u = v0x$, or

3. $\exists x \in \Sigma^\star$ with $v = u1x$.

Considering $\Sigma^\star$ as a tree with 0 to the left of 1, $u \sqsubset v$ means that either u is lexicographically less than v, u is on the left subtree with root v, or v is on the right subtree with root u. Then $(\Sigma^\star, \sqsubseteq)$ is a linear order. Indeed, for every u and v, let $p(u, v) \in \Sigma^\star$ be their longest common prefix. Then $u \sqsubset v$ is equivalent to $p(u, v)0$ is a prefix of u or $p(u, v)1$ is a prefix of v. Hence $\sqsubseteq$ linearly orders $\Sigma^\star$.

For every $v \in \Sigma^\star$, $v0 \sqsubset v \sqsubset v1$, so it has no endpoints. Let $u \sqsubset v$. Then in the first two cases, $u \sqsubset u1 \sqsubset v$ and in the third case $u \sqsubset v0 \sqsubset v$, so it is dense.

Both structures are automatic since they are definable in $\mathcal{W}_2$. $\lhd$

Example B.2.6 *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are automatically presentable linear orders then so is their ordered sum $\mathcal{L}_1 + \mathcal{L}_2$ and their ordered product $\mathcal{L}_1 \cdot \mathcal{L}_2$.*

For $i \in \{1, 2\}$ let $(D_i, \leq_i)$ be an automatic linear order over alphabet Σ_i. Suppose that Σ_1 is disjoint from Σ_2. Then $\mathcal{L}_1 + \mathcal{L}_2$ is definable from $\mathcal{L}_1$ and $\mathcal{L}_2$ as follows. The domain is $D_1 \times \{0\} \cup D_2 \times \{1\}$. The ordering is defined as $(x, \epsilon) \leq (y, \delta)$ if either $\epsilon < \delta$ or both $\epsilon = \delta$ and $x \leq_\epsilon y$. For $\mathcal{L}_1 \cdot \mathcal{L}_2$, the domain is $\{(x, y) \mid x \in L_1, y \in L_2\}$ and the ordering is defined as $(a, x) \leq (b, y)$ if either $x <_2 y$ or both $x = y$ and $a <_1 b$. $\lhd$

A *well ordered structure (well order)* $(D, \leq)$ is a linear order in which every non empty subset has a $\leq$–minimum element.

Well orders are closed under sums and products and we use the usual multiplication and exponential notation. For example, $\omega + \omega = \omega.2$, $\omega \cdot \omega = \omega^2$.

Example B.2.7 *Every well order $\alpha \in \omega^\omega$ is automatically presentable.*

Suppose $(D, <)$ is an automatic well order, with $D \subset \Sigma^\star$. Then for every $d \in D$, the induced well order with domain $\{x \in D \mid x < d\}$ is definable with a parameter for d, hence automatic. So it is sufficient to check that every ordinal of the form ω^n for $n < \omega$ is automatic.

First note that ω is automatically presentable. For a given n define the domain

$$D = \{(x_1, \cdots, x_n) \mid x_i \in \omega\}.$$

Define the ordering on this domain as $(x_1, \cdots, x_n) \prec (y_1, \cdots, y_n)$ if the greatest i for which $x_i \neq y_i$ satisfies $x_i < y_i$. Then $(D, \prec)$ is definable from an automatic copy of ω and hence automatic. Further it is isomorphic to the ordinal ω^n.

Alternatively apply the fact from the previous example that automatic linear orders are closed under finite products to an automatic copy of ω. $\lhd$

A *tree* $(D, \preceq)$ is a partial order with a $\preceq$–minimum and for which every set of the form $\{y \in D \mid y \preceq x\}$ is a finite linear order. Fix $\Sigma = \{0, \cdots, k-1\}$.

Example B.2.8 *The full k–ary tree $(\Sigma^\star, \preceq_p)$ is automatically presentable over a k–ary alphabet, for $k \in \mathbb{N}$.*
Recall that the structure $\mathcal{W}_k$,

$$(\Sigma^\star, (\sigma_i)_{i<k}, el, \prec_p)$$

where $\sigma_i(w) = wi$, $el(x, y)$ when $|x| = |y|$ and $x \preceq_p y$ when x is a prefix of y, is automatic over Σ. ◁

Example B.2.9 Let $R \subset \Sigma^\star$ be a regular language and let $Pref(R)$ be the set of prefixes of strings in R. Let $\prec_p$ be the prefix relation. Then the partial orders $(Pref(R), \prec_p)$ and $(R \cup \{\lambda\}, \prec_p)$ are automatic trees over Σ. ◁

Example B.2.10 Let R be a regular language. Consider the partial order $\mathcal{T} = (R \cup \{\lambda\}, \preceq)$, where $x \preceq y$ if and only if $x = y$ or $|x| < |y|$ and x is lexicographically smallest among all $x' \in R$ such that $|x| = |x'|$. Then $\mathcal{T}$ is an automatic tree. ◁

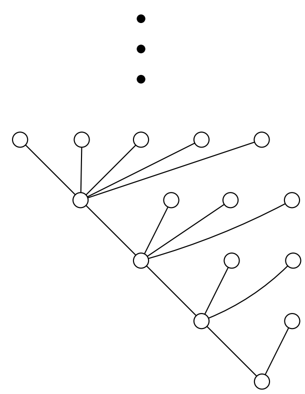

Figure B.3: Part of the tree $(0^\star 1^\star, \preceq)$.

A *semigroup* $(D, \cdot)$ consists of an associative binary operation $\cdot$ (considered as a ternary relation) on domain D. A *group* $(D, \cdot, \mathbf{1}, i)$ is a semigroup with identity **1** and an inverse operation i. Note that we consider groups in the signature containing one binary function for group multiplication $\cdot$.

Example B.2.11 The semigroup $(\mathbb{N}, +)$ and group $(\mathbb{Z}, +)$ are automatically presentable. ◁

Example B.2.12 *Every finitely generated abelian group $(A, +)$ is automatically presentable.*
Every finitely generated abelian group $(A, +)$ is isomorphic to a finite direct product of cyclic groups. That is, $\mathbb{Z}n \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_m}$ for some $n, m, k_1, \cdots, k_m \in \mathbb{N}$, where $\mathbb{Z}n$ is the direct product of n copies of $(\mathbb{Z}, +)$ and $\mathbb{Z}_j$ is the cyclic group of order i. Now note that the groups $\mathbb{Z}$ and $\mathbb{Z}_j$ are automatically presentable and that automatically presentable groups are closed under group sums. ◁

Example B.2.13 *The subgroup of* $(\mathbb{Q},+)/(\mathbb{Z},+)$ *generated by* $\{\frac{1}{b^i} \mid i \in \mathbb{N}\}$ *for a fixed* $b \in \mathbb{N} \setminus \{0,1\}$ *is automatically presentable.*
Let $\Sigma = \{0,\ldots,b-1\}$. Every non identity element of the group is a finite sum of generators, say $\Sigma_{i=1}^k a_i \frac{1}{b^i}$ where $0 \le a_i \le b-1$ and $a_k \ne 0$. Code this element as string $a_1 \ldots a_k$; and code the identity as 0. The domain D is then $\Sigma^\star(\Sigma \setminus \{0\}) \cup \{0\}$. Let $P \subset D^3$ be the graph of the addition. Now the reverse of $\otimes P$ is finite automaton recognisable since this is simply addition in base b as in Example B.1.8 except that the final carry is ignored and $\perp$ is interpreted as 0. But the reverse of a regular language is also regular, so $\otimes P$ is finite automaton recognisable. Hence (D,P) is the required automatic presentation. $\lhd$

The following extension of $(\mathbb{N},+)$ plays a central role. Recall that for $k \in \mathbb{N}$, the binary relation $|_k$ satisfies (x,y) when x is a power of k and x divides y.

Example B.2.14 *The structure* $(\mathbb{N},+,|_k)$ *is automatically presentable over the alphabet* $\Sigma = \{0,\cdots,k-1\}$.
Recall the automatic presentation $(A_k,+_k)$ from Example B.1.19. The formula

$$(\exists p)\,[p \in 0^\star \wedge x = \sigma_1 p \wedge p \prec_p y \wedge y \notin 0^\star],$$

where $\prec_p$ is the prefix relation, defines the image of the relation $|_k$ on domain A_k. $\lhd$

It is not known whether the group of rationals $(\mathbb{Q},+)$ is automatically presentable.

An *equivalence structure* (*equivalence relation*) (D,ρ) consists of a reflexive, symmetric, transitive binary relation ρ on D.

The next example show that automatic permutation structures are at least as complicated as automatic equivalence relations.

Example B.2.15 *Let* (E,ρ) *be an automatic equivalence relation. Then there is a bijection* g *on* E*, other than the identity, which is definable from* ρ *such that* $(e,g(e)) \in \rho$ *for all* $e \in E$.
Given (E,ρ), define g on E as follows. For $e \in E$, let $min(e)$ (respectively $max(e)$) be the $<_{llex}$-least (respectively greatest) element d such that $(e,d) \in \rho$. Then for $e = max(e)$ let $g(e) = min(e)$, and otherwise let $g(e)$ be the $<_{llex}$-least element $<_{llex}$-greater than e, say d, such that $(d,e) \in \rho$. Then (E,g) is the required permutation structure. $\lhd$

Example B.2.16 *Let* (E,ρ) *be an equivalence relation with no finite* ρ*–classes. Then it is automatically presentable. Further, if there are only finitely many infinite classes, then it is automatically presentable over a unary alphabet.*
Let $D = 0^*1^*$ and $(d,e) \in \rho$ when the number of 0's in d and in e are equal. Then (D,ρ) is an automatic equivalence relation with infinitely many infinite classes. Let $E = 1^*$ and $(d,e) \in \rho$ if $|d|$ is congruent to $|e|$ modulo n, for a fixed n. Then (E,ρ) is unary automatic and has exactly n infinite classes. $\lhd$

A *Boolean algebra* $(D, \cup, \cap, \setminus)$ is a structure where $\cup$, $\cap$ and $\setminus$ satisfy all the basic properties of set-theoretic union, intersection and relative complementation respectively.

Example B.2.17 *The Boolean algebra of finite and co-finite subsets of* $\mathbb{N}$ *is automatically presentable.*
Let $\Sigma = \{0,1\}$. Define the domain D as $\{0\} \cup \{0,1\}^{\star}1$. Let $X \subset \mathbb{N}$ be a non-empty finite set, and let m be the largest element in X. Define the characteristic string of X as $x_0 \cdots x_m$ where $x_i = 1$ if and only if $i \in X$. In case X is empty, define its characteristic string as λ. Code the finite set X as the string $c(X) = 0x_0 \cdots x_m$. Code the co-finite set $\mathbb{N} \setminus X$ as the string $c(Y) = 1x_0 \cdots x_m$. Note the empty set is coded as 0 and the universe $\mathbb{N}$ is coded as 1.

Let $X, Y \subset \mathbb{N}$. Then $c(\overline{X})$ is the same as $c(X)$ except for the first digit which is inverted. Treating $\perp$ as 0, $c(X \cup Y)$ is the bitwise $[c(X)\, or\, c(Y)]$ if X and Y are both finite, and the bitwise $c(X)\, and\, c(Y)$ if both are co-finite, and the bitwise $[not\, C(X)]\, and\, C(Y)$ if X is finite and Y is co-finite and the bitwise $C(X)\, and\, [not\, C(Y)]$ if X is co-finite and Y is finite. $\lhd$

There is a natural partial order on a Boolean algebra defined as $x \subset y$ when $x \cap y = x$. The partial order has a least element, written **0**, and a greatest element, namely B, written **1**.

An *atom* in a Boolean algebra is a non-zero element a such that there is no x with $\mathbf{0} \subset x \subset a$. A Boolean algebra is *atomic* if every non-zero element is $\supset$ an atom. A Boolean algebra is called *atomless* if it has no atoms. There is a unique countable atomless Boolean algebra which embeds every countable Boolean algebra.

Given a linear order $\mathcal{L}$ with a least element, one can form the *interval algebra* as follows. The domain D consists of all finite unions of half open intervals $[a, b)$. This forms a Boolean algebra with greatest element L. For example the interval algebra of $\mathcal{L} = [0,1] \cap \mathbb{Q}$ is the countable atomless Boolean algebra.

Example B.2.18 *For every* $n \in \mathbb{N}$*, the interval algebra formed from the ordinal* $\omega \cdot n$ *is automatic.*
The interval algebra formed from ordinal ω is isomorphic to the Boolean algebra of finite and co-finite sets of $\mathbb{N}$. But the Boolean algebra formed from $\omega \cdot n$ is definable from the one formed from ω. $\lhd$

Example B.2.19 *The configuration space of a Turing Machine is automatically presentable.*
Recall that a Turing machine $\mathcal{M}$ (say with only one tape, infinite in one direction, say to the right, and scanned by a single head) consists of an alphabet Σ (here Σ includes a symbol for the blank cell of the tape), with state set S, a set of initial states $I \subset S$, a set of final states $F \subset S$ and transition function $\delta : \Sigma \times S \to \Sigma \times S \times \{L, R\}$. A configuration of $\mathcal{M}$ is a tuple of the form (q, x_1, x_2) where $x_1 \cdot x_2 \in \Sigma^{\star}$ is the relevant content of the tape, $q \in S$ is the current state of $\mathcal{M}$ and the position of the head on the tape is $|x_1| + 1$ cells from the left. Write x_s for the symbol on the tape underneath the head, namely the first letter of x_2. An *initial* (*respectively*

final) *configuration* is one for which q is an initial (respectively final) state. The *configuration space* $C(\mathcal{M})$ is the directed graph with domain $\{(q, x_1, x_2) \mid x_1, x_2 \in \Sigma^\star, q \in S\}$ and an edge between (q_1, x_1, x_2) and (q_2, y_1, y_2) exactly when $\mathcal{M}$ has a transition from the first configuration to the second. That is, when $\delta(x_s, q_1) = (y_s, q_2, D)$ where $D = L$ and $|y_1| = |x_1| - 1$ or $D = R$ and $|y_1| = |x_1| + 1$. The configuration space $C(\mathcal{M})$ is an automatic structure over alphabet $\Sigma \cup S$. Indeed, the domain is definable from the regular languages $\Sigma^\star$ and S. The edge relation is definable by the formula

$$\bigvee_{\delta(a,q)=(b,r,D)} \mathrm{first}_a(x_2) \wedge \mathrm{first}_b(y_2) \quad \wedge \quad (D = L \rightarrow |y_1| = |x_1| - 1)$$
$$\wedge \quad (D = R \rightarrow |y_1| = |x_1| + 1).$$

But $\mathrm{first}_\sigma(x)$ and the relations $|x| = |y| - 1$ and $|x| = |y| + 1$ are FA recognisable (see Example B.1.27), hence so is the edge relation. Note also that the set of initial (final) configurations is FA recognisable.

Finally variations in the machine model, such as additional tapes, two-way heads etc., can also be treated in this way. For example, if $\mathcal{M}$ has two tapes then $\delta : \Sigma^2 \times S \rightarrow \Sigma^2 \times S \times \{L, R\}^2$ and a configuration of $\mathcal{M}$ is a tuple $(q, (\sigma_1, \sigma_2), (\delta_1, \delta_2))$ where $\sigma_1 \cdot \sigma_2$ is the content of the first tape and its tape head is on the $|\sigma_1| + 1$'st cell from the left, and $\delta_1 \cdot \delta_2$ is the content of the second tape and its tape head is on the $|\delta_1| + 1$'st cell from the left, and the machine $\mathcal{M}$ is in state q. Then similarly the configuration space $C(\mathcal{M})$ is automatic. $\triangleleft$

The content of this chapter is primarily from Khoussainov and Nerode [1995] and Blumensath [1999]. Theorem B.1.26 was first reported in Khoussainov, Rubin, and Stephan [2003b].

Chapter C

Characterisations of automatic structures

In classical automata theory the class of regular languages can be formalised in a variety of equivalent ways. Speaking loosely, deterministic automata give an operational view of regular languages. The addition of non-determinism or alternation allow succinct representations of regular languages. The Myhill-Nerode theorem gives a handle on the minimal deterministic automaton recognising a given regular language. Kleene's Theorem introduces regular expressions as alternative descriptions of regular languages. A particular formalisation may be suited to a particular application. This chapter presents the known characterisations of the FA recognisable relations and of automatic structures.

C.1 A decomposition theorem

This characterisation of FA recognisable relations is from Eilenberg, Elgot, and Shepherdson [1969, Theorem 11.1]. Roughly, it says that the convolution of an FA recognisable relation can be written as a finite union of regular languages, each of which is a product of factors satisfying a certain syntactic condition.

For instance, recall the prefix relation over $\Sigma = \{0, 1\}$ from Example B.1.10. It has regular expression $R_0 \cdot R_1$ where $R_0 = [\binom{1}{1} + \binom{0}{0}]^\star$ and $R_1 = [\binom{\perp}{1} + \binom{\perp}{0}]^\star$. So a machine reading a string from $R_0 \cdot R_1$ would, while processing a (non-empty) string from R_0, have non-blank symbols below both its heads. Subsequently while processing a (non-empty) string from R_1 it would have blank symbols under its first head. This is captured in the following definitions.

Let A be a non-empty subset of $\{1, \cdots, n\}$. Define the alphabet Σ_A as consisting of every element $w = (w_1, \cdots, w_n) \in (\Sigma_\perp)^n$ with the property that $w_i = \perp$ exactly when $i \notin A$. For example, if $\emptyset \neq A_1 \subset A_0 \subset \{1, \cdots, n\}$ then for all $v \in \Sigma_{A_1}, w \in \Sigma_{A_0}$ and $1 \leq i \leq n$, if $w_i = \perp$ then $v_i = \perp$.

Then in the example, $R_0 \subset \Sigma^\star_{\{1,2\}}$ and $R_1 \subset \Sigma^\star_{\{2\}}$.

Theorem C.1.1 (Decomposition) *Let $R \subset \Sigma^{\star n}$. Then R is FA recognisable if and only if $\otimes R$ is the finite union of sets which are products*

$$R_0 \cdots R_k, \qquad k \in \mathbb{N},$$

where each factor $R_i \subset (\Sigma_{A_i})^\star$ is FA recognisable and $A_k \subset \cdots \subset A_0$.

Proof Sufficiency is due to the closure of FA recognisable languages under concatenation and union. Necessity proceeds by induction on the complexity of recognisable languages. We use Kleene's theorem for $\Sigma_\perp^n$ restricted to sets of the form $\otimes R$. That is, $\otimes R$ is FA recognisable if and only if $\otimes R$ is a rational subset of $\Sigma_\perp^{n\star}$ in the sense that it can be formed by a finite number of applications of the operations union, product and Kleene star from the elements of $\Sigma_\perp^n$. Write $\mathcal{E}$ for the set of subsets of $\Sigma_\perp^{n\star}$ that are finite unions of products of the form stated. Clearly, $\mathcal{E}$ is closed under finite union.

To prove closure under product, suppose that $\otimes R, \otimes S \in \mathcal{E}$ and that $\otimes R \cdot \otimes S = \otimes Q$ for some $Q \subset \Sigma^{\star n}$. We show that $\otimes Q \in \mathcal{E}$. By distributivity it suffices to consider the products $\otimes R = R_0 \cdots R_k$ where $A_k \subset \cdots \subset A_0$ and $\otimes S = S_0 \cdots S_l$ where $B_l \subset \cdots \subset B_0$. Now for $x \in R_k$ and $y \in S_0$ the definition of convolution implies that if $x_i \in \{\perp\}^\star$ then $y_i \in \{\perp\}^\star$; so $i \notin A_k$ implies that $i \notin B_0$. Hence $B_0 \subset A_k$, so $R_0 \cdots R_k \cdot S_0 \cdots S_l$ has the required form, and hence $\otimes Q \in \mathcal{E}$.

To prove closure under Kleene star, suppose that $\otimes R \in \mathcal{E}$ and that $(\otimes R)^\star = \otimes Q$ for some $Q \subset \Sigma^{\star n}$. We show that $\otimes Q \in \mathcal{E}$. Again let $R_0 \cdots R_k$ and $S_0 \cdots S_l$ be two products of $\otimes R$. Then as above $R_0 \cdots R_k \cdot S_0 \cdots S_l \in (\otimes R)^\star$ has the required form with $B_0 \subset A_k$. Similarly $S_0 \cdots S_l \cdot R_0 \cdots R_k \in (\otimes R)^\star$ has the required form with $A_0 \subset B_l$. Hence $A_0 = \cdots = A_k = B_0 = \cdots B_l$, call it A say. Consequently $\otimes R$ is an FA recognisable subset of $\Sigma_A^\star$ and then so is $(\otimes R)^\star$. Hence $\otimes Q \in \mathcal{E}$.

For every word $w \in \Sigma^{\star n}$, the definition of convolution implies that $\otimes w$ can be written as $\sigma_1 \cdots \sigma_k$ for some $k \in \mathbb{N}$, where $\sigma_i \in \Sigma_{A_i}$ and $A_i = \{j \mid \text{the } j\text{-th letter of } \sigma_i \text{ is not a } \perp\}$. In this case $A_k \subset \cdots \subset A_1 \subset \{1, \cdots, n\}$ and so $\otimes\{w\} \in \mathcal{E}$. $\lhd$

C.2 Complete structures

Loosely speaking, a structure $\mathcal{C}$ is complete for a class of structures $\mathcal{K}$ in case that a structure belongs to $\mathcal{K}$ if and only if it is definable in $\mathcal{C}$ in a suitable logic. For example, the structure $(\mathbb{N}, +, \times)$ is complete for the class $\mathcal{K}$ of arithmetical structures using FO logic. Also the quantifier free fragment of $(\mathbb{N}, +, \times)$ is complete for the class $\mathcal{K}$ of computable structures.

Definition C.2.1 *Let $\mathcal{C}$ be a structure and $\mathcal{L}$ a logic. Suppose that*

1. *$\mathcal{C}$ is automatically presentable, and*

2. *$\mathcal{A}$ is automatically presentable if and only if $\mathcal{A}$ is $\mathcal{L}$ interpretable in $\mathcal{C}$, for every structure $\mathcal{A}$.*

Then $\mathcal{C}$ is called **complete** *for the class of automatically presentable structures.*

It turns out that there does indeed exist a complete structure $\mathcal{C}$ for the class $\mathcal{K}$ of automatic structures. This fact has been independently proved in various forms, summarised in the next theorem. Let $\Sigma = \{0, \cdots, k-1\}$ and recall the structure

$$\mathcal{W}_k = (\Sigma^\star, (\sigma_a)_{a \in \Sigma}, \preceq_p, el),$$

where $\sigma_a(w) = wa$, for $a \in \Sigma$, $w \preceq_p v$ if w is a prefix of v, and $el(w, v)$ if w and v have the same length.

Also recall the structure

$$\mathcal{N}_k = (\mathbb{N}, +, |_k),$$

where $x|_k y$ if x is a power of k and x divides y.

Finally $S : n \mapsto n + 1$ is the successor function on $\mathbb{N}$.

Theorem C.2.2 *Let $\mathcal{A}$ be a structure. Then the following are equivalent.*

1. *$\mathcal{A}$ is automatically presentable.*
2. *$\mathcal{A}$ is first order interpretable in $\mathcal{W}_k$ for some, equivalently all, $k \geq 2$.*
3. *$\mathcal{A}$ is first order interpretable in $\mathcal{N}_k$ for some, equivalently all, $k \geq 2$.*
4. *$\mathcal{A}$ is weak monadic second order interpretable in $(\mathbb{N}, S)$.*

We now outline the statements and proofs of these results in the present terminology.

Büchi [1960], Elgot [1961], Bruyère et al. [1994], Blumensath and Grädel [2000]

The existence of a complete structure for the FA recognisable relations is usually attributed to Büchi and Elgot who essentially proved the equivalence of 1 and 4 in the above theorem. However their respective proofs of the implication from $1 \to 4$ differ slightly. We first provide a Büchi-like proof, by coding runs of an automaton into a formula, of the equivalence between 1 and 2. Here the structure $\mathcal{W}_k$ deals directly with strings, as opposed to natural numbers.

Theorem C.2.3 *Suppose $k = |\Sigma| \geq 2$. Then a relation $R \subset \Sigma^{\star m}$ is FA recognisable if and only if R is first order definable in $\mathcal{W}_k$.*

Proof Sufficiency follows from observing that the structure $\mathcal{W}_k$ is automatic over Σ and then applying Theorem B.1.16.

To prove necessity, suppose $\otimes R$ is FA recognisable over Σ. We will construct a formula $\Phi_R(x_1, \cdots, x_m)$ of $\mathcal{W}_k$ so that for every $(a_1, \cdots, a_m)$,

$$\mathcal{W}_k \models \Phi_R(a_1, \cdots, a_m) \text{ if and only if } R(a_1, \cdots, a_m). \tag{C.1}$$

Let (S, q_1, Δ, F) be a deterministic automaton over alphabet $\Sigma_\perp^k$ recognising $\otimes R$. The idea is that Φ_R expresses that there exists a successful run of the automaton. It does this by stating the existence of $|S|$ many strings $r_1, \cdots, r_{|S|}$, where r_i has a 1 in the n-th place exactly when the automaton $\mathcal{A}$ is in state q_i when processing the n-th symbol of $\otimes(a_1, \cdots, a_m)$; that r_1 (corresponding to the initial state) starts with a 1 and that there is some $q_i \in F$ such that r_i ends in a 1.

Before defining Φ_R recall some definable relations of $\mathcal{W}_k$, see Example B.1.27. The statement $maxl(x_1, \cdots, x_m, l)$ says that the length of l is the largest of the lengths of the x_i. We also need to be able to pick out the symbol in a given position of a string. More precisely $co_a(x, p)$ holds when the $|p| + 1$'st symbol in x is a. This is definable in $\mathcal{W}_k$ by

$$\phi_a(x, p) : (\exists y \preceq_p x) el(p, y) \wedge \sigma_a y \preceq_p x,$$

for $a \neq 0$ and in case $a = 0$ we use $\phi_a \vee |p| \geq |x|$.

Then Φ_R is

$$(\exists r_1)(\exists r_2) \cdots (\exists r_{|S|})(\exists l)\, [\mathrm{STATE} \wedge \mathrm{INIT} \wedge \mathrm{TRANS} \wedge \mathrm{FINAL}].$$

where

1. STATE is
$$maxl(x_1, \cdots, x_m, l) \wedge \bigwedge_{1 \leq i \leq |S|} |r_i| = |l| \wedge \bigwedge_{q_i \neq q_j \in S} (\forall p \preceq_p l)\ [\neg\, (co_1(r_i, p) \wedge co_1(r_j, p))]\,,$$

2. INIT is $co_1(r_1, \lambda)$,

3. FINAL is $\bigvee_{r_i \in F} (\exists p \prec_p r_i) \bigvee_v [\sigma_v p = r_i \wedge co_1(r_i, p)]$ and

4. TRANS is
$$(\forall p \preceq_p l) \bigwedge_{\Delta(q_i, \sigma) = q_j} [\,[co_1(r_i, p) \wedge \bigwedge_{1 \leq h \leq m} co_{\sigma(h)}(a_h, p)] \rightarrow co_1(r_j, p1)\,],$$

   where $\sigma \in \Sigma_\perp^k$ and $\sigma(h)$ is the h-th component of σ except when this component is $\perp$, in which case $\sigma(h)$ is 0.

A minor technicality is that $\Phi_R(\otimes(\bar\lambda))$ never holds since for example INIT fails. So if the automaton accepts the empty tuple $\otimes(\bar\lambda)$ then the desired formula is $\Phi_R \vee \mathrm{EMPTY}$ where

5. EMPTY is $\bigwedge_{1\le i\le m} x_i = \lambda$.

For correctness we check that Φ_R satisfies Property C.1. To this end suppose that the state set of the automaton S is $\{q_1,\cdots,q_{|S|}\}$. Then if $\otimes(a_1,\cdots,a_m)$, a string of length l say, is accepted by the automaton and $(s_j)_{1\le j\le l}$ is the successful path, then $\Phi_R(a_1,\cdots,a_m)$ is true by defining r_i, for every $1\le i\le |S|$, as the string of length l consisting of a 1 in the j-th place if and only if $s_j = q_i$. Conversely if $\Phi_R(a_1,\cdots,a_m)$ then the automaton accepts the input $\otimes(a_1,\cdots,a_m)$ as witnessed by the run $(s_j)_{1\le j\le l}$ defined as $s_j = q_i$ if and only if r_i has a 1 in the j-th position. In fact STATE ensures that the r_i uniquely determine a sequence of states, INIT ensures that the first state is the initial state, TRANS ensures that the sequence respects the transition table, and FINAL ensures that the last state is a final state. This completes the proof. $\triangleleft$

Proposition C.2.4 *Suppose $\mathcal{A}$ is bi-interpretable with $\mathcal{W}_k$, with co-ordinate maps $\nu : A \to \Sigma^\star$ and $\mu : \Sigma^\star \to A$, and let $R \subset A^m$. Then relation R is first order definable in $\mathcal{A}$ if and only if $\nu(R)$ is FA recognisable over Σ.*

Proof For the forward direction let $R \subset A^m$ be first order definable via an $\mathcal{A}$-formula Ψ. Then for every $\overline{a} = (a_1,\cdots,a_m) \in A^m$,

$$\overline{a} \in R \iff \mathcal{A} \models \Psi(\overline{a}) \iff \mathcal{W}_k \models \Psi^\nu(\nu(\overline{a})),$$

where Ψ^ν is the corresponding $\mathcal{W}_k$-formula. So by Theorem C.2.3 there is some automaton $\mathcal{B}_R$ over Σ such that for every $\overline{x} \in \Sigma^{\star m}$,

$$\otimes(\overline{x}) \text{ is accepted by } \mathcal{B}_R \text{ if and only if } \mathcal{W}_k \models \Psi^\nu(\overline{x}).$$

But the set $\nu(A^m)$ is also first order definable in $\mathcal{W}_k$, say by formula Δ, and so again by Theorem C.2.3 there is an automaton $\mathcal{B}_\Delta$ over Σ such that for every $\overline{x} \in \Sigma^{\star m}$,

$$\otimes(\overline{x}) \text{ is accepted by } \mathcal{B}_\Delta \text{ if and only if } \mathcal{W}_k \models \Delta^\nu(\overline{x}).$$

So restricting the automaton $\mathcal{B}_R$ to the regular set $\mathcal{B}_\Delta$ yields the required automaton. Indeed, form $\mathcal{B}$ by intersecting $\mathcal{B}_R$ with the automaton $(\mathcal{B}_\Delta)^m$, the m-fold cross product of $\mathcal{B}_\Delta$. Then for every $\overline{a} \in A^m$,

$$\otimes(\nu(\overline{a})) \text{ is accepted by } \mathcal{B} \text{ if and only if } \mathcal{W}_k \models \Psi^\nu(\nu(\overline{a})).$$

For the reverse direction, fix $R \subset A^m$ so that $\nu(R)$ is FA recognisable over Σ. By Theorem C.2.3 there is a $\mathcal{W}_k$-formula Ψ_R such that for every $\overline{x} = \nu(\overline{a})$,

$$\overline{a} \in R \text{ if and only if } \mathcal{W}_k \models \Psi_R(\overline{x}).$$

So using the interpretation of $\mathcal{W}_k$ in $\mathcal{A}$, there is some $\mathcal{A}$-formula Ψ_R^μ such that for every $\overline{x}$,

$$\mathcal{W}_k \models \Psi_R(\overline{x}) \text{ if and only if } \mathcal{A} \models \Psi_R^\mu(\mu(\overline{x})).$$

By bi-interpretability, $\mu\nu$ is first order definable in $\mathcal{A}$, say via formula ϕ. So for every $\overline{a}$,

$$\overline{a} \in R \text{ if and only if } \mathcal{A} \models \Psi_R^\mu(\phi(\overline{a})).$$

Hence R is first order definable in $\mathcal{A}$ as required. $\lhd$

For $k > 1$, recall the structure $\mathcal{N}_k = (\mathbb{N}, +, |_k)$ where $m|_k n$ if m is a power of k and m divides n. The following theorem implies that $\mathcal{N}_k$ is complete, and is known as the Büchi–Bruyère Theorem, see Bruyère et al. [1994, Theorem 6.1].

Recall Theorem B.1.29 in which it is proved that $\mathcal{N}_k$ and $\mathcal{W}_k$ are bi-interpretable, where the co-ordinate mapping ν from $\mathcal{N}_k$ to $\mathcal{W}_k$ is defined as sending n to $\mathrm{base}_k(n)$, the shortest least-significant-digit-first base-k representation of n. So for $R \subset \mathbb{N}^m$ define $\otimes_k R \subset \Sigma^{m\star}$ as the set

$$\{\otimes(\mathrm{base}_k(n_1), \cdots, \mathrm{base}_k(n_m)) \mid (n_1, \cdots, n_m) \in R\}.$$

Theorem C.2.5 *A relation $R \subset \mathbb{N}^m$ is first order definable in $\mathcal{N}_k = (\mathbb{N}, +, |_k)$ if and only if $\otimes_k R$ is FA recognisable over $\Sigma = \{0, \cdots, k-1\}$.* [1]

Proof Follows immediately from Theorem B.1.29 and Proposition C.2.4. $\lhd$

As an aside, we note that Michaux and Point [1986] prove the reverse direction of the previous theorem by induction on the complexity of a regular expression for $\otimes_k R$. It is also worth mentioning that Bruyère et al. [1994, Theorem 5.1] state alternative characterisations of the relations definable in $(\mathbb{N}, +, |_k)$ in terms of notions called k–substitution and k–algebraicity. These will not be pursued here.

Theorem C.2.6 [Blumensath and Grädel - 2000, Theorem 5.4] *Let $\mathcal{A}$ be a structure. The following are equivalent:*

1. *$\mathcal{A}$ is automatically presentable.*

2. *$\mathcal{A}$ is FO interpretable in $\mathcal{W}_k$ for some, equivalently all, $k \geq 2$.*

3. *$\mathcal{A}$ is FO interpretable in $\mathcal{N}_k$ for some, equivalently all, $k \geq 2$.*

[1] In Bruyère et al. [1994] the structure considered is $(\mathbb{N}, +, V_k)$ where $V_k(m) = n$ if n is the highest power of k that divides m. Note that $|_k$ is definable in $(\mathbb{N}, +, V_k)$ and V_k is definable in $(\mathbb{N}, +, |_k)$ and so the formulation above is equivalent.

Proof The equivalence of the last two items follows from the mutual interpretability of $\mathcal{W}_k$ and $\mathcal{N}_j$, Theorem B.1.29. The equivalence of the first two items is as follows. Suppose $\mathcal{A}$ is automatically presentable. Then it is isomorphic to some structure $\mathcal{A}'$ that is automatic over Σ. By taking isomorphic copies, we may assume that $\Sigma = \{0, 1, \cdots, j-1\}$ for some $j \geq 2$. Then by Theorem C.2.3 $\mathcal{A}'$ is FO definable in $\mathcal{W}_j$, and so by Theorem B.1.29, $\mathcal{A}'$ if FO interpretable in $\mathcal{W}_k$ for every $k \geq 2$. Conversely, suppose $\mathcal{A}$ is FO interpretable with dimension j in $\mathcal{W}_k$ for some $k \geq 2$. Let $\mathcal{A}'$ be the isomorphic quotient structure that is FO definable in $\mathcal{W}_k$. Then for every atomic relation of $\mathcal{A}'$, and the domain and the congruence, there is an automaton over alphabet $\Sigma = \{0, 1, \cdots, k-1\}$ that accepts it. Hence $\mathcal{A}'$ is automatic (by Proposition B.1.22 over alphabet $\Sigma_\perp^j$), and so $\mathcal{A}$ is automatically presentable. $\lhd$

We remark that taking $k = 2$ implies that a binary alphabet suffices for automaticity, see Proposition B.1.21.

Elgot [1961], Eilenberg, Elgot, and Shepherdson [1969]

This section presents an alternative proof of the translation from automata to formulae. Elgot [1961] originally gave this proof in the setting of weak monadic second order logic of $(\mathbb{N}, S)$, though we follow an equivalent first order presentation from Eilenberg et al. [1969].

Theorem C.2.7 *Suppose $k = |\Sigma| \geq 2$. Then a relation $R \subset \Sigma^{\star n}$ is FA recognisable if and only if R is first order definable in $\mathcal{W}_k$.*

Proof Sufficiency follows from observing that the structure $\mathcal{W}_k$ is automatic over Σ and then applying Theorem B.1.16.

For necessity, we follow the proof in Eilenberg et al. [1969]. Define the *basic local* subsets of $\Sigma^\star$ as $\{\lambda\}, \sigma\Sigma^\star, \Sigma^\star\sigma$ and $\Sigma^\star\sigma\delta\Sigma^\star$ for $\sigma, \delta \in \Sigma$. The proof is in two steps. The first step is to prove that if $R \subset \Sigma^{\star n}$ is FA recognisable then R is first order definable in terms of basic local sets. The second step is to prove that the basic local sets are first order definable in $\mathcal{W}_k$.

Let $R \subset \Sigma^{\star n}$ be FA recognisable, say $\otimes R$ is recognised by automaton $\mathcal{A} = (S, \iota, \Delta, F)$ over alphabet $(\Sigma_\perp^n)^\star$. Since $|\Sigma| \geq 2$, there exists r such that S embeds into Σ^r, say via embedding j. Consider a new alphabet $\Omega = \Sigma_\perp^{n+r}$. Let f be the projection $f : \Omega^\star \to \Sigma_\perp^{n\star}$ extending the mapping sending $(\sigma_1, \cdots, \sigma_{n+r})$ to $(\sigma_1, \cdots \sigma_n)$, for $\sigma_i \in \Sigma_\perp$. Define a language K over Ω as the intersection of the following basic local sets. For every $\sigma, \delta \in \Sigma_\perp^n$,

a) $(\sigma, j(s))\Omega^\star$ when $s = \iota$,

b) $\Omega^\star(\sigma, j(s))$ when $\Delta(s, \sigma) \in F$, and

c) $\Omega^\star(\sigma, j(s))(\delta, j(t))\Omega^\star$ when $\Delta(s, \sigma) = t$.

Note that K is regular since it is a finite intersection of regular terms. Then $\otimes R = f(K)$. Indeed, $v = \sigma_1 \cdots \sigma_n \in \otimes R$ if and only if there is a successful run $s_0, \cdots, s_n$ of v on $\mathcal{A}$, that is $w = (\sigma_1, j(s_0)) \cdots (\sigma_n, j(s_{n-1})) \in K$, and so $f(w) = v$. Conversely, if $w \in K$, then there is a successful run of $\mathcal{A}$ on $f(w)$, and so $f(w) \in \otimes R$.

So $(a_1, \cdots, a_n) \in R$ if and only if $\otimes(a_1, \cdots, a_n) \in \otimes R$ if and only if there exists $b_1, \cdots, b_r \in \Sigma^\star$ such that $\otimes(a_1, \cdots, a_n, b_1, \cdots, b_r) \in K$ if and only if there exists $b_1, \cdots, b_r \in \Sigma^\star$ such that $(a_1, \cdots, a_n, b_1, \cdots, b_r) \in L$ where $L \subset (\Sigma^\star)^{n+r}$ has the property that $\otimes L = K$. Note that L is well defined since $(a_1, \cdots, a_n, b_1, \cdots, b_r) \in K$ implies that $(a_1, \cdots, a_n) \in (\otimes R) \cdot \{\perp\}^{n\star}$ and $(b_1, \cdots, b_r) \in (\otimes \Sigma^{\star r}) \cdot \{\perp\}^{n\star}$. In summary we have

$$R = \{(a_1, \cdots, a_n) \mid (\exists b_1, \cdots, \exists b_r)(a_1, \cdots, a_n, b_1, \cdots, b_r) \in L\},$$

and this completes the first step. So to prove that R is definable in W_k it suffices to show that for every $m \in \mathbb{N}$, if $L \subset \Sigma^{\star m}$ satisfies $\otimes L = K$ for some basic local subset K of $\Sigma_\perp^m$, then L is definable in W_k.

If $K = \{\perp\}^m$ then $L = \{\lambda\}^m$ which is definable by

$$\bigwedge_{1 \leq i \leq m} x_i = \lambda,$$

where $x = \lambda$ is definable by $(\forall y \preceq_p x) y = x$. For $a_i \in \Sigma \cup \{\lambda\}$, if $K = (a_1, \cdots, a_m) \cdot \Sigma_\perp^{m\star}$ then

$$L = \{(x_1, \cdots, x_m) \mid a_i \preceq_p x_i\}.$$

If $K = \Sigma_\perp^{m\star} \cdot (a_1, \cdots, a_m)$ then

$$L = \{(x_1, \cdots, x_m) \mid (\exists z \preceq_p x_i)\, \sigma_{a_i} z = x_i\}$$

with the convention that $\sigma_\lambda(z) = z$. Finally if $K = \Sigma_\perp^{m\star} \cdot (a_1, \cdots, a_m) \cdot (b_1, \cdots, b_m) \cdot \Sigma_\perp^{m\star}$ then

$$L = \{(x_1, \cdots, x_m) \mid (\exists z \preceq_p x_i)\, \sigma_{b_i} \sigma_{a_i} z \preceq_p x_i\}.$$

This completes the proof. $\lhd$

Nabebin [1976], Blumensath [1999]

This section presents a complete structure for automatic structures whose domain consists of strings over a unary alphabet, namely $|\Sigma| = 1$. Theorem C.2.7 does not hold for $k = 1$ since for instance the set of strings of even length is regular though not definable in $\mathcal{W}_1$.

Note that $\mathcal{W}_1$ is isomorphic to $(\mathbb{N}, S, \leq, =)$ where $S(n) = n + 1$ and $\leq$ and $=$ have their usual interpretations. By adding predicates that enable defining the missing relations like $\{n \in \mathbb{N} \mid n \text{ is even}\}$ one gets a structure complete for those automatic structures whose domain consists of strings over a unary alphabet.

First we characterise those relations that are FA recognisable over a unary alphabet. Define $\nu : \mathbb{N} \to \{1\}^\star$ by $n \mapsto 1^n$ and notice that this is an isomorphism between the monoids $(\mathbb{N}, +)$ and $(\{1\}^\star, \cdot)$. Write $\equiv_p$ for the binary relation of congruence modulo p, namely

$$\{(x, y) \mid p \text{ divides } y - x\}.$$

A set $F \subset \mathbb{N}$ is *eventually periodic* if there exists a *threshold* $t \in \mathbb{N}$ and a *period* $p \in \mathbb{N}$ such that for $x \geq t$, it holds that $x \in F$ if and only if $x + p \in F$. Note that if F has period p then it also has period kp for every $k > 1$. The following fact is elementary.

Fact C.2.8 *For $\Sigma = \{\sigma\}$, $R \subset \Sigma^\star$ is regular if and only if*

$$\nu^{-1}(R) = \{k \in \mathbb{N} \mid \sigma^k \in R\}$$

is eventually periodic.

The following characterisation is due to Nabebin [1976] and Blumensath [1999].

Theorem C.2.9 *Let $R \subset \mathbb{N}^n$ and $\Sigma = \{1\}$. Then $\nu(R)$ is FA recognisable over Σ if and only if R is first order definable in the structure*

$$\mathcal{U} = (\mathbb{N}, \leq, (\equiv_p)_{p \in \mathbb{N}}).$$

Proof For the duration of this proof, definable means first order definable in $\mathcal{U}$. The structure $\nu(\mathcal{U})$ is automatic over $\{1\}$. Hence if R is definable then by Theorem B.1.16, $\nu(R)$ is FA recognisable over $\{1\}$.

Conversely suppose $R \subset \mathbb{N}^n$ is such that $\nu(R)$ is FA recognisable over $\{1\}$. By the decomposition theorem, R is a finite union of sets of the form $R_0 R_1 \cdots R_k$ where $k \in \mathbb{N}$ and each $\nu(R_i)$ regular over alphabet $\{1\}_{A_i}$. Note that we may assume that the A_i's are distinct since if $\nu(R_i)$ and $\nu(R_{i+1})$ are regular subsets of $\{1\}_{A_l}$ then so is $\nu(R_i \cdot R_{i+1})$. Hence we have that $k < n$. Furthermore since the union is finite, it is sufficient to establish that $R = R_0 \cdots R_k$ is definable. In this case since $A_i \supset A_{i+1}$, there is a bijection $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$ such that for every $\overline{x} = (x_1, \cdots, x_n) \in R$ it holds that

$$x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}.$$

Write $\pi(R)$ for the relation obtained by permuting the co-ordinates of R via π. Note that it is also FA recognisable over $\{1\}$. It is sufficient to establish that $\nu(\pi(R))$ is definable since in this case so is $\nu(R)$ by Theorem B.1.5. In other words, we may assume that π is the identity $j \mapsto j$ and so each A_i is a final segment of $\{1, \cdots, n\}$.

We proceed by induction on n, the arity of R, and prove that there exists $p \in \mathbb{N}$ and for $1 \leq i \leq n$ there exists $F_i \subset \mathbb{N}$, eventually periodic sets with period p, with the following property. For every $\overline{x}$, it holds that $\overline{x} \in R$ if and only if for each i there exists $a_i \in F_i$ such that

$$x_1 = a_1 \text{ and for } 1 \leq k < n,$$
$$x_{k+1} = x_k + a_{k+1}.$$

For fixed p and F_i's this can easily be expressed as a first order definition in $\mathcal{U}$, as required.

case $n = 1$**.** Then $\nu(R)$ is a regular subset of $\{1\}^\star$. Hence by the Fact R is eventually periodic, as required.

case $n > 1$**.** Say $|A_0| = j$ in which case A_0 being a final segment, is equal to $\{n-j+1, \cdots, n\}$. If $j < n$ then consider the j-ary relation

$$S = \{(x_1, \cdots, x_j) \mid (y_1, \cdots, y_n) \in R \wedge \bigwedge_{1 \leq i \leq j} x_i = y_{n-j+i}.\}$$

Since $\nu(S)$ is FA recognisable over $\{1\}$, (it is first order definable from R), by induction (which is applicable since $j < n$ and $S = S_0 \cdots S_k$ where each S_i is a final segment of $\{1, \cdots, j\}$) it satisfies the claim for some p, and F_i, for $i \leq j$. Hence R too satisfies the claim with p and sets G_i defined as F_{n-j+i} for $i \leq j$ and $G_i = \{0\}$ for $i > j$.

So assume that $A_0 = \{1, \cdots, n\}$. Then $\nu(R_0)$ is a regular subset of $\{1\}^{n\star}$ and so there exists $p_0 \in \mathbb{N}$ and eventually periodic set $G_0 \subset \mathbb{N}$ with period p_0 such that

$$R_0 = \{(l, \cdots, l) \mid l \in G_0\}.$$

Since $|A_1| < n$ as before $R_1 \cdots R_k$ satisfies the claim for some p' and G_i for $1 \leq i \leq n$. Make the periods uniform by replacing both p_0 and p' by $q = \mathrm{lcm}(p_0, p')$.

Now $(x_0, \cdots, x_n) \in R$ is equivalent to the condition that there is some $(l, \cdots, l) \in R_0$ and $(y_1, \cdots, y_n) \in R_1 \cdots R_k$ such that $x_i = l + y_i$ for every $1 \leq i \leq n$. By assumption there exists $a_0 \in G_0$ and $a_i \in G_i$ such that

$$\begin{aligned} &l = a_0 \text{ and } y_1 = a_1, \text{ and for } 1 \leq k < n, \\ &y_{k+1} = y_k + a_{k+1}. \end{aligned}$$

So define $F_1 = \{a_0 + a_1 \mid a_0 \in G_0 \wedge a_1 \in G_1\}$ and $F_i = G_i$ for $i \neq 1$. Then R satisfies the claim with respect to $q \in \mathbb{N}$ and eventually periodic sets F_i. This completes the proof. $\triangleleft$

Corollary C.2.10 *A structure $\mathcal{A}$ is automatically presentable over a unary alphabet if and only if it is interpretable* (*with dimension* 1) *in the structure* $(\mathbb{N}, \leq, (\equiv_n)_{n \in \mathbb{N}})$.

Weak monadic second order theory of $(\mathbb{N}, S)$

This section presents a complete structure for automatic structures in terms of second order logic. Precisely the weak monadic second order WMSO logic consists of individual variables $x, y, \cdots$ that range over elements of the domain and weak monadic variables $X, Y, \cdots$ that range over **finite subsets** of the domain.[2] Quantification is possible over individual and weak monadic

[2] Note that a formula with free set variables, say $\Phi(X, Y)$, defines a relation in which the elements are finite subsets of the domain. This should be distinguished from an alternative definition of WMSO in which it is merely the quantified set variables that are restricted to finite subsets.

variables. There is always a symbol $\in$ where $x \in Y$ is interpreted as meaning that individual element x is contained in the set Y.

The following theorem was essentially proved in Büchi [1960] and Elgot [1961], directly as in the proofs of Theorems C.2.3 and C.2.7 respectively. However we follow Elgot and Rabin [1966] and show that $\mathrm{WMSO}(\mathbb{N}, S)$ is mutually interpretable with FO$(\mathbb{N}, +, |_2)$. We first define these notions.

If $\Phi(\overline{x}, \overline{X})$ is a WMSO formula of structure $\mathcal{B}$, here $\overline{x}$ are the free individual variables, and $\overline{X}$ are the free weak monadic variables, then Φ^B is defined as $\{(\overline{x}, \overline{X}) \mid \mathcal{B} \models \Phi(\overline{x}, \overline{X})\}$. This relation is then said to be WMSO *definable* in $\mathcal{B}$. As in the first order case, if Δ, Φ_{R_i} and ϵ are $\mathcal{B}$-formulae, and $\Phi^B_{R_i}$ is a relation on Δ^B, and ϵ^B is a congruence relation on the structure

$$(\Delta^B, \Phi^B_{R_1}, \cdots),$$

then the quotient structure is called WMSO-definable in $\mathcal{B}$. If $\mathcal{A}$ is isomorphic to this quotient structure, say via map ν, then $\mathcal{A}$ is WMSO-*interpretable* in $\mathcal{B}$. In this case we write $\mathcal{A} \leq^{\nu}_{\mathrm{WMSO}} \mathcal{B}$.

Now if Δ contains only first order variables, then for every WMSO formula Φ of $\mathcal{A}$ one may effectively find a WMSO formula Φ^ν of $\mathcal{B}$ such that for every $\overline{x}$ and $\overline{X}$,

$$\mathcal{A} \models \Phi(\overline{x}, \overline{X}) \iff \mathcal{B} \models \Phi^\nu(\nu(\overline{x}, \overline{X})).$$

However if Δ contains some weak monadic second order variables, then we can only conclude that for every **FO** formula Φ of $\mathcal{A}$ one may effectively find a WMSO formula Φ^ν of $\mathcal{B}$ such that for every $\overline{x}$,

$$\mathcal{A} \models \Phi(\overline{x}) \iff \mathcal{B} \models \Phi^\nu(\nu(\overline{x})).$$

In this case we write FO$(\mathcal{A}) \leq^\nu \mathrm{WMSO}(\mathcal{B})$.

Similarly write $\mathrm{WMSO}(\mathcal{B}) \leq^\mu$ FO$(\mathcal{A})$ to mean that $\mathcal{B} \leq^{\mu}_{\mathrm{FO}} \mathcal{A}$ and furthermore that for every **WMSO** formula Φ of $\mathcal{B}$ one may effectively find a FO formula Φ^μ of $\mathcal{A}$ such that for every $\overline{x}$ and $\overline{X}$,

$$\mathcal{B} \models \Phi(\overline{x}, \overline{X}) \iff \mathcal{A} \models \Phi^\mu(\mu(\overline{x}, \overline{X})).$$

In case FO$(\mathcal{A}) \leq^\nu \mathrm{WMSO}(\mathcal{B})$ and $\mathrm{WMSO}(\mathcal{B}) \leq^\mu$ FO$(\mathcal{A})$ we say that FO$(\mathcal{A})$ and $\mathrm{WMSO}(\mathcal{B})$ are *mutually interpretable*.

Theorem C.2.11 *A structure $\mathcal{A}$ is automatically presentable if and only if it is weak monadic second order interpretable in $(\mathbb{N}, S)$.*

Proof We will shortly establish that FO$(\mathbb{N}, +, |_2)$ and $\mathrm{WMSO}(\mathbb{N}, S)$ are mutually interpretable. First notice that $\mathcal{A} \leq_{\mathrm{FO}} (\mathbb{N}, +, |_2)$ and FO$(\mathbb{N}, +, |_2) \leq \mathrm{WMSO}(\mathbb{N}, S)$ imply that FO$(\mathcal{A}) \leq \mathrm{WMSO}(\mathbb{N}, S)$. So assume $\mathcal{A}$ is automatically presentable. Then it is first order interpretable in $(\mathbb{N}, +, |_2)$, and so FO($\mathcal{A}$) $\leq$ $\mathrm{WMSO}(\mathbb{N}, S)$, and in particular $\mathcal{A}$ is WMSO interpretable in $(\mathbb{N}, S)$, as required.

Conversely notice that

$$\mathcal{A} \leq_{\mathrm{WMSO}} (\mathbb{N}, S) \text{ and } \mathrm{WMSO}(\mathbb{N}, S) \leq \mathrm{FO}(\mathbb{N}, +, |_2)$$

imply that $\mathrm{WMSO}(\mathcal{A}) \leq \mathrm{FO}(\mathbb{N}, +, |_2)$. So assume $\mathcal{A}$ is WMSO interpretable in $(\mathbb{N}, S)$. Then $\mathrm{WMSO}(\mathcal{A}) \leq \mathrm{FO}(\mathbb{N}, +, |_2)$, and in particular $\mathcal{A}$ is isomorphic to a structure that is first order definable in $(\mathbb{N}, +, |_2)$; hence $\mathcal{A}$ is automatically presentable as required.

FO$(\mathbb{N}, +, |_2) \leq^{\nu}$ **WMSO**$(\mathbb{N}, S)$: Consider the bijection ν from $\mathbb{N}$ onto the finite subsets of $\mathbb{N}$ where $m \in \nu(n)$ if and only if the m-th position of the shortest binary expansion of n contains a 1 (counting the least significant digit as the zero'th position). For instance $\nu(0) = \emptyset$, $\nu(1) = \{0\}$, $\nu(2) = \{1\}$, $\nu(3) = \{0, 1\}$, and $\nu(4) = \{2\}$. Note that $\nu(2^n) = \{n\}$ for every $n \in \mathbb{N}$.

Let Φ be a first order $(\mathbb{N}, +, |_2)$-formula. Proceed by induction on the complexity of Φ to construct a weak monadic second order $(\mathbb{N}, S)$-formula Φ' such that

$$\Phi(u_1, \cdots, u_k) \iff \Phi'(\nu(u_1), \cdots, \nu(u_k)). \tag{C.2}$$

We will use the following abbreviations:

$$\begin{array}{rcl}
\{0\} & \text{for} & \neg(\exists x)\,[S(x) = z] \\
\mathrm{Sing(X)} & \text{for} & (\exists x)(\forall y)\,[x \in X \wedge (y \in X \rightarrow y = x)] \\
\mathrm{X \subset Y} & \text{for} & (\forall z)\,[z \in X \rightarrow z \in Y] \\
\mathrm{Pred(z) = y} & \text{for} & S(y) = z \\
\mathrm{Pref(X)} & \text{for} & (\forall z)\,[z \in X \rightarrow \mathrm{Pred(z) \in X}] \\
x \leq y & \text{for} & (\forall Z)\,[\mathrm{Pref(Z) \rightarrow (y \in Z \rightarrow x \in Z)}] \\
\mathrm{Min(X) = y} & \text{for} & y \in X \wedge (\forall z)\,[z \in X \rightarrow y \leq z].
\end{array}$$

The atomic relations are defined as follows: $\mathrm{Div(N, M)}$ is defined as $\mathrm{Sing(N) \wedge Min(N, M)}$ and $\mathrm{Sum(D, E, F)}$ is defined as

$$(\exists C)\left[\{0\} \subset C \wedge (\forall p) \bigvee_{(d,e,c,f,g) \in S} p \in^d D \wedge p \in^e E \wedge p \in^c C \wedge p \in^f F \wedge S(p) \in^g C\right],$$

where $S = \{(d, e, c, f, g) \mid d + e + c = f + 2g\}$ for $d, e, c, f, g \in \{0, 1\}$ and $\in^k$ interpreted as $\in$ if $k = 1$ and $\notin$ if $k = 0$. Then

$$\begin{array}{rcl}
n|_2 m & \iff & \mathrm{Div}(\nu(\mathrm{n}), \nu(\mathrm{m})), \text{ and} \\
d + e = f & \iff & \mathrm{Sum}(\nu(\mathrm{d}), \nu(\mathrm{e}), \nu(\mathrm{f})).
\end{array}$$

Suppose Ψ and Ξ are formula with corresponding formulae Ψ' and Ξ' satisfying condition C.2. If $\Phi = \Psi \vee \Xi$ then defining $\Phi' = \Psi' \vee \Xi'$ satisfies the condition. If $\Phi = \neg\Psi$ then defining $\Phi' = \neg\Psi'$ satisfies the condition. If $\Phi = (\exists x)\,[\Psi(x, \overline{y})]$ then defining $\Phi' = (\exists X)\,[\Psi'(X, \overline{Y})]$ satisfies the condition. This completes the induction. Also note that $\nu(\mathbb{N}, +, |_2) = (\mathrm{Delta}, \mathrm{DIV}, \mathrm{SUM})$ where $\Delta(X)$ is the formula $X = X$.

WMSO$(\mathbb{N}, S)$ $\leq^\mu$ **FO**$(\mathbb{N}, +, |_2)$: Let μ be the bijection from finite subsets of $\mathbb{N}$ onto $\mathbb{N}$ where $\mu(X)$ is defined as $\sum_{x \in X} 2^x$. For instance $\mu(\emptyset) = 0$, $\mu(\{0\}) = 1$, $\mu(\{1\}) = 2$ and $\mu(\{0, 1\}) = 3$. Extend μ to individual elements and so define $\mu(n)$ for $\mu(\{n\})$, for $n \in \mathbb{N}$. Note that $\mu(n) = 2^n$.

Let Φ be a weak monadic second order $(\mathbb{N}, S)$-formula. Proceed by induction on the complexity of Φ to construct a first order $(\mathbb{N}, +, |_2)$-formula $\Phi^\dagger$ such that

$$\Phi(X_1, \cdots, X_k, x_1, \cdots, x_l) \iff \Phi^\dagger(\mu(X_1), \cdots, \mu(X_k), \mu(x_1), \cdots, \mu(x_l)). \tag{C.3}$$

Recall the following definable relations (see Exercise B.1.28): $P_2(n)$ says that n is a power of 2, and $\epsilon_j(m, n)$, for n a power of 2, $j \in \{0, 1\}$, holds if and only the coefficient of n in the binary representation of m is j.

The atomic relations are defined as follows:

$$\begin{array}{rll} \mathrm{In(n, m)} & \text{is defined as} & P_2(n) \wedge \epsilon_1(m, n) \\ \mathrm{Succ(n) = m} & \text{is defined as} & P_2(n) \wedge P_2(m) \wedge \neg(\exists c)\,[P_2(c) \wedge n < c < m] \end{array}$$

Then

$$\begin{array}{rcl} x \in X & \iff & \mathrm{In}(\mu(\mathrm{x}), \mu(\mathrm{X})), \text{ and} \\ S(x) = y & \iff & \mathrm{Succ}(\mu(\mathrm{x})) = \mu(\mathrm{y}). \end{array}$$

Suppose Ψ and Ξ are formulae with corresponding formulae $\Psi^\dagger$ and $\Xi^\dagger$ satisfying condition C.3. Then $[\Psi \vee \Xi]^\dagger$ is defined as $\Psi^\dagger \vee \Xi^\dagger$, $[\neg\Psi]^\dagger$ is defined as $\neg\Psi^\dagger$, $[(\exists y)\Psi(y, \cdots)]^\dagger$ as $(\exists y)\,[P_2(y) \wedge \Psi^{\text{-}}(y, \cdots)$ and $[(\exists Y)\Psi(Y, \cdots)]^\dagger$ as $(\exists y)\Psi^\dagger(y, \cdots)$. The resulting formulae satisfy condition C.3. This completes the induction. Also note that $\mu(\mathbb{N}, S) = (\Delta^{\mathcal{N}_2}, \mathrm{Succ})$ where $\Delta(x)$ is the formula $x = x$. $\triangleleft$

Corollary C.2.12 *The weak monadic second order theory of* $(\mathbb{N}, S)$ *is decidable.*

Proof Let Φ be a WMSO-sentence of $(\mathbb{N}, S)$. Then Φ^μ is a FO-sentence of $(\mathbb{N}, +, |_2)$ such that

$$(\mathbb{N}, S) \models \Phi \text{ if and only if } (\mathbb{N}, +, |_2) \models \Phi^\mu.$$

Now use the procedure deciding $(\mathbb{N}, +, |_2)$ to settle the latter. $\triangleleft$

C.3 A Myhill-Nerode type theorem

This characterisation is due to Khoussainov and Nerode [1995, Theorem 3.2] where it is stated in slightly different notation. As in the last paragraph of the proof of Theorem C.1.1, recall that for $w \in \Sigma^{\star n}$, the definition of convolution gives us that $\otimes w = r_0 \cdots r_k$. Define A_i as the set $\{j \mid \text{the } j\text{-th position of } r_i \text{ is not } \perp\}$ so that $r_i \in (\Sigma_{A_i})^\star$. Then $A_k \subset \cdots \subset A_0 \subset \{1, \cdots, n\}$. Define the *sort* of w as A_k.

An *n-congruence* η on $\Sigma^{\star n}$ is an equivalence relation satisfying the following two properties:

1. If $w \,\eta\, v$ then w and v have the same sort.

2. For every $u \in \Sigma^{\star n}$, say with sort A, and every w, v of the same sort, say B, if $A \subset B$ then $w \,\eta\, v \iff (w \cdot u) \,\eta\, (v \cdot u)$.

Theorem C.3.1 *Let $R \subset \Sigma^{\star n}$. Then R is FA recognisable if and only if R is the union of equivalence classes of some n-congruence η on $\Sigma^{\star n}$ of finite index.*

Proof Suppose $\otimes R$ is recognised by the deterministic automaton (S, ι, Δ, F). Then define the n-congruence η_R as satisfying (w, v) exactly when w and v have the same sort, say B, and for every $u \in \Sigma^{\star n}$ of sort $A \subset B$, we have $wu \in R \iff vu \in R$. Then η_R has finite index since the mapping sending $[w]_{\eta_R}$ (the equivalence class of w) to the state $\Delta(\iota, \otimes w)$ is one-to-one. Further $R = \{[w]_{\eta_R} \mid \Delta(\iota, \otimes w) \in F\}$.

For the converse, suppose R is the union of equivalence classes of some finite n-congruence η of finite index. If $\otimes w = r_0 \cdots r_k$ as above, then $[w]_\eta = [r_0]_\eta \cdots [r_k]_\eta$. So $\otimes R$ is the finite union of sets which are products as in the decomposition theorem. Hence R is FA recognisable.

Alternatively, define an automaton accepting $\otimes R$ as follows. The state set is $[w]_\eta$ for $w \in \Sigma^{\star n}$. The initial state is $\{\lambda\}^n$. The final states are those $[w]_\eta \subset R$. There is a transition from state $[w]_\eta$ to $[v]_\eta$ on input $\sigma \in \Sigma_\perp^n$ exactly when $w \cdot \delta \in [v]_\eta$, where $\otimes\delta = \sigma$. Then $v \in L(\mathcal{A})$ if and only if $\mathcal{A}$ has a successful run $[w_0]_\eta, \cdots, [w_n]_\eta$ on $\otimes v$ if and only if $v = w_n \in R$. $\triangleleft$

As an immediate corollary we have

Theorem C.3.2 *Let $\mathcal{A}$ be a relational structure with $A \subset \Sigma^\star$. Then $\mathcal{A}$ is automatic over Σ if and only if for the domain and every atomic relation R of $\mathcal{A}$ there is some n-congruence η of finite index (here n is the arity of R) so that R is the union of equivalence classes of η.* $\triangleleft$

C.4 Reduction to automatic graphs

This section shows that it is sufficient, in some sense, to study automatic graphs. We describe a procedure that given an automatic structure $\mathcal{A}$ (of finite signature) produces an automatic undirected graph $\mathcal{G}(\mathcal{A})$ so that $\mathcal{A}$ and $\mathcal{G}(\mathcal{A})$ are mutually interpretable. The transformation of $\mathcal{A}$ into $\mathcal{G}(\mathcal{A})$, for arbitrary structures, is from Hodges [1993, Theorem 5.5.1]. Then as a consequence of Proposition C.2.4 we have that $\mathcal{A}$ is automatically presentable if and only if $\mathcal{G}(\mathcal{A})$ is automatically presentable. However we directly check that the interpretation preserves automaticity (and regularity of embeddings), rather than appealing indirectly to interpretations in complete structures.

Theorem C.4.1 *For every structure $\mathcal{A}$ there is a graph $\mathcal{G}(\mathcal{A})$ with the following properties.*

1. *$\mathcal{A}$ is automatic if and only if $\mathcal{G}(\mathcal{A})$ is automatic. And an automatic presentation of $\mathcal{G}(\mathcal{A})$ can be constructed in linear time in the size of an automatic presentation of $\mathcal{A}$.*

2. *$\mathcal{A}$ and $\mathcal{G}(\mathcal{A})$ are mutually interpretable.*

3. *The interpretation of $\mathcal{A}$ into $\mathcal{G}(\mathcal{A})$ preserves embeddings. That is $\mathcal{A}$ embeds in $\mathcal{B}$ if and only if $\mathcal{G}(\mathcal{A})$ embeds in $\mathcal{G}(\mathcal{B})$. Moreover the interpretation preserves FA recognisable embeddings.*

We remark that the proof implies that if $\mathcal{A}$ is automatic over Σ then $\mathcal{G}(\mathcal{A})$ is automatic over a possibly larger alphabet. So by Proposition B.1.21, $\mathcal{G}(\mathcal{A})$ is automatically presentable over a binary alphabet.

An n-*tag*, where $n > 1$, is an undirected graph isomorphic to $(\{0, 1, \ldots, n, c\}, E)$, where the set E of edges consists of all pairs $\{i, i+1\}$ for $0 \leq i < n$, $\{n, 1\}$ and $\{2, c\}$. The vertex 0 is the *start* of the n-tag. The element c is needed to make the tag rigid, that is a structure without nontrivial automorphisms.

With each element $a \in A$ we associate a 5-tag denoted by $T(a)$ so that the vertices of $T(a)$ are the words $ac, a, a1, \ldots, a5$ and edges $\{a, a1\}$, $\{ak, a(k+1)\}$ for $1 \leq k \leq 4$, $\{a5, a1\}$ and $\{a2, ac\}$. Here a is the start vertex of the tag $T(a)$. We say that every element of this tag is *indexed* by a.

List the relations of $\mathcal{A}$ as $R_1, \cdots, R_p$. Then code relation R_i of arity n_i as follows. Firstly, with each tuple $\bar{a} = (a_1, \ldots, a_{n_i})$ for which $R_i(\bar{a})$ is true we associate a $(5+i)$-tag $T(i, \bar{a})$ with vertices $\bar{a}$, $\bar{a}1$, …, $\bar{a}(5+i)$, $\bar{a}c$ and edges $\{\bar{a}, \bar{a}1\}$, $\{\bar{a}k, \bar{a}(k+1)\}$ for $1 \leq k \leq i+4$, and $\{\bar{a}(5+i), \bar{a}\}$ and $(\bar{a}2, \bar{a}c)$. We say that every element of this tag is *indexed* by $\bar{a}$. Secondly, with each tuple $\bar{a} = (a_1, \ldots, a_{n_i})$ for which $R_i(\bar{a})$ is true and where the kth element of this tuple is a_k, we associate the graph $L(i, \bar{a}, k)$ consisting of the k vertices a_k, $\bar{a}k1$, $\bar{a}k2$, $\bar{a}k3$, …, $\bar{a}kk$, $\bar{a}$ and edges appearing between every consecutive pair in this list. Thus, $L(i, \bar{a}, k)$ establishes a path of length $k+1$ between a and $\bar{a}$ in case a is indeed the kth element of the tuple $\bar{a}$. We say that every element on this path is *indexed* by $\overline{a}$.

Lemma C.4.2 *If the domain A and the predicate R_i of the structure $\mathcal{A}$ are regular over Σ then the following relations are regular.*

1. *The language* $T(A) = \bigcup_{a \in A} T(a)$ *and the binary relation*

$$E_1(A) = \{(x, y) \mid (\exists a \in A)\, [\{x, y\} \textit{ is an edge in } T(a)]\}.$$

2. *The language* $T(R_i) = \bigcup_{\bar{a} \in R_i} T(i, \bar{a})$ *and the binary relation*

$$E_2(R_i) = \{(x, y) \mid (\exists \bar{a} \in R_i)\, [\{x, y\} \textit{ is an edge in } T(i, \bar{a})]\}.$$

3. *The language* $L(i) = \bigcup_{\bar{a} \in R_i, 1 \leq k \leq n_i} L(i, \bar{a}, k)$ *and the binary relation*

$$E_3(R_i) = \{(x, y) \mid \{x, y\} \textit{ is an edge in some } L(i, \bar{a}, k)\}.$$

Proof Let r be the largest arity amongst the p-many predicates of R_i and define the alphabet $\Omega = \{c, 1, 2, \cdots, r + p + 5\}$. Assume that Σ is disjoint from Ω and denote their union by Σ'. Then the specified relations are regular over Ω. We illustrate part 1, the other two being similar.

$$T(A) = A \cdot \{\lambda, c, 1, 2, 3, 4, 5\}$$

and $\otimes E_1(A)$ is the union of the convolutions of

$$\{A \cdot i\} \times \{A \cdot j\}$$

for $(i, j) \in \{(\lambda, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (2, c)\}$. $\triangleleft$

Analysis of this proof shows that the sizes of the automata that recognise the languages $T(A)$ and $E_1(A)$ are linear in the size of the automaton recognising A. Similarly, for the rest of the relations.

Now define $\mathcal{G}(\mathcal{A}) = (V(\mathcal{A}), E(\mathcal{A}))$, where $V(\mathcal{A})$ is

$$T(A) \cup \bigcup_{1 \leq i \leq p} T(R_i) \cup \bigcup_{1 \leq i \leq p} L(i)$$

and $E(\mathcal{A})$ is

$$E_1(A) \cup \bigcup_{1 \leq i \leq p} E_2(R_i) \cup \bigcup_{1 \leq i \leq p} E_3(R_i).$$

Here is the proof of Theorem C.4.1.

Proof Lemma C.4.2 shows that if $\mathcal{A}$ is automatic over Σ then $\mathcal{G}(\mathcal{A})$ is automatic over Σ'. And the remarks after the lemma show that one can construct an automatic presentation of $\mathcal{G}(\mathcal{A})$ from one of $\mathcal{A}$ in linear time.

We now prove that $\mathcal{A}$ is first order definable in $\mathcal{G}(\mathcal{A})$. The property of a vertex being on the start of an n-tag, for a fixed n, is definable by the formula

$$\mathrm{Start}_n(x) : (\exists x_0) \cdots (\exists x_n)(\exists c)[x = x_0 \wedge \{(x_2, c), (x_n, x_1)\} \subset E(\mathcal{A}) \bigwedge_{0 \leq i < n} (x_i, x_{i+1}) \in E(\mathcal{A})].$$

Given $k \in \mathbb{N}$, the property stating that there is a path of length k between x and y is definable by the formula

$$\mathrm{Path}_k(x, y) : (\exists x_0) \cdots (\exists x_k)\, [x = x_0 \wedge y = x_k \wedge \bigwedge_{0 \leq i < k} (x_i, x_{i+1}) \in E(\mathcal{A})].$$

Define a mapping ν sending $a \in \mathcal{A}$ to the unique element $x \in V(A)$ that is the start of the 5-tag indexed by a. Define a domain $D = \{x \in V(\mathcal{A}) \mid \mathcal{G}(A) \models \mathrm{Start}_5(x)\}$. For relation R_i of $\mathcal{A}$ define the relation $P_i(x_1, \ldots, x_{n_i})$ by the formula

$$(\exists x)\, [\mathrm{Start}_{5+i}(x) \wedge \bigwedge_{1 \leq k \leq n_i} \mathrm{Path}_{k+1}(x_k, x)],$$

saying that there is a path of length $k + 1$ from x_k to the start of some $5 + i$ tag. Then $\nu(\mathcal{A}) = (D, R_1, \cdots, R_p)$ is definable in $\mathcal{G}(\mathcal{A})$. So if $\mathcal{G}(\mathcal{A})$ is automatic over Σ' then $\mathcal{A}$ is automatic over Σ (since $D \subset \Sigma^\star$).

For the converse we sketch a proof that $\mathcal{G}(\mathcal{A})$ is interpretable in $\mathcal{A}$ (see Hodges [1993] for a slightly fuller treatment). Recall that in the construction of $\mathcal{G}(\mathcal{A})$ there are three main types of vertices. There are those that occur on 5-tags, indexed by elements of $\mathcal{A}$. There are those that occur on $(5 + i)$-tags, indexed by tuples $\overline{x} \in R_i$. There are those that occur on paths of length $\leq n_j + 1$, where n_j is the arity of R_i, indexed by tuples $\overline{x} \in R_i$. Within each main type, there are a variety of *species*, which can be thought of as formulae in the language of undirected graphs. For instance one species consists of those vertices that are the start of some 5-tag. Formally species can be represented by formulae $E_i(x)$ in the language of undirected graphs where $E_i(x)$ says

$$(\exists x_0) \cdots (\exists x_{i-1})(\exists x_{i+1}) \cdots (\exists x_{n+1})\, [\mathrm{Tag}_n(x_0, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{n+1})]$$

where $\mathrm{Tag}_n(y_0, y_1, \cdots, y_n, z)$ expresses that $\{y_0, y_1, \cdots, y_n, z\}$ forms an n-tag with start y_0, edges $\{y_i, y_{i+1}\}$ for $0 \leq i < n$ and edges $\{y_n, y_1\}$ and $\{y_2, z\}$. Likewise there are species (formulae) for elements occurring at specific positions on paths that are indexed by some tuple $\overline{x}$.

Let m be the number of distinct species; the exact value is not important, merely note that it depends only on the number p of predicates of $\mathcal{A}$, and the largest of their arities. Let a, b be two fixed distinct elements of A. The $\mathcal{A}$-formula

$$\mathrm{Choose}_i(y_1, \cdots, y_m) : y_i = a \wedge \bigwedge_{j \neq i} y_j = b,$$

is used to pick out the i-th species. Define a map μ sending a vertex $e \in V(\mathcal{A})$, of species i say, to the unique tuple $(\overline{x}, \overline{y})$ in $\mathcal{A}$ of length $p + m$ with the property that if e is of species i, then e is indexed by $\overline{x}$ and $\mathcal{A} \models \mathrm{Choose}_i(\overline{y})$. Furthermore, the edge relation between pairs of $p + m$ tuples is given by the relationship between species as in the construction of $\mathcal{G}(\mathcal{A})$. So μ is a $p + m$-dimensional interpretation of $\mathcal{G}(\mathcal{A})$ in $\mathcal{A}$.

Finally an embedding $f : \mathcal{A} \to \mathcal{B}$ naturally extends to an embedding $g : \mathcal{G}(\mathcal{A}) \to \mathcal{G}(\mathcal{B})$. Moreover if f is FA recognisable then so is g since, for example, if $f(x) = y$ then $g(xj) = yj$, a condition that can be checked utilising the finite automaton for f.

Conversely if $f : \mathcal{G}(\mathcal{A}) \to \mathcal{G}(\mathcal{B})$ is an embedding then the restriction of f to D is an embedding of $\mathcal{A}$ into $\mathcal{B}$. Indeed, let R_i be an atomic relation in $\mathcal{A}$, and $\overline{y} = g\overline{x}$. Then $\mathcal{A} \models R(\overline{x})$ if and only if $\overline{x}$ is the index of a $(5+i)$-tag if and only if $f\overline{x}$ is the index of a $(5+i)$-tag (since $\mathcal{G}(\mathcal{A})$ has no nontrivial automorphisms) if and only if $\mathcal{A} \models R(\overline{y})$. Further if f is FA recognisable, then so is its restriction to the regular domain D. $\triangleleft$

Chapter D

Unary vs. non-unary

This chapter presents an initial step in classifying classes automatic structures. First some classes of unary automatic structures are classified. Then the complexity involved in the more general case is investigated.

D.1 Unary automatic structures

As an initial step in classifying automatic structures, we consider the unary automatic graphs. That is, those automatic structures $\mathcal{G}$ satisfying the following constraints:

1. $\mathcal{G} = (G, E)$ is a graph.

2. The domain G consists of unary strings.

We define a procedure $\mathcal{U}$ that is used to describe the isomorphism types of the unary automatic graphs. The procedure $\mathcal{U}$ takes finite graphs and finite relations as input parameters, collectively denoted by $\mathbb{P}$ say, and yields a possibly infinite graph $\mathcal{U}(\mathbb{P})$, that we call an *unwinding* of the parameters $\mathbb{P}$. Varying the input parameters $\mathbb{P}$ over all syntactically valid possibilities, yields exactly the required class of graphs. Unwindings are formally defined in Definition D.1.7. The main result is then:

Theorem D.1.1 (Unwinding) *A graph is automatically presentable over the alphabet* $\{1\}$ *if and only if it is isomorphic to an unwinding* $\mathcal{U}(\mathbb{P})$ *of some parameter set* $\mathbb{P}$.

The name *unwinding* stems from the pictorial view of the procedure $\mathcal{U}$. In a simple case, $\mathcal{U}$ takes a finite graph $\mathcal{B} = (B, E_B)$ and a relation $R \subset B^2$ as parameters. It then "unwinds" $\mathcal{B}$ by spreading countably many disjoint copies of it in a line, and placing edges between vertices of these copies as prescribed by the binary relation R. That is, the unwinding has domain $\mathbb{N} \times B$

and for every $n \in \mathbb{N}$ an edge from (n, b) to (n, c) when $(b, c) \in E_B$, and for every $n \in \mathbb{N}$ an edge from (n, b) to $(n + 1, c)$ when $(b, c) \in R$.

We begin by describing the 2–tape automata over a unary alphabet. In the decomposition theorem (Theorem C.1.1) with $n = 2$ and $\Sigma = \{1\}$, each Σ_{A_i} is one of $\{\binom{1}{1}\}$, $\{\binom{\perp}{1}\}$ or $\{\binom{1}{\perp}\}$. So each of the factors R_i is recognised by a 1–tape unary automaton. But the structure of these automata are easily understood.

The following will be a simple running example to help illustrate the concepts.

Example D.1.2 The structure $(\mathbb{Z}, S)$ is automatically presentable over a unary alphabet. Indeed the structure with domain $\{1\}^\star$ and relation R given by the regular expression

$$\binom{1}{\perp}\binom{1}{\perp} \cup \binom{\perp}{1} \cup \binom{1}{1}[\binom{1}{1}\binom{1}{1}]^\star\binom{\perp}{1}\binom{\perp}{1} \cup [\binom{1}{1}\binom{1}{1}]^\star\binom{1}{\perp}\binom{1}{\perp},$$

is an automatic presentation where λ codes 0, and for $n \in \mathbb{N} \setminus \{0\}$, the string 1^{2n+1} codes n and 1^{2n} codes $-n$. $\lhd$

Let $\Sigma = \{\sigma\}$. Since the monoids $\Sigma^\star$ (with concatenation) and $\mathbb{N}$ (with addition) are isomorphic, we implicitly may interchange a unary string σ^n with its length n.

A set $A \subset \mathbb{N}$ has *period* p ($p \in \mathbb{N}$) if for every $n \in \mathbb{N}$, it holds that $n \in A$ if and only if $n + p \in A$. A set is *periodic* if it has some period p. If A has period p then A also has period $k \times p$ for every $k \in \mathbb{N} \setminus \{0\}$.

The following facts are elementary.

Fact D.1.3

1. Every regular unary language $A \subset \{\sigma\}^\star$ can be expressed as $T \cup P$ where T is finite and P is periodic.
2. Say $T \subset \{\sigma\}^\star$ be finite and let $P \subset \{\sigma\}^\star$ be periodic. For every $t \in \mathbb{N}$ at least as large as $\max_{w \in T} |w|$ and every period p of P, there is an automaton $\mathcal{B}$ recognising $T \cup P$ of the following form: The states S of $\mathcal{B}$ are $\{q_0, \cdots, q_{t+p-1}\}$, the initial state is q_0, the transitions are $\Delta(q_i, \sigma) = q_{i+1}$ for $0 \leq i < t + p$, and $\Delta(q_{t+p}, \sigma) = q_t$, and accepting states $\{q_i \mid i \in T \cup P \text{ and } 0 \leq i \leq t + p - 1\}$.

Note that it is not assumed that T and P are disjoint. So t, p, and F uniquely characterise the automaton $\mathcal{B}$. Call the parameter set (t, p) the *looping constant* of the automaton.

If $\mathcal{A}_1, \cdots, \mathcal{A}_k$ are unary automata with looping constants (t_i, p_i) respectively, then there exists automata $\mathcal{B}_1, \cdots, \mathcal{B}_k$ where for each i the automaton $\mathcal{B}_i$ is equivalent to $\mathcal{A}_i$ but $\mathcal{B}_i$ has looping constant (t, p) where $t = \max_i t_i$ and $p = (t + 1) \times \prod_i p_i$.

Note D.1.4 Let R be as in the Decomposition Theorem for $n = 2$ and $|\Sigma| = 1$. Say $\mathcal{L}(\mathcal{A}) = \otimes R$ for some automaton $\mathcal{A}$. Then each factor R_i is recognised by a 1–tape unary automaton $\mathcal{A}_i$. As in the previous paragraph we can simultaneously choose these automata so that each has the same looping constant, say (t, p). In fact since $p \geq t$, we can choose automata so that each has looping constant (p, p). Write $p(\mathcal{A})$ for p.

Continuing Example D.1.2 let $\mathcal{A}$ be an automaton recognising R. It is not hard to see that $\mathcal{A}$ may be chosen with $p(\mathcal{A}) = 2$.

We now introduce some additional notation that considerably eases the readability of what follows.

Definition D.1.5 *Given a period p define*

$$T = \{n \in \mathbb{N} \mid 0 \leq n < p\}$$

and for $i \in \mathbb{N}$ define

$$P_i = \{n \in \mathbb{N} \mid p(i+1) \leq n < p(i+2)\}.$$

In particular $P_0 = \{n \in \mathbb{N} \mid p \leq n < 2p\}$. For $n \in P_0$, let $n_i \in \mathbb{N}$ be a shorthand for $n + ip$ so that $n_i \in P_i$.

Lemma D.1.6 *Let $\mathcal{A}$ be a 2–tape automaton over alphabet $\{1\}$ with $p = p(\mathcal{A}) \in \mathbb{N}$ chosen as in note D.1.4. Then $L(\mathcal{A}) = B \cup D$ where D is a finite set and B is a finite union of sets of the following type (where $a, b \in P_0$ and $d \in T$):*

a. $\{\otimes(d, b_{i+1}) \mid i \in \mathbb{N}\}$,

b. $\{\otimes(b_{i+1}, d) \mid i \in \mathbb{N}\}$,

c. $\{\otimes(a_i, b_i) \mid i \in \mathbb{N}\}$,

d. $\{\otimes(a_i, b_{i+1}) \mid i \in \mathbb{N}\}$,

e. $\{\otimes(b_{i+1}, a_i) \mid i \in \mathbb{N}\}$,

f. $\{\otimes(a_i, b_{i+2+j}) \mid i, j \in \mathbb{N}\}$,

g. $\{\otimes(b_{i+2+j}, a_i) \mid i, j \in \mathbb{N}\}$.

Proof By the decomposition theorem and the previous remarks we see that $L(\mathcal{A})$ consists exactly of words of the form $\binom{1}{1}^x (\delta)^y$ where $(\delta) \in \{\binom{\perp}{1}, \binom{1}{\perp}\}$ and either

1. $x, y \in T$ or

2. $x \in T$ and $y = e_i$ for some $e \in P_0$ and all $i \in \mathbb{N}$, or

3. $x = e_i$ and $y \in T$ for some $e \in P_0$ and all $i \in \mathbb{N}$, or

4. $x = e_i$ and $y = f_j$ for some $e, f \in P_0$ and all $i, j \in \mathbb{N}$.

In case 1. there are only finitely many such words, so these are placed in D.

In case 2. say $\{\binom{1}{1}^d\binom{\perp}{1}^{e_i} \mid i \in \mathbb{N}\} \subset L(\mathcal{A})$. Then $\otimes(d, d+e_i) \in L(\mathcal{A})$ for every $i \in \mathbb{N}$. But $d + e_0 = b_\epsilon$ for some $a_\epsilon \in P_0 \cup P_1$. So $\{\otimes(d, b_{i+\epsilon}) \mid i \in \mathbb{N}\} \subset L(\mathcal{A})$ which corresponds to type a. and if $\epsilon = 0$ then there are finitely many words of the form $\otimes(d, b_0)$ which can be placed in D. The case of $\binom{1}{1}^d\binom{1}{\perp}^{e_i}$ is symmetrical.

In case 3. say $\{\binom{1}{1}^{e_i}\binom{\perp}{1}^{d} \mid i \in \mathbb{N}\} \subset L(\mathcal{A})$. Then $\otimes(e_i, d+e_i) \in L(\mathcal{A})$ for every $i \in \mathbb{N}$. But $d + e_0 = b_\epsilon$ for some $b_\epsilon \in P_0 \cup P_1$. So $\{\otimes(e_i, b_{i+\epsilon}) \mid i \in \mathbb{N}\} \subset L(\mathcal{A})$ corresponding to cases c. or d. as required. The case of $\binom{1}{1}^{e_i}\binom{1}{\perp}^{d}$ is symmetrical.

In case 4. say $\{\binom{1}{1}^{e_i}\binom{\perp}{1}^{f_j} \mid i \in \mathbb{N}\} \subset L(\mathcal{A})$. Then as before $\otimes(e_i, f_j + e_i) \in L(\mathcal{A})$ for every $i, j \in \mathbb{N}$ and $f_0 + e_0 = b_\epsilon \in P_1 \cup P_2$ so in particular $\{\otimes(e_i, b_{i+\epsilon+j}) \mid i, j \in \mathbb{N}\} \subset L(\mathcal{A})$ corresponding to cases f. or d. as required. The case of $\binom{1}{1}^{e_i}\binom{1}{\perp}^{f_j}$ is symmetrical. $\triangleleft$

Continuing Example D.1.2 for $p(\mathcal{A}) = 2$, the set $T = \{0, 1\}$ and $P_0 = \{2, 3\}$. Then $\mathcal{L}(\mathcal{A})$ viewed as a subset of $\mathbb{N}^2$ can be expressed as $B \cup D$ where $D = \otimes\{(2, 0), (0, 1), (1, 3)\}$ and letting $a = 2$ and $b = 3$, B is the union of following sets:

- $\{\otimes(a_{i+1}, a_i) \mid i \in \mathbb{N}\}$, (corresponding to item e. in the Lemma), and
- $\{\otimes(b_i, b_{i+1}) \mid i \in \mathbb{N}\}$, (corresponding to item d.).

Unwinding construction for graphs

The following informal description should serve as sufficient introduction to the definition of the unwinding procedure which is central to the characterisation of the unary automatic graphs.

The unwinding procedure takes a set of parameters $\mathbb{P}$ as input, consisting of two finite graphs $\mathcal{B}$ and $\mathcal{D}$ and binary relations R_j and L_j where $1 \leq j \leq 4$. It yields a graph $\mathcal{U}(\mathbb{P})$ which copies these relations with certain regularity. In particular this graph, called an *unwinding*, contains one copy of $\mathcal{D}$ and countably many copies of $\mathcal{B}$, where we denote the ith copy by $\mathcal{B}^i$. Pictorially, the procedure "unwinds" the copies of $\mathcal{B}$ in a line, from left to right say, so that the B^ith copy precedes the B^{i+1}th copy for all $i \in \mathbb{N}$, and $\mathcal{D}$ precedes $\mathcal{B}^0$. All that remains is to specify the edges amongst the $\mathcal{B}^i$'s and between the $\mathcal{B}^i$'s and $\mathcal{D}$. The relations R_i and L_i suggest edges directed to the right and left respectively: the edges between $\mathcal{D}$ and its adjacent copy in the line, namely $\mathcal{B}^0$, are realised by the parameters R_1 and L_1; while edge relations between $\mathcal{D}$ and the copies of $\mathcal{B}$ not adjacent to $\mathcal{D}$ (namely $\mathcal{B}^i$ for $i \geq 1$) are prescribed by the parameters R_2 and L_2. Analogously, edge relations between adjacent copies of $\mathcal{B}$ are prescribed by parameters R_3 and L_3. Finally edge relations between non-adjacent copies of $\mathcal{B}$ are prescribed by parameters R_4 and L_4.

If $\mathcal{B} = (B, E_B)$ is a graph, then for $i \in \mathbb{N}$, the *i-th copy* of $\mathcal{B}$ is the graph with vertex set $B^i = B \times \{i\}$. We write b^i for the ordered pair (b, i). The edge relation of $\mathcal{B}^i$ is defined as $E_B^i = \{(a^i, b^i) \mid (a, b) \in E_B\}$.

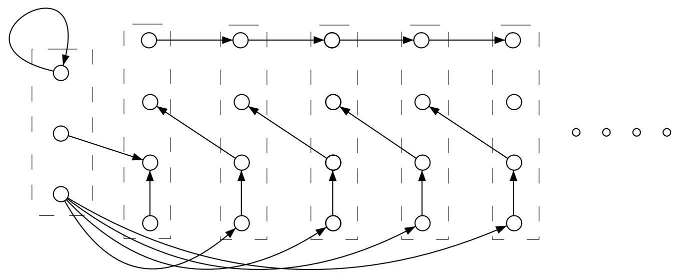

Figure D.1: An unwinding of some parameter set.

In preparation for the formal definition, let B and D be disjoint finite sets of symbols. Choose a parameter set $\mathbb{P} = (\mathcal{D}, \mathcal{B}, R_i, L_i)_{i=1,2,3,4}$ consisting of graph parameters and relational parameters. That is, $\mathcal{D} = (D, E_D)$ and $\mathcal{B} = (B, E_B)$ are (possibly empty) graphs. The relational parameters are possibly empty relations where $R_i \subset D \times B$ and $L_i \subset B \times D$ for $i = 1, 2$ and $R_i, L_i \subset B \times B$ for $i = 3, 4$. The parameters $\mathcal{D}, R_i, L_i$ for $i = 1, 2$ are called *extension parameters*.

Definition D.1.7 (unwinding) *The* **unwinding** *of the parameter set, written* $\mathcal{U}(\mathbb{P})$*, is the graph with vertex set* $\cup_{i\in\mathbb{N}} B^i \cup D$*. The edge relation consists of* $\cup_{i\in\mathbb{N}} E_B^i$*,* E_D *and the following edges. For* $a, b \in B$*,* $d \in D$

$$\begin{array}{llllll}
(d, b^0) & \textit{when} & (d,b) \in R_1; & (b^0, d) & \textit{when} & (b,d) \in L_1; \\
(d, b^{i+1}) & \textit{when} & (d,b) \in R_2; & (b^{i+1}, d) & \textit{when} & (b,d) \in L_2, \quad i \in \mathbb{N}; \\
(a^i, b^{i+1}) & \textit{when} & (a,b) \in R_3; & (a^{i+1}, b^i) & \textit{when} & (a,b) \in L_3, \quad i \in \mathbb{N}; \\
(a^i, b^{i+2+j}) & \textit{when} & (a,b) \in R_4; & (a^{i+2+j}, b^i) & \textit{when} & (a,b) \in L_4, \quad i, j \in \mathbb{N}\,.
\end{array}$$

A graph is called an unwinding *if it is isomorphic to* $\mathcal{U}(\mathbb{P})$ *for some* $\mathbb{P}$*.*

Continuing Example D.1.2 define $\mathbb{P}$ as follows. Let $\mathcal{D}$ be the graph with vertex set $\{d, e\}$ and edge (d, e). Let $\mathcal{B}$ be the graph with vertex set $\{a, b\}$ and no edges. Let $R_1 = \{(e, b)\}$, $L_1 = \{(a, d)\}$, $R_3 = \{(b, b)\}$ and $L_3 = \{(a, a)\}$. Let R_i and L_i ($i = 2, 4$) be empty. Then $\mathcal{U}(\mathbb{P})$, the unwinding of the parameter set $\mathbb{P}$, yields a graph isomorphic to $(\mathbb{Z}, S)$.

Here are some other examples of graphs that are unwindings. One half of the Theorem D.1.1 says that each such graph is automatically presentable over a unary alphabet.

Example D.1.8 Every finite graph is an unwinding.
Let $\mathcal{D}$ be the finite graph, and let all the other parameters be empty. $\triangleleft$

Example D.1.9 $(\mathbb{N}, S)$ *is an unwinding.*
Let $\mathcal{B}$ have vertex set $\{x\}$ and no edges, let $R_3 = \{(x, x)\}$ and let all the other parameters be empty. Then unwinding $\mathcal{U}(\mathbb{P})$ is isomorphic to $(\mathbb{N}, S)$. $\triangleleft$

Example D.1.10 *Every ordinal $< \omega^2$ is an unwinding.*

Every ordinal $< \omega^2$ is of the form $\omega\beta + \alpha$ for some $\beta, \alpha \in \omega$. Let $\mathcal{D}$ be the graph with vertex set $\{d_1, \cdots, d_\alpha\}$ and edge relation $\{(d_i, d_j) \mid 1 \leq i \leq j \leq \alpha\}$. Let $\mathcal{B}$ be the graph with vertex set $\{b_1, \cdots, b_\beta\}$ and edge relation $\{(b_i, b_j) \mid 1 \leq i \leq j \leq \beta\}$. Let $L_1 = L_2 = \{(b_i, d_j) \mid 1 \leq i \leq \beta, 1 \leq j \leq \alpha\}$. Let $R_3 = R_4 = \{(b_i, b_j) \mid 1 \leq i \leq j \leq \beta\}$. Let all the other parameters be empty. Then $\mathcal{U}(\mathbb{P})$ is isomorphic to the ordinal $\omega\beta + \alpha$. $\triangleleft$

Example D.1.11 *The finite disjoint union of unwindings is an unwinding.*

Let $\mathbb{P}(1)$ and $\mathbb{P}(2)$ be two parameter sets and define $\mathbb{P}$ as their disjoint union (taken component-wise). Then $\mathcal{U}(\mathbb{P})$ is the disjoint union of $\mathcal{U}(\mathbb{P}(1))$ and $\mathcal{U}(\mathbb{P}(2))$. $\triangleleft$

Example D.1.12 *The countable disjoint union of a finite graph is an unwinding.*

Let $\mathcal{B}$ be the finite graph and let all the other parameters be empty. Then $\mathcal{U}(\mathbb{P})$ is a countable disjoint union of copies of the graph $\mathcal{B}$. $\triangleleft$

Proof of Theorem D.1.1(Unwinding)

Proof If the domain of G is finite then it is both an unwinding and automatically presentable over $\{1\}$. So assume that the domain of $\mathcal{G}$ is infinite.

$(\rightarrow)$: Without loss of generality suppose that $\mathcal{G} = (G, E)$ is an automatic graph with domain $\{1\}^\star$. Let $\mathcal{A}$ be a 2–tape automaton recognising the convolution $\otimes E$. The domain $\{1\}^\star$ can be partitioned into T and $\cup_{i \in \mathbb{N}} P_i$ (Definition D.1.5). Let $\mathcal{P}_0$ and $\mathcal{T}$ be the subgraphs $\mathcal{G}$ with vertex set P_0 and T respectively. We are required to find a parameter set $\mathbb{P} = (\mathcal{D}, \mathcal{B}, R_i, L_i)_{i=1,2,3,4}$ and show that $\mathcal{G}$ is isomorphic to $\mathcal{U}(\mathbb{P})$.

Define the graph parameters as follows: $\mathcal{D} = \mathcal{T}$ and $\mathcal{B} = \mathcal{P}_0$. We now define the relational parameters R_i and remark that the parameters L_i are symmetrically defined. For all $a, b \in P_0$ and $d \in T$,

$$\begin{array}{llllll} (d,b) \in R_1 & \iff & (d, b_0) \in E \ , & (d,b) \in R_2 & \iff & (d, b_1) \in E, \\ (a,b) \in R_3 & \iff & (a_0, b_1) \in E \ , & (a,b) \in R_4 & \iff & (a_0, b_2) \in E \ . \end{array}$$

Now, $\mathcal{U}(\mathbb{P}) = (U, E_U)$ is isomorphic to $\mathcal{G}$ under the isomorphism which sends a^i to a_i for $a \in P_0$, and fixes T. Indeed, for all $d \in T$, $a, b \in P_0$ and $i, j \in \mathbb{N}$, we have

$$\begin{array}{lllllll} (d, b^0) \in E_U & \iff & (d,b) \in R_1 & \iff & (d, b_0) \in E, & & \\ (d, b^{i+1}) \in E_U & \iff & (d,b) \in R_2 & \iff & (d, b_1) \in E & \iff & (d, b_{i+1}) \in E, \\ (a^i, b^{i+1}) \in E_U & \iff & (a,b) \in R_3 & \iff & (a_0, b_1) \in E & \iff & (a_i, b_{i+1}) \in E, \\ (a^i, b^{i+2+j}) \in E_U & \iff & (a,b) \in R_4 & \iff & (a_0, b_2) \in E & \iff & (a_i, b_{i+2+j}) \in E \ . \end{array}$$

In each row, the first equivalence follows from the unwinding definition and the second equivalence follows from the definitions of R_i above. In the last three rows, the third equivalences follow from Lemma D.1.6 parts $1, 2$ and 3 respectively. Analogous statements are true about the parameters L_i. These cases account for all the edges in $\mathcal{U}(\mathbb{P})$ and $\mathcal{G}$. So we have described a parameter set $\mathbb{P}$ and exhibited that the unwinding $\mathcal{U}(\mathbb{P})$ is isomorphic to the given automatic graph $\mathcal{G}$, as required.

$(\leftarrow)$: Let $\mathcal{U}(\mathbb{P}) = (U, E_U)$ be an unwinding for some choice $\mathbb{P} = (\mathcal{D}, \mathcal{B}, R_i, L_i)_{i=1,2,3,4}$ of parameters. We show that $\mathcal{U}$ is isomorphic to an automatic graph $\mathcal{G}$ over the domain $\{1\}^\star$. We first define $\mathcal{G}$. Suppose $D = \{d0, \cdots, dm\}$ and $B = \{b1, \cdots, bn\}$. Then by the notation preceding the unwinding definition, $B^i = \{(b1, i), \cdots, (bn, i)\}$. Define the bijective mapping Φ from the domain U onto $\{1\}^\star$ as sending di to 1^{i-1} for $0 \leq i \leq m$ and (bj, i) to 1^{m+j+ni} for $1 \leq j \leq n$. Then define $\mathcal{G} = (\{1\}^\star, E)$ where E is the image under Φ of the relation E_U, namely $\Phi(E_U) = \{(\Phi(x), \Phi(y)) \mid (x, y) \in E_U\}$. We finish the proof by showing that $\mathcal{G}$ is automatic. Indeed, we show there exists an automaton $\mathcal{A}$ which recognises the convolution $\otimes E$. It follows from the unwinding definition that we can write E_U as $E_U = \cup_{i\in\mathbb{N}} E_B^i \cup E_D \cup U_1 \cup U_2 \cup U_3 \cup U_4$ where

$$\begin{array}{rcll} U_1 &=& \{(d, b^0) \mid d \in D, b \in B\} & \cup \ \ \{(b^0, d) \mid d \in D, b \in B\}, \\ U_2 &=& \{(d, b^{i+1}) \mid d \in D, b \in B, i \in \mathbb{N}\} & \cup \ \ \{(b^{i+1}, d) \mid d \in D, b \in B, i \in \mathbb{N}\}, \\ U_3 &=& \{(a^i, b^{i+1}) \mid a, b \in B, i \in \mathbb{N}\} & \cup \ \ \{(a^{i+1}, b^i) \mid a, b \in B, i \in \mathbb{N}\}, \\ U_4 &=& \{(a^i, b^{i+j+2}) \mid a, b \in B, i, j \in \mathbb{N}\} & \cup \ \ \{(a^{i+j+2}, b^i) \mid a, b \in B, i, j \in \mathbb{N}\} \ . \end{array}$$

So $E = \Phi(E_U) = \Phi(\cup_{i\in\mathbb{N}} E_B^i) \cup \Phi(E_D) \cup \Phi(U_1) \cup \Phi(U_2) \cup \Phi(U_3) \cup \Phi(U_4)$. But the set of all FA recognisable n–ary relations is closed under union. Hence it is sufficient, and routine, to show that each relation in the above expression is FA recognisable over $\{1\}$. So we have constructed an automatic graph $\mathcal{G}$ which is isomorphic to the given unwinding $\mathcal{U}(\mathbb{P})$, as required. $\lhd$

Application of the unwinding theorem

The following results can be used to show that a particular structure does not have a unary automatic presentation. For our purposes, all graph-theoretic properties are in the underlying undirected graph of G. For example, a path is a sequence of vertices with the property that successive vertices are connected by an edge in either direction.

Lemma D.1.13 *If a graph is an unwinding then it contains a finite number of infinite connected components and there is a bound on the size of the finite connected components.*

Proof Consider an unwinding $\mathcal{U}(\mathcal{D}, \mathcal{B}, R_i, L_i)_{i=1,2,3,4}$. Then since the graph $\mathcal{D}$ is finite it is sufficient to show the claim for graphs with empty extension parameters. Let $\mathcal{H} = (H, E)$ be such a graph. Let $\mathcal{F} = (F, E_F)$ be a finite connected component of $\mathcal{H}$. Then for all $a \in B$, if $a^i \in F$ and $a^j \in F$ then $i = j$. For otherwise, by the unwinding definition, a^i would be connected to infinitely many a^k, contradicting the assumption that $\mathcal{F}$ is finite. Hence the number of vertices in the finite connected components of $\mathcal{H}$ is bounded by $|B|$.

Consider an infinite component $\mathcal{I}$ of $\mathcal{H}$. Then by the unwinding definition there is some $c \in B$ such that I contains infinitely many c^i. But if c^i and c^{i+k} are in I then so is c^{i+k+n} for every $n \in \mathbb{N}$. Hence there are at most $|B|$ many distinct infinite components. $\lhd$

In order to characterise the isomorphism types of a particular class of automatic graphs, by Lemma D.1.13 it is sufficient to characterise the isomorphism types of the infinite connected automatic graphs in that class.

Equivalence structures

Theorem D.1.14 [also Blumensath - 1999] *An equivalence structure is automatically presentable over a unary alphabet if and only if it has finitely many infinite equivalence classes and there is a finite bound on the sizes of the finite equivalence classes.*

Proof Necessity is just an application of Lemma D.1.13. For sufficiency note that each infinite equivalence class is an unwinding, the countable disjoint union of a finite equivalence class is an unwinding by Example D.1.12 and by assumption the equivalence relation consists of a finite union of unwindings, so by Example D.1.11 is itself an unwinding. $\lhd$

Injections and permutations

Theorem D.1.15 *An injection structure is automatically presentable over a unary alphabet if and only if it has finitely many infinite orbits and there is a finite bound on the sizes of the finite orbits.*

Theorem D.1.16 [also Blumensath - 1999] *A permutation structure is automatically presentable over a unary alphabet if and only if it has finitely many infinite orbits and there is a finite bound on the sizes of the finite orbits.*

Linear orders

Write ω for the (linear order of the) type of the positive natural numbers, ω^* for the type of the negative natural numbers, and $\mathbf{n}$ for the type of the finite linear order with exactly $n \in \mathbb{N}$ elements. Each has an automatic presentation over a unary alphabet. Also, if linear orders $\mathcal{L}_1$ and $\mathcal{L}_2$ are unwindings then the ordered sum $\mathcal{L}_1 + \mathcal{L}_2$ is also an unwinding.

Proposition D.1.17 *If $\mathcal{L} = (L, \leq)$ is automatically presentable over a unary alphabet then $\mathcal{L}$ contains a finite number of elements with no immediate successors.*

Proof Suppose that $(L, \leq)$ is isomorphic to an unwinding, say $\mathcal{U}(\mathbb{P})$, where $\mathbb{P}$ is as in Definition D.1.7. Write s for the partial function on $\mathcal{L}$ that sends x to its successor (if defined). Let $b \in B$ and suppose b^{i_0} has no successor for some $i_0 > 0$. Then from the unwinding definition $s(b^1)$ is undefined. Suppose, without loss of generality, that $b^1 < b^2$. Then again from the unwinding definition we see that for every $i > 0$, $b^i < b^{i+1}$ and $s(b^i)$ is undefined. So for every $i > 0$ there exist infinitely many $z \in L$ between b^i and b^{i+1}. In particular, for every i there exists $c \in B$ such that $b^i < c^k < b^{i+1}$ for every $k \geq 2$. But since B is finite, there exists $i < j$ and $c \in B$ such that $b^i < c^2 < b^{i+1}$ and $b^j < c^2 < b^{j+1}$, contradicting the fact that $b^{i+1} \leq b^j$. Hence if $x \in L$ has no successor then $x \in D \cup B^0$, a finite set. $\lhd$

Corollary D.1.18 *The order $\mathbb{Q}$ is not automatically presentable over a unary alphabet.* $\lhd$

Suppose a linearly ordered structure $\mathcal{L} = (L, \leq)$ is automatic over a unary alphabet. Define a binary relation ρ as satisfying those pairs (x, y) where there are only finitely many elements between x and y. Indeed there are finitely many elements in $[x, y]$ (for $x < y$) if and only if

$$(\exists z \in [x, y])(\forall z' \in [x, y])\, |z'| \leq |z|.$$

Hence ρ is FA recognisable in the given unary presentation of $\mathcal{L}$. But ρ is an equivalence relation, so by Theorem D.1.14 there are finitely many infinite ρ–equivalence classes. The following are now immediate.

Theorem D.1.19 [also Blumensath - 1999] *A linear order $\mathcal{L} = (L, \leq)$ has an automatic presentation over a unary alphabet if and only if it is isomorphic to a finite sum of linear orders chosen from ω, ω^* or $\mathbf{n}$, for $n \in \mathbb{N}$.*

Corollary D.1.20 [cf. Rubin - 1999] [also Blumensath - 1999] *A well order is automatically presentable over a unary alphabet if and only if it is $< \omega^2$.*

Corollary D.1.21 [Blumensath - 1999] *The unary automatic structures are not closed under n-dimensional interpretations for $n > 1$.*

Proof The ordinal ω^2 is 2-dimensionally definable in terms of the unary automatic ordinal ω. $\lhd$

Proposition D.1.22 Blumensath *The monadic second order theory of every unary automatic structure is decidable.*

Proof Recall that a structure is unary automatic if and only if it is first order interpretable (with dimension 1) in the structure $\mathcal{U} = (\mathbb{N}, \leq, (\equiv \text{ modulo } n)_{n \in \mathbb{N}})$. We establish that $\mathcal{U}$ is monadic second order definable in the structure $(\mathbb{N}, S)$. Then since $(\mathbb{N}, S)$ has decidable monadic second order theory Büchi [1962], so does every unary automatic structure.

The formula $\mathrm{Seg}(X)$ defined as

$$(\forall x \in X)(\exists y \in X)\, S(y) = x$$

says that X is an initial segment of $\mathbb{N}$. So the formula $\mathrm{Leq}(X)$ defined as

$$(\forall X)\, [(\mathrm{Seg}(X) \wedge m \in X) \to n \in X]$$

says that $n \leq m$. The formula $M_n(X)$ defined as

$$\lambda \in X \wedge (\forall x) \left[x \in X \iff \wedge_{1 \leq i < n} S^i(x) \notin X \wedge S^n(x) \in X\right]$$

says that X consists of those x that are multiples of n.

Then the formula $\mathrm{Mod}_n(x, y)$ defined as

$$\bigvee_{1 \leq i \leq n} (\forall X) \ [M_i(X) \rightarrow (x \in X \iff y \in X)]$$

says that $x \equiv y$ modulo n. So $(\mathbb{N}, \mathrm{Leq}, (\mathrm{Mod}_n)_{n \in \mathbb{N}})$ is a monadic second order definition of $\mathcal{U}$ in $(\mathbb{N}, S)$. $\lhd$

Most of the content of this section has been reported in Khoussainov and Rubin [2001], Rubin [1999] and independently by Blumensath [1999]. Indeed, the latter proves the unwinding characterisation of the unary automatic graphs essentially via a method similar to the proof of Theorem C.2.9. Note that if $\mathcal{A}$ is a unary automatic structure then by Theorem C.4.1 there is a mutually interpretable automatic graph $\mathcal{G}(\mathcal{A})$. However the proof of $\mathcal{G}(\mathcal{A})$ does not give us that $\mathcal{G}(\mathcal{A})$ is automatic over a unary alphabet. See Blumensath [1999] for an algebraic characterisation of relations of arbitrary arity that are FA recognisable over a unary alphabet.

D.2 Automatic equivalence and injection structures

The isomorphism types of unary automatic equivalence relations and injection structures are identified in Theorems D.1.14, D.1.16 and D.1.15. This section illustrates the complexities involved in characterising classes of automatic structures in the non-unary case. The classes of equivalence and injection structures are chosen since classically their isomorphism types are easily described in terms of natural invariants. Also, a full characterisation of the automatic equivalence relations will probably precede one for trees, linear orders etc.

Automatic equivalence relations

An *equivalence structure* $\mathcal{E}$ is of the form (E, ρ) where ρ is an equivalence relation on domain E. Define the *height* $h_{\mathcal{E}}$ of $\mathcal{E}$ as the function $\mathbb{N} \cup \{\omega\} \rightarrow \mathbb{N} \cup \{\omega\}$ satisfying $h_{\mathcal{E}}(n) = m$ if and only if m is the number of ρ-equivalence classes of size n. The value ω means that the number or size is infinite.

Note that equivalence structures $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic if and only if $h_{\mathcal{E}_1} = h_{\mathcal{E}_2}$.

So to characterise the automatic equivalence structures one should describe the behaviour of $h_{\mathcal{E}}$ exactly when $\mathcal{E}$ is automatic.

Theorem D.1.14 says that $\mathcal{E}$ is *unary* automatic if and only if $h_{\mathcal{E}}(\omega)$ is finite and for all but finitely many $n \in \mathbb{N}$, $h_{\mathcal{E}}(n) = 0$. The results of this section show that the situation in the non-unary case is complex.

For an equivalence structure $\mathcal{E}$ we define $\mathcal{E}_\omega$ (respectively $\mathcal{E}_f$) as the restriction of $\mathcal{E}$ to those elements that are in infinite (respectively finite) equivalence classes.

Lemma D.2.1 *If the equivalence structure $\mathcal{E}$ is automatic then so are $\mathcal{E}_f$ and $\mathcal{E}_\omega$. Moreover, $\mathcal{E}$ is automatically presentable if and only if $\mathcal{E}_f$ is.*

Proof The set of elements that are in infinite ρ-classes is definable by $\exists^\infty y\, \rho(x, y)$. So if $\mathcal{E}$ is automatic then so is $\mathcal{E}_\omega$ and $\mathcal{E}_f$. Furthermore $\mathcal{E}_\omega$ only consists of infinite equivalence classes. There are either finitely many or infinitely many of them. But such equivalence relations are always automatically presentable as in Example B.2.16. $\triangleleft$

Thus, in characterising automatic equivalence structures $\mathcal{E}$, we can assume that each equivalence class is finite. Therefore, from now on assume that $h_{\mathcal{E}}(\omega) = 0$.

Lemma D.2.2 *If $\mathcal{E}$ is an automatic equivalence structure then it has an automatic presentation (E', ρ') satisfying the property that if $(x, y) \in \rho'$ then $|x| = |y|$.*

Proof Suppose $\mathcal{E}$ is automatic over Σ. Recall the length lexicographic ordering $<_{llex}$ is automatic over Σ, Example B.1.10. It has order type ω and if $x <_{llex} y$ then $|x| \leq |y|$. Define a new domain E' over alphabet $((\Sigma \cup \{1\})^\star)^2$ as the set of pairs $(x, 1^n)$ where x is in the domain of $\mathcal{E}$ and n is the length of the $<_{llex}$-longest word in the ρ-equivalence class containing x. Note that E' is FA recognisable. Define a new equivalence relation ρ' containing pairs $((x, 1^n), (y, 1^m))$ if and only if $(x, y) \in \rho$ and $n = m$. Then (E', ρ') is an automatic equivalence relation isomorphic to $\mathcal{E}$. $\triangleleft$

Next we build equivalence structures from languages $L \subset \Sigma^*$. Define an equivalence structure $\mathcal{E}(L) = (L, \sim_L)$ where $\sim_L$ (x, y) exactly when $|x| = |y|$. If L is regular over Σ then $\mathcal{E}(L)$ is an automatic equivalence relation over Σ.

We would like to characterise automatic equivalence structures in terms of $h_{\mathcal{E}}$. The next series of results provide several examples and standard constructions for building automatic equivalence structures whose height function exhibits nontrivial behaviour.

Let L be a language over Σ. The growth of L is the function g_L defined as $g_L(n) = |\Sigma^n \cap L|$ for $n < \omega$. The following is implicit in Szilard, Yu, Zhang, and Shallit [1992].

Lemma D.2.3 *For any polynomial function p whose coefficients are positive integers there is a regular language L_p whose growth function is p.*

Proof Note that if L_1 and L_2 have growth rates p_1 and p_2, respectively, and $L_1 \bigcap L_2 = \emptyset$ then their union has growth rate $p_1 + p_2$. So it is sufficient to exhibit for each $k \in \mathbb{N}$ a language L_{n^k} with growth rate n^k.

For $w \in \Sigma^\star$, write w^+ for $ww^\star$. Note that $A_k = 0^+1^+ \cdots k^+$ has growth $\binom{n-1}{k}$. Consider the languages $B_k = 0^+1^+ \cdots (k-1)^+k^*$. Then $B_k = A_{k-1} \cup A_k$. Hence the growth of B_k is

$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$. Consider the languages C_k defined as the disjoint union of $k!$ copies of B_k. Then C_k has growth $n(n-1)\cdots(n-k+1)$ which we write as $n^{\underline{k}}$. We now make use of the standard identity $x^k = \Sigma_{i=0}^k S(k,i)x^{\underline{i}}$ where the $S(k,i)$ are Stirling numbers of the first kind; that is the number of ways of partitioning a set of size k into i non-empty subsets. So $L_{n^k} = \cup_{i=0}^k \cup_{S(k,i)} C_i$, where the unions are taken to be disjoint, has the required growth. $\triangleleft$

Lemma D.2.4 *For any exponential function $e(n)$ of the form k^{an+b}, where $2 \le k$ and a, b are positive integers, there exists a regular language whose growth function is exactly e.*

Proof Let $\Sigma = \{1, 2, \cdots, k^a\}$. Then $L = \Sigma^*$ has growth k^{an}. The disjoint union of k^b many copies of L has growth k^{an+b}. $\triangleleft$

Theorem D.2.5 *For every function f which is either a polynomial p whose coefficients are positive integers or an exponential function k^{an+b}, where $k \ge 2$ and a, b are fixed positive integers, there exists an automatic equivalence relation $\mathcal{E}$ such that $h_{\mathcal{E}}(f(n)) = c$ for all $n \in \mathbb{N}$ and where $c \le \omega$ a constant.*

Proof From Lemma D.2.3 and Lemma D.2.4 there exists a regular language L whose growth function is identical to f. So $\mathcal{E}(L)$ is automatic and the desired equivalence structure. The theorem for case $c = 1$ is proved. Now note that for $c \le \omega$, the c-fold disjoint union of automatic equivalence structures is automatic, by Proposition B.1.22. $\triangleleft$

Consider two functions f, g with domain and range $\mathbb{N}$. Their *Dirichlet convolution* is

$$(f \star g)(n) = \sum_{ab=n} f(a)g(b),$$

and their *Cauchy product* is

$$(f\#g)(n) = \sum_{a+b=n} f(a)g(b).$$

Proposition D.2.6 *Let $\mathcal{H}$ be the class of height functions of automatic equivalence structures. Then $\mathcal{H}$ is closed under addition, Dirichlet convolution and Cauchy product.*

Proof Let $\mathcal{E}_i = (E_i, \rho_i)$ for $i = 1, 2$ be two automatic equivalence structures with height functions f and g respectively. Without loss of generality, assume that the domains E_1 and E_2 are disjoint. Define the automatic equivalence structure $\mathcal{E}_{E_1 \cup E_2}$ as their disjoint union; that is, the domain is $E_1 \cup E_2$ and the relation is $\rho = \rho_1 \cup \rho_2$. Then the height function of $E_{E_1 \cup E_2}$ is $f + g$.

For the Dirichlet convolution define the direct product $\mathcal{E}_1 \times \mathcal{E}_2$ as the equivalence structure with domain $E_1 \times E_2$. Define two pairs (x_1, y_1) and (x_2, y_2) to be related if $(x_1, x_2) \in \rho_1$ and $(y_1, y_2) \in \rho_2$. Then this equivalence structure is automatic and has height $f * g$.

For the Cauchy product, let $T_i \subset E_i$ be the unary predicate that picks out the length lexicographically least element from each equivalence class of $\mathcal{E}_i$. Define an equivalence structure $\mathcal{E}$ with domain $(T_1 \times E_2) \cup (T_2 \times E_1)$. Define two pairs (x_1, y_1) and (x_2, y_2) to be related if either $[x_1 = x_2$ and $(y_1, y_2) \in \rho_1 \cup \rho_2]$ or $[(x_1, y_2) \in \rho_1$ and $(x_2, y_1) \in \rho_2]$. Then this is an equivalence structure and is automatic. We now check that it has height $f\#g$. Suppose X is a ρ_1-equivalence class of size a, containing $x' \in T_1$, and Y is a ρ_2-equivalence class of size b, containing $y' \in T_2$. Associate with the pair (X, Y) the $\mathcal{E}$-class C containing (x', y'). Then by definition, C comprises $\{(x', y) \mid y \in Y\}$ and $\{(y', x) \mid x \in X\}$. Hence C has size $a + b$. Conversely, every $\mathcal{E}$ class $\mathcal{C}$ of size n contains some (x', y') for $x' \in T_1$ and $y' \in T_2$, and so uniquely determines the corresponding ρ_1 and ρ_2 classes, say X and Y respectively, for which $|X| + |Y| = n$. $\lhd$

Example D.2.7 *There is an automatic equivalence structure $\mathcal{E}$ so that $h_{\mathcal{E}}(n)$ is the number of divisors of n.*

Let $\mathcal{I}$ be an equivalence structure with $h_{\mathcal{I}}(n) = 1$. It is automatically presentable by Theorem D.2.5. Then by Proposition D.2.6 there is an automatic structure $\mathcal{E}$ with height $h_{\mathcal{I}} \star h_{\mathcal{I}}(n)$; that is, $h_{\mathcal{E}}(n)$ is the number of divisors of n. $\lhd$

We have not been able to characterise the automatic equivalence structures. We think that the characterisation is closely related to the question of whether the isomorphism problem for automatic equivalence relations is decidable.

Automatic permutation structures

An *injection structure* $\mathcal{A}$ is of the form (A, f) where f is a one-to-one map of A. Writing f^{-1} for the inverse of f, an orbit of f is defined as the set $\{f^i(a) \mid i \in \mathbb{Z}\}$ for some $a \in A$. Note that an orbit of an injection structure may be finite or isomorphic to (ω, S) or $(\mathbb{Z}, S)$.

If f is a bijection then (A, f) is called a *permutation structure*. As for equivalence structures define the height $h_{\mathcal{A}} : \mathbb{N} \cup \{\omega\} \to \mathbb{N} \cup \{\omega\}$ where $h_{\mathcal{A}}(n)$ is the number of orbits of size n. In Theorems D.1.15 and D.1.16 it is shown that an injection (or permutation) $\mathcal{A}$ has an automatic presentation over a *unary* alphabet if and only if $h_{\mathcal{A}}(\omega)$ is finite and $h_{\mathcal{A}}(n) = 0$ for all but finitely many $n \in \mathbb{N}$.

An automatic equivalence structure $\mathcal{E}$ over $\Sigma^\star$ can be turned into an automatic permutation structure $\mathcal{A}(\mathcal{E})$ as follows. Consider the automatic linear $<_{llex}$ on $\Sigma^\star$ type ω; see Example B.1.10. For each $x \in E$ we proceed as follows. If x is not the maximal element in its equivalence class, then define $f(x)$ as the minimal y ρ-equivalent to x. Otherwise define $f(x)$ as the minimal element in the equivalence class containing x. Note that if $x \in \mathcal{E}_\omega$ then f transforms the equivalence class into $\mathbb{Z}$-type orbit, namely the structure isomorphic to $(\mathbb{Z}, S)$ where S is the successor function on the integers. Clearly then $h_{\mathcal{E}} = h_{\mathcal{A}(\mathcal{E})}$. Hence, we can replace equivalence structures with permutation structures in Theorem D.2.5.

We now show that the height functions $h_{\mathcal{A}}$ of automatic permutation structures can be related to the running times of Turing machines. Let $\mathcal{M}$ be a Turing machine over input alphabet Σ. Its configuration graph $C(\mathcal{M})$ consists of the set of all configurations of $\mathcal{M}$, with an edge from c to d if T can move from c to d in a single transition. Recall that the configuration graph is automatic, see Example B.2.19.

Definition D.2.8 *A Turing machine $\mathcal{R}$ is* **reversible** *if every vertex in $C(\mathcal{R})$ has both indegree and outdegree at most one.*

Bennett [1973] proved that every deterministic Turing machine $\mathcal{M}$ can be simulated by a reversible Turing machine $\mathcal{R}$.

Lemma D.2.9 *For every deterministic Turing machine $\mathcal{M}$ there is an equivalent reversible Turing machine $\mathcal{R}$.*

Proof Suppose $\mathcal{M}$ is a deterministic machine. The idea is that machine $\mathcal{R}$ simulates $\mathcal{M}$ and records the transitions that $\mathcal{M}$ performed on a separate 'history' tape. Recall Example B.2.19 for notation of configuration spaces. Formally, machine $\mathcal{R}$ has the same state set as $\mathcal{M}$, the same initial state and the same halting states. Suppose that the transitions of $\mathcal{M}$ are numbered $\{1, 2, \cdots, k\}$. Then the transitions of $\mathcal{R}$ are of the form

$$\Delta_R((\sigma_1, \sigma_2), q) = ((\delta_1, \delta_2), q', (d_1, d_2))$$

where $\Delta_M(\sigma_1, q) = (\delta_1, q', d_1)$ is a transition of $\mathcal{M}$ numbered $1 \leq \delta_2 \leq k$ and $d_2 = R$ and $\sigma_2 = \lambda$ is the blank tape symbol. That is, if $\mathcal{R}$ is in state q and has σ_1 under its first (simulating-tape) head and nothing under its second (history-tape) head, then it does exactly what $\mathcal{M}$ would have done on its first tape, and records (the number of) that transition of $\mathcal{M}$ on a cell of its history tape and moves the history tape-head one cell to the right. Note that $\mathcal{M}$ and $\mathcal{R}$ accept the same language.

Configurations of $\mathcal{R}$ are of the form (c, y) where c is a configuration of $\mathcal{M}$ and $y = (y_1, y_2)$ is the content of the second tape of $\mathcal{R}$ with the head placed $|y_1| + 1$ cells from the left. If there is no edge leaving c in the configuration space $C(\mathcal{M})$ of $\mathcal{M}$ then there is no edge leaving (c, y) in $C(\mathcal{R})$. If there is such an edge in $C(\mathcal{M})$ then since $\mathcal{M}$ is deterministic it is unique. Now since $\mathcal{R}$ also ignores the contents of its second tape, there is a unique edge leaving (c, y) in $C(\mathcal{R})$. Hence every configuration of $C(\mathcal{R})$ has outdegree at most one.

Now suppose that (c, y), with $y = (y_1, y_2)$, configuration of $\mathcal{R}$ that has indegree more than one. Say it has predecessors are (d, x) and (d', x'). Write t (respectively t') for the unique transition t of $\mathcal{M}$ that sends c to d (respectively d'). Since this transition is recorded as the first symbol of y_2 it follows that $t = t'$. Furthermore t uniquely determines the state and content of the tape under the heads. So if $d = (q_1, a_1, a_2)$ and $d' = (q_1', a_1', a_2')$ then $q_1 = q_1'$ and the first symbols of a_2 and a_2' are equal (namely those symbols under the respective heads). Similarly the tape contents not under the head are recorded in c and hence determined, so $a_1 = a_1'$ and $a_2 = a_2'$. Hence $d = d'$.

Now if $x = (x_1, x_2)$ and $x' = (x_1', x_2')$ then by construction of $\mathcal{R}$, $y_1 = x_1 \cdot t = x_1' \cdot t$ so $x_1 = x_1'$. Similarly $x_2 = x_2'$ since their first symbol is the blank and the rest are the same as the rest of y_2. Hence $x = x'$ and so (c, y) has a unique predecessor in $C(\mathcal{R})$.

So every configuration of $C(\mathcal{R})$ has indegree at most one and so $\mathcal{R}$ is reversible as required. $\lhd$

Let $Time_T(w)$ be the number of steps T takes to halt on w, so that $Time_T(w) = \omega$ if T does not halt on w.

Theorem D.2.10 *For every reversible Turing machine R, there corresponds an automatic permutation structure $\mathcal{A}(R)$ such that for every $w \in \Sigma^\star$ there is an orbit in $\mathcal{A}(R)$ of size $Time_R(w)$.*

Proof Let the configuration graph of R be $\mathcal{C}(\mathcal{R}) = (C, E)$ over alphabet Σ. We may assume that the unique initial state of R is not a final state and that once the machine leaves the initial state it never returns to it. Then $\text{Time}(w) > 0$ for all w. Let $I, O \subset C$ respectively be the set of configurations with indegree 0 and with outdegree 0. These sets being definable are automatic. Let $\mathcal{C}'$ be a disjoint isomorphic copy of $\mathcal{C}$ over alphabet Σ' as in Remark B.1.20 and write c' for $\delta(c)$ where $\delta : \mathcal{C} \to \mathcal{C}'$ is the isomorphism. Define a permutation function f on domain $C \cup C'$ as follows. If $(c, d) \in E$ then $f(c) = d$ and $f(d') = c'$. If $c \in I$ then $f(c') = c$. If $d \in O$ then $f(d) = d'$.

The structure $(C \cup C', E, I, O, \delta)$ is automatic over alphabet $\Sigma \cup \Sigma'$. Conclude that the structure $(C \cup C', f)$ is also automatic. Factor this structure by the congruence relation satisfying pairs of the form (c, c') and (c', c) for $c \in I \cup O$. Write (D, g) for the resulting automatic permutation structure. If $Time_T(w) = n$ then (D, g) has a corresponding orbit of length $2n$. And if $Time_T(w) = \omega$ then (D, g) has (two) corresponding $\mathbb{Z}$-type orbits. Note that (D, g) may have $\mathbb{Z}$-type orbits corresponding to configurations of $C(\mathcal{R})$ that are not in any of the orbits generated by words w. The desired structure is $\mathcal{A}(T) = (D, h)$ where $h = g \circ g$. $\lhd$

The following is essentially from Blumensath and Grädel [2002].

Theorem D.2.11 *It is undecidable whether two automatic permutation structures are isomorphic.*

Proof Define an automatic permutation structure $\mathcal{F}$ as consisting of infinitely many $\mathbb{Z}$-type orbits, see Example B.2.2. For a deterministic Turing machine $\mathcal{M}$, construct an equivalent reversible machine $\mathcal{R}$ and then the automatic structure $\mathcal{A}(\mathcal{R})$. Then $\mathcal{M}$ halts on no word if and only if $\mathcal{A}(\mathcal{R})$ is isomorphic to $\mathcal{F}$. So the complement of the halting problem is reduced to the isomorphism problem for automatic permutation structures. $\lhd$

Blumensath and Grädel [2002] originally proved undecidability of the isomorphism problem for automatic structures by an implicit construction of reversible Turing Machines. Most of the content of this section has been reported in Ishihara, Khoussainov, and Rubin [2002] and Khoussainov and Rubin [2003].

Chapter E

Automatic linear orders and trees

An interesting problem concerns characterising the isomorphism types of classes of automatic structures. The main results in this chapter (Theorems E.2.7 and E.5.9) are necessary conditions for the automaticity of trees and linear orders. The conditions are stated in terms of a natural rank function on linear orders and trees, closely related to the classical Cantor-Bendixson rank of topological spaces. It then follows that the isomorphism problem for automatic ordinals is decidable, Corollary E.3.3. In fact, the Cantor normal form can be extracted from a presentation of an automatic ordinal. Also presented are automatic versions of König's lemma.

E.1 Linear order preliminaries

All classical definitions and results on linear orders can be found in Rosenstein [1982].

A *partial order* is a pair $(A, \preceq)$ such that $\preceq$, the *partial ordering*, is a reflexive, transitive and anti-symmetric binary relation on the domain A.

A *linear order* $\mathcal{L}$ is a partial order $(L, \leq)$ in which $\leq$ is total, that is $(\forall x \forall y)\,[x \leq y \vee y \leq x]$. If $\mathcal{L}$ is a linear order, then unless specified we denote its domain by L and ordering by $\leq_L$ or simply $\leq$. Similarly if $S \subset L$ then we write $\mathcal{S} = (S, \leq_S)$ for the order with domain S and ordering $\leq$ restricted to S. In this case we say that $\mathcal{S}$ is a suborder of $\mathcal{L}$.

Recall that the lexicographic order $\leq_{llex}$ and the length-lexicographic order $\leq_{llex}$ are FA recognisable (Definition B.1.9). The linear orders $(\Sigma^\star, \leq_{lex})$ and $(\Sigma^\star, \leq_{llex})$ are automatic.

Note E.1.1 *If $\mathcal{A} = (A, (R_i)_i)$ is an automatic structure over Σ, then $(A, (R_i)_i, \leq_{llex})$ is also automatic over Σ. Consequently every automatic presentation of $\mathcal{A}$ can be expanded to include the regular relations $\leq_{lex}$ and $\leq_{llex}$ (restricted to the domain A). This fact will be used repeatedly.*

Other examples of automatically presentable linear orders are $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$ and the order on rationals $(\mathbb{Q}, \leq)$. Moreover, if $\mathcal{L}_1 = (L_1, \leq_1)$ and $\mathcal{L}_2 = (L_2, \leq_2)$ are automatic linear orders then so are their sum and product. Hence the ordinals ω^n for every $n \in \mathbb{N}$ are automatically presentable.

Classically linear orders are characterised in terms of scattered and dense linear orders. We say that $\mathcal{L}$ is *dense* if for all distinct a and b in L with $a < b$ there exists an $x \in L$ with $a < x < b$. There are only five types of countable dense linear orders up to isomorphism: the order of rational numbers with or without least and greatest elements, and the order type of the trivial linear order with exactly one element. We say that $\mathcal{L}$ is *scattered* if it does not contain a nontrivial dense suborder.

Write ω for the (order) type of the positive integers, $\omega^\star$ for the negative integers, ζ for the integers, η for the rationals and $\mathbf{n}$ for the finite order on n elements. The empty order is written $\mathbf{0}$ and the order with exactly one element is written $\mathbf{1}$. A suborder $\mathcal{S}$ of $\mathcal{L}$ is an *interval* if for every $x, y \in S$ with $x <_L y$ it is the case that $z \in S$ for every $z \in L$ satisfying $x <_L z <_L y$. An interval is *closed* if it is of the form $\{z \in L \mid x \leq z \leq y\}$ if $x \leq y$ and $\{z \in L \mid y \leq z \leq x\}$ otherwise; either way the interval is written $[x, y]$.

Definition E.1.2 *Consider a linear order $\mathcal{I}$ as an index set for a set of linear orders $\{\mathcal{A}_i\}_{i \in I}$. The $\mathcal{I}$-***sum**

$$\mathcal{L} = \Sigma\{\mathcal{A}_i \mid i \in I\}$$

is the linear order with domain $\cup_i A_i$ (we may assume that the domains A_i are pairwise disjoint). For $x \in A_i, y \in A_j$ define $x \leq_L y$ if $(i <_I j) \vee (i = j \wedge x \leq_{A_i} y)$.

We refer to the case when I is dense as a *dense sum*. If every $\mathcal{A}_i$ is scattered and $\mathcal{I}$ is scattered then the sum is scattered. If $\mathcal{A}_i = \mathcal{B}$ for every $i \in I$, then the sum is written as a product $\mathcal{B}\mathcal{I}$. For instance $\omega 2$ is $\omega + \omega$. The classical characterisation, whose proof is given below, says that:

Theorem E.1.3 [Hausdorff - 1908] *Every countable linear order $\mathcal{L}$ can be represented as a dense sum of countable scattered linear orders.*

In turn the scattered linear orders can be characterised inductively, where to each linear order one associates an ordinal ranking, called the VD–rank. VD stands for very discrete.

Definition E.1.4 *For each countable ordinal α, define the set VD_α of linear orders inductively as*

1. $\mathrm{VD}_0 := \{\mathbf{0}, \mathbf{1}\}$.
2. $\mathrm{VD}_\alpha :=$ *all linear orders formed as $\mathcal{I}$–sums where $\mathcal{I}$ is of the type $\omega, \omega^\star, \zeta$ or $\mathbf{n}$ for some $n < \omega$ and every $\mathcal{A}_i$ is a linear order from $\bigcup\{\mathrm{VD}_\beta \mid \beta < \alpha\}$.*

Define the class VD as the union of the VD_α's. The **VD–rank** *of a linear order $\mathcal{L} \in \mathrm{VD}$, written $\mathrm{VD}(\mathcal{L})$, is the least ordinal α such that $\mathcal{L} \in \mathrm{VD}_\alpha$.*

Example E.1.5 Let $\mathcal{L}_1 = \Sigma\{\zeta + \mathbf{n} \mid n \in \omega\}$, $\mathcal{L}_2 = (\zeta \cdot \zeta) \cdot \zeta$. Then $\mathrm{VD}(\mathcal{L}_1) = 2$, $\mathrm{VD}(\mathcal{L}_2) = 3$ and $\mathrm{VD}(\mathcal{L}_1 + \mathcal{L}_2) = 4$. $\triangleleft$

In general, if $\alpha = \max(\mathrm{VD}(\mathcal{L}_1), \mathrm{VD}(\mathcal{L}_2))$, then $\alpha \leq \mathrm{VD}(\mathcal{L}_1 + \mathcal{L}_2) \leq \alpha + 1$, (see Rosenstein [1982], Lemma 5.15).

Example E.1.6 Let α, β be countable ordinals. Then $\mathrm{VD}(\beta) \leq \alpha$ if and only if $\beta \leq \omega^\alpha$. In particular, $\mathrm{VD}(\omega^\alpha) = \alpha$. $\triangleleft$

Theorem E.1.7 [Hausdorff - 1908] *A countable linear order $\mathcal{L}$ is scattered if and only if $\mathcal{L}$ is in VD.*

There is an alternative definition of ranking that generalises VD–rank and assigns an ordinal rank to non-scattered linear orders as well. We proceed with the definitions.

Definition E.1.8 *A* **condensation (map)** *of $\mathcal{L}$ is a mapping c from L to non-empty intervals of L such that $c(y) = c(x)$ whenever $y \in c(x)$. The* **condensation** *of $\mathcal{L}$ is the linear order $c[\mathcal{L}]$ whose domain consists of the collection of non-empty intervals $c(x)$ for $x \in L$ ordered by $c(x) \trianglelefteq c(y)$ if $c(x) = c(y)$ or $(\forall x' \in c(x))(\forall y' \in c(y))[x' <_L y']$.*

The relation x *is condensed (by c) to* y defined as $x \in c(y)$ is an equivalence relation.

As an illustration of the definition we prove that every countable linear ordering can be represented as a dense sum of scattered linear orderings (Theorem E.1.3).

Proof The mapping $c_S : x \mapsto \{y \in L \mid [x, y] \text{ is scattered}\}$ is a condensation since if $y \in c_S(x)$ then for all a, $[y, a]$ does not contain a dense subordering if and only if $[x, a]$ does not contain a dense subordering. Now $\mathcal{L} = \sum\{a \mid a \in c[\mathcal{L}]\}$ and each $a = c_S(x) \in c[\mathcal{L}]$ is scattered. Finally $c_S[\mathcal{L}]$ is dense since for $c_S(x) \triangleleft c_S(y)$, if there is no z with $c_S(x) \triangleleft c_S(z) \triangleleft c_S(y)$ then $[x, y]$ is scattered. $\triangleleft$

Definition E.1.9 *Define* $\boldsymbol{c_{FC}(x)}$ *as $\{y \in L \mid [x, y]$ is a finite interval of $\mathcal{L}\}$.*

For every ordinal α define a condensation map c^α_{FC} of $\mathcal{L}$ inductively:

$$c^\alpha_{FC}(x) = \{y \in L : y = x \vee (\exists \beta < \alpha)\,[c_{FC}(c^\beta_{FC}(y)) = c_{FC}(c^\beta_{FC}(x))]\,\}.$$

In the expression $c_{FC}(c^\beta_{FC}(y))$ the term c_{FC} is a condensation map of the linear order $c^\beta[\mathcal{L}]$; hence $c_{FC}(c^\beta_{FC}(y))$ is the set of elements of $c^\beta[\mathcal{L}]$ that are condensed to the element $c^\beta_{FC}(y)$. Note that this definition implicitly gives $c^0_{FC}(x) = \{x\}$ and $c^1_{FC}(x) = c_{FC}(x)$.

Here FC stands for finite condensation and indeed c^α_{FC} is a condensation map of $\mathcal{L}$. The idea is that $c^1_{FC}(x)$ is the set of elements of $\mathcal{L}$ that are only finitely far away from x; $c^2_{FC}(x)$ is the set of elements of $\mathcal{L}$ that are in intervals of $c_{FC}[\mathcal{L}]$ which themselves are only finitely far away in $c_{FC}[\mathcal{L}]$ from the interval $c^1_{FC}(x)$, etc.

Definition E.1.10 *The least ordinal α such that $c^{\beta}_{FC}(x) = c^{\alpha}_{FC}(x)$ for all $x \in L$ and $\beta \geq \alpha$ is called the* **FC–rank** *of $\mathcal{L}$, written* $\mathrm{FC}(\mathcal{L})$.

For instance a non-empty linear order $\mathcal{L}$ is dense if and only if its FC–rank is 0. A dense sum of orders of FC–rank α has FC–rank α. **From now on we write c for $\mathbf{c_{FC}}$.**

Example E.1.11 The FC–rank of $\mathcal{L}$ is the least ordinal α such that $c^{\alpha}[\mathcal{L}]$ is dense. So $\mathcal{L}$ is scattered if and only if $c^{\alpha}[\mathcal{L}] \simeq \mathbf{1}$ for some ordinal α. $\lhd$

The following theorem connects FC–ranks and VD–ranks of scattered linear orderings.

Theorem E.1.12 [Hausdorff - 1908] *If $\mathcal{L}$ is scattered then its VD–rank equals its FC–rank.* $\lhd$

For instance the ordinal ω^n is scattered and has VD–rank and FC–rank n.

Given linear order $\mathcal{L}$ and $A \subset L$ we will use c to denote the condensation of $\mathcal{L}$ and c_A to denote the condensation of the linear order $\mathcal{A}$. For instance if $a \in A$ then $c_A(a) = \{y \in A \mid [a, y] \cap A \text{ is finite}\}$. Here are some useful properties.

Lemma E.1.13

1. [Rosenstein - 1982, Lemma 5.14] *If $\mathcal{L}$ is scattered and $M \subset L$ then* $\mathrm{FC}(\mathcal{M}) \leq \mathrm{FC}(\mathcal{L})$.

2. [Rosenstein - 1982, Lemma 5.13 (2)] $\mathrm{FC}(c^{\alpha}(x)) \leq \alpha$ *and $c^{\alpha}(x)$ is a scattered interval of $\mathcal{L}$ for every ordinal α and $x \in L$.*

3. [Rosenstein - 1982, Exercise 5.12 (1)] *If I is an interval of $\mathcal{L}$ then $c^{\alpha}_{I}(x) = c^{\alpha}(x) \cap I$ for every ordinal α and $x \in I$.*

4. *For every $x, y \in L$, if the interval $[x, y]$ is scattered then $c^{\alpha}_{[x,y]}(x) = c^{\alpha}_{[x,y]}(y)$ if and only if* $\mathrm{FC}([x, y]) \leq \alpha$.

Proof We prove the last item. Let $x, y \in L$ and α be an ordinal. Then by definition $\mathrm{FC}([x, y]) \leq \alpha$ means that (†) for every $z \in [x, y]$, $c^{\alpha}_{[x,y]}(z) = c^{\alpha+1}_{[x,y]}(z)$, which necessarily equals $[x, y]$ since $[x, y]$ is scattered. Denote the condition $c^{\alpha}_{[x,y]}(x) = c^{\alpha}_{[x,y]}(y)$ by (††).

Then (†) clearly implies (††) by considering $z \in \{x, y\}$. For the converse suppose (††). We first claim that $c^{\alpha}_{[x,y]}(x) = [x, y]$. Indeed (††) implies that $y \in c^{\alpha}_{[x,y]}(x)$ since $c^{\alpha}_{[x,y]}$ is a condensation, which means that $[x, y]$ is a subset of the interval $c^{\alpha}_{[x,y]}(x)$. But also $c^{\alpha}_{[x,y]}(x) \subset [x, y]$ by item (3). Hence $c^{\alpha}_{[x,y]}(x) = [x, y]$ as claimed. So if $z \in [x, y] = c^{\alpha}_{[x,y]}(x)$ then $c^{\alpha}_{[x,y]}(x) = c^{\alpha}_{[x,y]}(z)$ by the property of being a condensation. Hence $z \in [x, y]$ implies that $c^{\alpha}_{[x,y]}(z) = [x, y]$. In particular then also $c^{\alpha+1}_{[x,y]}(z) = [x, y]$ which implies (†) as required. $\lhd$

E.2 Ranks of automatic linear orders

We now prove the central technical result, Theorem E.2.7, via three propositions that generalise the ideas in Delhommé [2001a]. As a matter of convenience we introduce the following variation of VD–rank.

Definition E.2.1 *If $\mathcal{L}$ is scattered define its* **$\mathbf{VD}_*$–rank**, *written* $\mathrm{VD}_*(\mathcal{L})$, *as the least ordinal α such that $\mathcal{L}$ can be written as a finite sum of orderings of VD–rank $\leq \alpha$.*

For example it is not hard to check that $\mathrm{VD}(\omega) = \mathrm{VD}_*(\omega) = 1$ and that $\omega 2 + 1$ has VD–rank 2 but VD_*–rank 1. We list some basic properties.

Property E.2.2 *Suppose $\mathcal{L}$ is scattered.*

1. *$c^\alpha[\mathcal{L}]$ is a finite linear order if and only if $\mathrm{VD}_*(\mathcal{L}) \leq \alpha$. So $\mathrm{VD}_*(\mathcal{L})$ is the least ordinal such that $c^\alpha[\mathcal{L}]$ is a finite linear order.*
2. *If $M \subset L$ then $\mathrm{VD}_*(\mathcal{M}) \leq \mathrm{VD}_*(\mathcal{L})$.*
3. $\mathrm{VD}_*(\mathcal{L}) \leq \mathrm{VD}(\mathcal{L}) \leq \mathrm{VD}_*(\mathcal{L}) + 1$.
4. $\mathrm{VD}_*(\mathcal{L}) = \alpha$ *implies that $\mathcal{L}$ contains an interval, say M, with $\mathrm{VD}(\mathcal{M}) = \alpha$ and $\mathrm{VD}_*(\mathcal{M}) = \alpha$.*

Proof For the first item observe that for every α, $\mathcal{L} = \Sigma\{a \mid a \in c^\alpha[\mathcal{L}]\}$. Each a is an interval of the form $c^\alpha(x)$ for some $x \in L$. So by Lemma E.1.13 every a has VD–rank at most α. So if $c^\alpha[\mathcal{L}]$ is finite then $\mathrm{VD}_*(\mathcal{L}) \leq \alpha$. Conversely let $\mathcal{L} = \mathcal{L}_1 + \cdots + \mathcal{L}_k$ and $\mathrm{VD}(\mathcal{L}_i) \leq \alpha$. Then $L_i \subset c^\alpha(x)$ if $x \in L_i$. Since for $x \in L$ the $c^\alpha(x)$ are pairwise disjoint, $c^\alpha[\mathcal{L}]$ is finite.

For the second item suppose $\mathcal{L}$ can be expressed as a finite sum $\mathcal{L}_1 + \cdots + \mathcal{L}_K$ where $\mathrm{VD}(\mathcal{L}_i) \leq \alpha$. Define $M_i = M \cap L_i$. By Lemma E.1.13 (1) the VD–rank of M_i is at most α. But $\mathcal{M} = \mathcal{M}_1 + \cdots + \mathcal{M}_i$ so $\mathrm{VD}_*(\mathcal{M}) \leq \alpha$.

The third item follows from item (1) above and the property that $\mathrm{VD}(\mathcal{L})$ is the least ordinal β such that $c^\beta[\mathcal{L}]$ is isomorphic to $\mathbf{1}$.

For the last item suppose $\mathrm{VD}_*(\mathcal{L}) = \alpha$. Then $\mathcal{L}$ can be expressed as a finite sum of orders of VD–rank (and by item (2) also VD_*–rank) at most α. There is a summand with VD_*–rank α for otherwise every summand can be written as a finite sum of linear orders of VD–rank $< \alpha$, and hence $\mathcal{L}$ could be written as a finite sum of linear orders of VD–rank $< \alpha$, contrary to assumption. Finally by item (3) if a summand has VD_*–rank α then it has VD–rank α. $\lhd$

Lemma E.2.3 *Suppose $\mathcal{L}$ is a scattered linear order containing $\Sigma_{i \in I} A_i$ as a subordering and $\mathrm{VD}_*(\mathcal{A}_i) = \beta$, where $\mathcal{I}$ has order type ω or $\omega^\star$ and each A_i is non-empty. Then $\mathrm{VD}_*(\mathcal{L}) > \beta$.*

Proof By Property E.2.2 (4) we can assume without loss of generality that $\mathrm{VD}_*(\mathcal{A}_i) = \mathrm{VD}(\mathcal{A}_i) = \beta$. This means that A_i can not be written as a finite sum of orders of VD–rank $< \beta$. Suppose that $\mathcal{I}$ has order type ω, the other case being similar. Let $\mathcal{A} = \Sigma_i \mathcal{A}_i$. For every i choose some $x_i \in A_i$.

Suppose that $c_A^\beta(x_i) = c_A^\beta(x_{i+2})$ for some i. In other words x_i is condensed to x_{i+2} in at most β steps. Then $\beta > 0$ since $c_A^0(x) = \{x\}$ for every x. Moreover there is some $\gamma < \beta$ so that there are finitely many elements in $c_A^\gamma[\mathcal{A}]$ between $c_A^\gamma(x_i)$ and $c_A^\gamma(x_{i+2})$. These finitely many elements are of the form $c_A^\gamma(x)$ for $x \in A$ and so have VD–rank at most γ. In particular $\mathcal{A}_{i+1}$ can be written as a finite sum of orders of VD–rank at most γ, contrary to assumption. We conclude that $c_A^\beta(x_i) \neq c_A^\beta(x_{i+2})$ for every i.

Hence $c_A^\beta[\mathcal{A}]$ is infinite and so $\mathrm{VD}_*(\mathcal{A}) > \beta$ by Property E.2.2 (1). So $\mathrm{VD}_*(\mathcal{L}) > \beta$ by Property E.2.2 (2). $\lhd$

Proposition E.2.4 *Suppose $\mathcal{L}$ is a scattered linear ordering and consider a finite partition of the domain $L = A_1 \cup A_2 \cup \ldots \cup A_k$. Then there exists some $\delta \in \{1, \ldots, k\}$ with* $\mathrm{VD}_*(\mathcal{A}_\delta) = \mathrm{VD}_*(\mathcal{L})$.

Proof The proof is done by induction. So assume that $\mathrm{VD}_*(\mathcal{L}) = \alpha$ is the ordinal to be addressed and that the proposition holds for all $\beta < \alpha$. Let $L, k, A_1, \ldots, A_k$ be as in the statement of the proposition.

If $\alpha = 0$ then $\mathrm{VD}_*(\mathcal{L}) = \mathrm{VD}_*(\mathcal{A}_\epsilon) = 0$ for every nonempty subset A_ϵ of L.

Otherwise by Property E.2.2 item (4), there is some interval of $\mathcal{L}$, say $\mathcal{M}$, with $\mathrm{VD}(\mathcal{M}) = \mathrm{VD}_*(\mathcal{M}) = \alpha$. Then $\mathcal{M}$ is an $\mathcal{I}$–sum of linear orders $\{\mathcal{M}_i\}$ of VD–rank $< \alpha$, where $\mathcal{I}$ is an infinite linear order of the type $\omega, \omega^\star$ or ζ. So suppose that $\mathcal{I}$ is of type ω (the other two order types are similar).

Suppose α is a successor ordinal, say $\alpha = \beta + 1$. There are infinitely many i such that $\mathrm{VD}_*(\mathcal{M}_i) = \beta$, for otherwise we could write $\mathcal{M}$ as a finite sum of orders of VD–rank β, and conclude that $\mathrm{VD}_*(\mathcal{M}) \leq \beta$. For each such i let $A_{\delta,i} = M_i \cap A_\delta$, where $\delta \in \{1, \cdots, k\}$. Applying the induction hypothesis to every $\mathcal{M}_i$ we see that there is an $\epsilon \in \{1, \cdots, k\}$ and infinitely many j such that $\mathrm{VD}_*(\mathcal{A}_{\epsilon,j}) = \mathrm{VD}_*(\mathcal{M}_j) = \beta$. Hence A_ϵ contains an ω-sum of linear orders of VD_*–rank β. By Lemma E.2.3 $\mathrm{VD}_*(\mathcal{A}_\epsilon) = \alpha$.

Suppose that α is a limit ordinal. The supremum of the VD–ranks of the $\mathcal{M}_i$ is α. Using the notation of the case above, and applying induction, we see that there is an $\epsilon \in \{1, \cdots, k\}$ and infinitely may j such that $\mathrm{VD}_*(\mathcal{A}_{\epsilon,j}) = \mathrm{VD}_*(\mathcal{M}_j)$, and the supremum of the VD_*–ranks of these $\mathcal{A}_{\epsilon,j}$ is α. Then $\mathrm{VD}_*(\mathcal{A}_\epsilon) = \alpha$ as required. $\lhd$

Proposition E.2.5 *Let $\mathcal{L}$ be a scattered order with VD–rank at least α. Then for every $\beta < \alpha$ there exists a closed interval of $\mathcal{L}$ of VD–rank $\beta + 1$.*

Proof Recall that VD–ranks and FC–ranks coincide on scattered linear orders, Theorem E.1.12. Fix $\beta < \alpha$. Since $\mathcal{L}$ has FC–rank $> \beta$, by definition there is some $x \in L$ such that $c^{\beta}(x) \neq c^{\beta+1}(x)$. Pick $y \in c^{\beta+1}(x) \setminus c^{\beta}(x)$. Then $c^{\beta}(x) \neq c^{\beta}(y)$ and $c^{\beta+1}(x) = c^{\beta+1}(y)$. Recall that $c^{\beta}_{[x,y]}$ is the condensation mapping c^{β} within the interval $[x, y]$. Hence by Lemma E.1.13 (3), $c^{\beta}_{[x,y]}(x) \neq c^{\beta}_{[x,y]}(y)$ and $c^{\beta+1}_{[x,y]}(x) = c^{\beta+1}_{[x,y]}(y)$. By Lemma E.1.13 (4) the first fact implies that $\mathrm{FC}([x, y]) > \beta$ and the second fact implies that $\mathrm{FC}([x, y]) \leq \beta + 1$. So the FC–rank of $[x, y]$ is exactly $\beta + 1$. $\lhd$

Proposition E.2.6 *The VD–rank of every automatic scattered linear ordering is finite.*

Proof Given an automatic scattered linear order $\mathcal{L}$ over $\Sigma^{\star}$ let $(Q_{\leq}, \iota_{\leq}, \Delta_{\leq}, F_{\leq})$ be a deterministic 2–tape automaton recognising the ordering of $\mathcal{L}$. Similarly let $(Q_A, \iota_A, \Delta_A, F_A)$ be a deterministic 3–tape automaton recognising the definable relation $\{(x, z, y) \mid x \leq z \leq y\}$. We assume the state sets Q_A and $Q_{\leq}$ are disjoint.

For $x, y \in L$ and $v \in \Sigma^{\star}$, define $[x, y]_v$ as the set of all $z \in L$ such that $x \leq z \leq y$ and z has prefix v. For $|v| \geq |x|, |y|$ define $I(x, v, y) \in Q_A$ and $J(x) \in Q_{\leq}$ as follows. $I(x, v, y)$ is the state in Q_A that results from the initial state ι_A after reading the convolution of (x, v, y), namely $(x\perp^n, v, y\perp^m)$ where $n, m \geq 0$ are chosen so that the length of each component is exactly $|v|$. That is define $I(x, v, y) := \Delta_A(\iota_A, \otimes(x, v, y))$. Similarly define $J(v) := \Delta_{\leq}(\iota_{\leq}, \otimes(v, v))$. Write $K(x, v, y)$ for the ordered pair $(I(x, v, y), J(v))$.

Now if $K(x, v, y) = K(x', v', y')$ then the subordering with domain $[x, y]_v$ is isomorphic to the subordering with domain $[x', y']_{v'}$ via the map $vw \mapsto v'w$ for $w \in \Sigma^{\star}$. Indeed the domains are isomorphic since for every $w \in \Sigma^{\star}$,

$$vw \in [x, y]_v$$

if and only if

$$\Delta_A(\Delta_A(\iota_A, \otimes(x, v, y)), \otimes(\epsilon, w, \epsilon)) \in F_A$$

if and only if

$$\Delta_A(\Delta_A(\iota_A, \otimes(x', v', y')), \otimes(\epsilon, w, \epsilon)) \in F_A$$

if and only if

$$v'w \in [x', y']_{v'}.$$

The map preserves the ordering since for $w_1, w_2 \in \Sigma^{\star}$ such that $vw_1, vw_2 \in [x, y]_v$ and $v'w_1, v'w_2 \in [x', y']_{v'}$ we have

$$vw_1 \leq vw_2$$

if and only if

$$\Delta_{\leq}(\Delta_{\leq}(\iota_{\leq}, \otimes(v, v)), \otimes(w_1, w_2)) \in F_{\leq}$$

if and only if

$$\Delta_{\leq}(\Delta_{\leq}(\iota_{\leq}, \otimes(v', v')), \otimes(w_1, w_2)) \in F_{\leq}$$

if and only if

$$v'w_1 \leq v'w_2.$$

Hence the number of isomorphism types of suborderings with domain $[x, y]_v$ for $|v| \geq |x|, |y|$ is bounded by the number of distinct pairs $K(x, v, y)$ which is at most $|Q_A| \times |Q_{\leq}|$, denoted by d. In particular (†) there are at most d many VD_*–ranks among suborderings with domain of the form $[x, y]_v$ for $|v| \geq |x|, |y|$.

Now suppose there exists a closed interval $[x, y]$ of $\mathcal{L}$ with VD–rank at least $2(d + 2)$. Using Proposition E.2.5, for every $1 \leq i \leq 2(d + 2)$, the interval $[x, y]$ contains a closed interval, say $[x_i, y_i]$, of VD–rank i. So by Property E.2.2 (3) at least $d + 2$ many of these intervals have different VD_*–ranks; and at least $d + 1$ many of these intervals have non-zero VD_*–rank. Say $[x_j, y_j]$ is one of these $d + 1$ many intervals. Set $n = \max\{|x_j|, |y_j|\}$ and partition $[x_j, y_j]$ into the set $[x_j, y_j] \cap \Sigma^{<n}$ and the finitely many sets of the form $[x_j, y_j]_v$ where $|v| = n$. Since the finite set $[x_j, y_j] \cap \Sigma^{<n}$ has VD_*–rank 0, by Proposition E.2.4 there is some v_j with $|v_j| = n$ so that the subordering on domain $[x_j, y_j]_{v_j}$ has the same VD_*–rank as $[x_j, y_j]$. Hence there are at least $d+1$ many intervals of the form $[x_j, y_j]_{v_j}$ all with different VD_*–ranks. This contradicts (†) and so we conclude that the VD–rank of every closed interval $[x, y]$ of $\mathcal{L}$ is at most $e = 2(d+2)$. So for every $x, y \in L$, $c^e(x) = c^e(y)$ and so $\mathrm{VD}(\mathcal{L}) \leq e$ as required. $\lhd$

As a corollary of the proposition just proved we derive the following result for all automatic linear orderings:

Theorem E.2.7 *The FC–rank of every automatic linear order is finite.*

Proof Let $\mathcal{L}$ be a linear order and write it as $\sum\{\mathcal{L}_i \mid i \in D\}$ where $\mathcal{D}$ is dense and each $\mathcal{L}_i$ is scattered. We will show that for every $i \in D$ and every $a, b \in L_i$, the VD–rank of $[a, b]$ is uniformly bounded. Let $(Q_{\leq}, \iota_{\leq}, \Delta_{\leq}, F_{\leq})$ be a deterministic 2–tape automaton recognising the ordering of $\mathcal{L}$. Let $(Q_A, \iota_A, \Delta_A, F_A)$ be a deterministic 3–tape automaton recognising the definable relation $\{(x, z, y) \mid x \leq z \leq y\}$. Now consider an interval $[a, b]$ of $\mathcal{L}_i$ for some $i \in D$. The proof of the previous theorem ensures that the VD–rank of the scattered interval $[a, b]$ is at most e, where the constant e does not depend on $[a, b]$ or i but only on $|Q_A|$ and $|Q_{\leq}|$. Therefore the VD–rank of interval $[a, b]$ is at most e. Hence $\mathrm{VD}(\mathcal{L}_i) \leq e$ for every $i \in D$ and so $\mathrm{FC}(\mathcal{L}) \leq e$. $\lhd$

Remark E.2.8 This result is a necessary though not sufficient condition for a linear order to be automatically presentable. Indeed there are linear orders of rank 2 that are not automatically presentable. For instance if $R \subset \mathbb{N}$ is a non-computable set, then the linear order $\Sigma_{n \in R}(\zeta + \mathbf{n})$ does not have decidable first order theory and so is not automatically presentable.

Corollary E.2.9 [Delhommé - 2001a] *An ordinal α is automatically presentable if and only if $\alpha < \omega^\omega$.*

Proof Suppose α is an automatically presentable ordinal. Then by Theorem E.2.7 it has finite FC–rank and so by Example E.1.6 $\alpha < \omega^\omega$ as required.

Conversely given $\alpha < \omega^\omega$ there exists $n < \omega$ such that $\alpha < \omega^n$. But ω^n is automatically presentable. Say $(W, \leq)$ is an automatic presentation. Let $p \in W$ be the string corresponding to α. So the suborder of $\mathcal{W}$ on the definable domain $\{x \in W \mid x < p\}$ is isomorphic to α. Hence α is automatically presentable. $\triangleleft$

Proposition E.2.10 *Suppose $\mathcal{L}$ is an automatic linear order.*

1. *The FC–rank of $\mathcal{L}$ is computable from $\mathcal{L}$.*
2. *It is decidable whether or not $\mathcal{L}$ is scattered.*
3. *If $\mathcal{L}$ is not scattered then a regular dense subordering is effectively computable from $\mathcal{L}$.*

Proof Let $\mathcal{L}$ be an automatic linear order. The condensation c_{FC}, viewed as the equivalence relation x related to y if $x \in c_{FC}(y)$, is FO$^\infty$-definable in $\mathcal{L}$ since $c_{FC}(x) = c_{FC}(y)$ if and only if $[x, y]$ is finite. Since the ordering on $c_{FC}[\mathcal{L}]$ is FO definable in $\mathcal{L}$ (see Definition E.1.8) the linear orders $c^i_{FC}[\mathcal{L}]$ are FO$^\infty$-definable for every $i \in \mathbb{N}$, and so by Theorem B.1.26 each is automatic. So successively compute automatic presentations for $c^1_{FC}[\mathcal{L}], c^2_{FC}[\mathcal{L}], \cdots$ and stop the first time $c^r_{FC}[\mathcal{L}]$ is dense for some $r \in \mathbb{N}$. This procedure must terminate since, being automatic, the FC–rank of $\mathcal{L}$ is finite. Moreover since denseness is a FO property the procedure is effective. So the FC–rank of $\mathcal{L}$ is r. Now by Example E.1.11 the linear order $c^r_{FC}[\mathcal{L}]$ is isomorphic to **1** if and only if $\mathcal{L}$ is scattered. Again testing whether an automatic linear order is isomorphic to **1** can be decided by using the sentence $(\exists x)(\exists y)\,[x < y]$. In case $c^r_{FC}[\mathcal{L}]$ is not the singleton it must be an infinite dense ordering. Viewing c^r_{FC} as an automatic equivalence relation on $\mathcal{L}$ (the $c^r_{FC}(x)$ partition $\mathcal{L}$), the $<_{llex}$-smallest representatives from every equivalence class form a dense subordering of $\mathcal{L}$ that is a regular subset of L. $\triangleleft$

E.3 Decidability results for automatic ordinals

Theorem E.2.7 can now be applied to prove decidability results for automatic ordinals. Contrast this with the fact that the set of computable structures that are well orderings is Π^1_1-complete Rogers [1967].

Proposition E.3.1 *Let $\mathcal{L} = (L, \leq)$ be an automatic structure. It is decidable whether $\mathcal{L}$ is isomorphic to an ordinal.*

Proof First check that $\leq$ linearly orders L, by testing whether $\mathcal{L}$ is reflexive, transitive and anti-symmetric – all first order axioms and hence computable properties. Although being a well-order is not first order expressible, see for instance Theorem 13.13 in Rosenstein [1982], the following algorithm can be used.

1. **Input** the presentation $(L, \leq)$ of $\mathcal{L}$.

2. Let $D = L$.

3. **While** $(D, \leq)$ is not dense and $(\forall x \in D)\, [\omega^*$ does not embed in the interval $c(x)]$

   Do Replace $(D, \leq)$ by a presentation for $c[\mathcal{D}]$.

4. **End While**

5. If $\mathcal{D}$ is isomorphic to **1** then **Output $\mathcal{L}$ is an ordinal**,
   else **Output $\mathcal{L}$ is not an ordinal**.

Every step in the algorithm is computable. Indeed the equivalence relation on pairs (x, y) satisfying $c(x) = c(y)$ is definable as $(\neg\exists^\infty z)\, [x < z < y]$. So a presentation for $c[\mathcal{D}]$ is computed by factoring $\mathcal{D}$ by c. The while test is expressible as

$$\neg(\forall x \neq y)\, (\exists z)\, [x < z < y]$$

and

$$(\forall x)(\neg\exists^\infty y)\, (c(x) = c(y) \wedge y < x).$$

The final test is expressible by $(\exists x)\, (\forall y)\, [x = y]$.

The FC–rank of $\mathcal{L}$ is finite, say k, because the structure is automatic. So the algorithm terminates after at most $k+1$ many while-loop tests. If $\mathcal{L}$ is an ordinal then $c[\mathcal{L}]$ is an ordinal and for every $x \in L$, $c(x)$ is either finite or isomorphic to ω. By induction on k, for every $0 \leq i \leq k$, $c^i[\mathcal{L}]$ passes the $(i+1)$-th while-test. The resulting order $\mathcal{D} = c^k[\mathcal{L}]$ is isomorphic to **1** as required.

If $\mathcal{L}$ is not an ordinal then there exists an infinite decreasing sequence of elements. Suppose there exists such a sequence $x_1 > x_2 > x_3 \ldots$ and an $n_0 \in \mathbb{N}$ such that for all $i \geq n_0$ $c(x_i) = c(x_{n_0})$. Then the while-test fails the first time it is executed and the resulting order $\mathcal{D} = \mathcal{L}$ is not isomorphic to the ordinal **1**. If there is no such sequence (x_i) and n_0 then there exists a sequence, say $y_1 > y_2 > y_3 > \ldots$ such that $c(y_{i+1}) \lhd c(y_i)$ for all $i \in \mathbb{N}$; this is an infinite decreasing sequence of elements in $c[\mathcal{L}]$. Continue inductively in this way with $c[\mathcal{L}]$ in place of $\mathcal{L}$. Suppose the while-test fails the m-th time for some $1 \leq m \leq k$. If it fails because there is some $x \in c^{m-1}[\mathcal{L}]$ for which $\omega^\star$ embeds in $c(x)$ then $\mathcal{D} = c^{m-1}[\mathcal{L}]$ is infinite and so not isomorphic to **1**. If there is no such m, then the while-test must fail the $(k+1)$'st time. In this case $\mathcal{D} = c^k[\mathcal{L}]$ is dense but as before there is a sequence $y_1 > y_2 > y_3 > \ldots$ with $c^k(y_{i+1}) \lhd c^k(y_i)$ for every $i \in \mathbb{N}$. In this case $\mathcal{D}$ is not isomorphic to **1**. $\lhd$

We now show that the isomorphism problem for automatic ordinals is decidable. Recall that by Cantor's Normal Form Theorem if α is an ordinal then it can be uniquely decomposed as $\omega^{\alpha_1}n_1 + \omega^{\alpha_2}n_2 + \ldots + \omega^{\alpha_k}n_k$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are ordinals satisfying $\alpha_1 > \alpha_2 > \ldots > \alpha_k$ and $k, n_1, n_2, \ldots, n_k$ are natural numbers. The proof of deciding the isomorphism problem for automatic ordinals is based on the fact that Cantor's normal form can be extracted from automatic presentations of ordinals.

Theorem E.3.2 *If α is an automatic ordinal then its normal form is computable from an automatic presentation of α.*

Proof Let $(R, \leq_{ord})$ be an automatic presentation over Σ of the ordinal α. Recall that the unknown ordinal is of the form $\alpha = \omega^m n_m + \omega^{m-1} n_{m-1} + \ldots + \omega^2 n_2 + \omega n_1 + n_0$ where $m, n_m, n_{m-1}, \ldots, n_1, n_0$ are natural numbers. Now one can compute the values $m, n_0, n_1, \ldots$ by the following algorithm.

1. **Input** the presentation $(R, \leq_{ord})$.
2. Let $D = R$, $m = 0$, $n_m = 0$.
3. **While** $D \neq \emptyset$ **Do**
4. **If** D has a maximum u

   Then Let $n_m = n_m + 1$, let $D = D - \{u\}$.

   Else Let $L \subset D$ be the set of limit ordinals in D; that is L is the set of all $x \in D$ with no immediate predecessor in D. Replace D by L, let $m = m + 1$, let $n_m = 0$.
5. **End While**
6. **Output** the formula

$$\omega^m n_m + \omega^{m-1} n_{m-1} + \ldots + \omega^2 n_2 + \omega n_1 + n_0$$

   using the current values of $m, n_0, \ldots, n_m$.

The algorithm is computable since the set of limit ordinals of an automatic linear order $\mathcal{L}$ is first order definable, and hence computable, in $\mathcal{L}$ by the formula

$$\{x \mid (\exists y)\, [y < x \land (\forall z)\, [y \leq z < x \to y = z]\,]\}.$$

Removing the maximal element from D reduces the ordinal represented by D by 1 while the corresponding n_m is increased by 1. Replacing D by the set of its limit ordinals is like dividing the ordinal represented by D by ω; the set of limit ordinals (including 0) strictly below $\omega^m a_m + \ldots + \omega^1 a_1$ has order type $\omega^{m-1} a_m + \ldots + \omega^1 a_2 + a_1$. So the next coefficient can start to be computed. Based on this it is easy to verify that the algorithm computes the coefficients $n_0, n_1, \ldots$ in this order. The algorithm eventually terminates since m is finite. $\lhd$

The following is immediate.

Corollary E.3.3 *The isomorphism problem for automatic ordinals is decidable.*

Compare this with the fact that the isomorphism problem for permutation structures is not decidable, Theorem D.2.11. It is not known whether the isomorphism problem for automatic linear orders is decidable.

E.4 Automatic tree preliminaries

The remainder of this chapter deals with trees viewed as partial orders. Theorems E.5.6 and E.5.9 give a necessary condition for certain trees to be automatic. The condition is similar to that for linear orders and says that the Cantor-Bendixson rank (Definition E.5.1) of the tree be finite.

A *tree* $\mathcal{T} = (T, \preceq)$ is a partial order that has a least element r, called the root, and in which $\{y \in T \mid y \preceq x\}$ is a finite linear order for each $x \in T$. So we think of trees as growing upwards. Write $x \| y$ if $x \not\preceq y$ and $y \not\preceq x$. A partial order $(T, \preceq)$ is a *forest* if there is a partition of the domain $T = \cup T_i$ such that every $(T_i, \preceq)$ is a tree. The subtree rooted at x, written $\mathcal{T}(x)$, has domain $T(x) = \{y \in T \mid x \preceq y\}$ with order $\prec$ restricted to this domain. The set $S(x)$ of immediate successors of x is defined as

$$S(x) = \{y \in \mathcal{T} \mid x \prec y \wedge (\forall z)[x \preceq z \preceq y \rightarrow (z = x \vee z = y)]\}.$$

A tree $\mathcal{T}$ is *finitely branching* if $S(x)$ is finite for each $x \in T$. A *path* of a tree $(T, \preceq)$ is a subset $P \subset T$ which is linearly ordered (by $\preceq$) and maximal (under set-theoretic inclusion) with this property. A path with finitely many elements is called a *finite path*; otherwise it is called an *infinite path*.

Recall that $\leq_{llex}$, the length lexicographic order on $\Sigma^\star$, has order type ω. Thus if $\mathcal{T}$ is an automatic tree with $T \subset \Sigma^\star$ then the length-lexicographic order on $\Sigma^\star$ is inherited by each set $S(x)$. This permits one to talk about the first, second, third, ... successor of x.

The *Kleene-Brouwer* ordering of a tree $(T, \preceq)$ is written $<_{kb}$ and defined as follows. Let x, y be nodes on T. Then $x <_{kb} y$ if and only if either $y \prec x$ or there are u, v, w such that $v, w \in S(u)$, $v \preceq x$, $w \preceq y$ and $v <_{llex} w$. In words, $x <_{kb} y$ if and only if either x is above y in the tree or x is to the left of y (with respect to $<_{llex}$ restricted to immediate successors). Note that $\leq_{kb}$ linearly orders T and $(T, \leq_{kb})$ is first order definable from $(T, \preceq, \leq_{llex})$. Write $\mathcal{KB}_T$ for the structure $(T, \leq_{kb})$. We remark that a tree $\mathcal{T}$ has no infinite path if and only if $(T, <_{kb})$ is well-ordered.

Finally recall that all trees $\mathcal{T}$ are assumed to be countable.

E.5 Ranks of automatic trees

Our approach to proving facts about trees is to associate a linear order with a tree, in such a way that the tree is automatic if and only if the linear order is automatic. Then by Theorem E.2.7 the linear order has finite rank which it turns out implies that the rank of the tree is finite. More precisely, in this section it is shown that every automatic tree has finite Cantor-Bendixson rank.

Given a tree $\mathcal{T}$, define a subset of T as consisting of those nodes $x \in T$ with the property that there exist at least two distinct infinite paths in the subtree of $\mathcal{T}$ rooted at x. It follows from downward closure that this sub-partial order, $d(\mathcal{T})$, is a subtree of $\mathcal{T}$ with the same root.

For each ordinal α define the iterated operation $d^\alpha(\mathcal{T})$ inductively as follows.

1. $d^0(\mathcal{T}) = \mathcal{T}$.
2. $d^{\alpha+1}(\mathcal{T})$ is $d(d^\alpha(\mathcal{T}))$.
3. If α is a limit ordinal, then $d^\alpha(\mathcal{T})$ is $\cap_{\beta<\alpha} d^\beta(\mathcal{T})$.

Definition E.5.1 *The* **Cantor-Bendixson rank** *of a tree $\mathcal{T}$, written $\mathrm{CB}(\mathcal{T})$, is the least ordinal α such that $d^\alpha(\mathcal{T}) = d^{\alpha+1}(\mathcal{T})$.*

Remark E.5.2 The Cantor-Bendixson rank of an arbitrary topological space X is defined as above, using D given as $D\,X = \{P \in X \mid p \text{ is not isolated}\}$ instead of d. Recall that P is isolated if $\{P\}$ is an open set. So given a tree $\mathcal{T} = (T, \preceq)$, consider the following topological space. The set of elements are the infinite paths in $\mathcal{T}$, written $[\mathcal{T}]$. For $P \in [\mathcal{T}]$ and $x \in T$ write $x \prec P$ if $x \in P$ and say that x is on P. The basic open sets are of the form $\{P \in [\mathcal{T}] \mid x \prec P\}$ for every $x \in T$. Then the Cantor-Bendixson rank of this topological space, $\mathrm{CB}[\mathcal{T}]$, is just the least ordinal α such that $D^{\alpha+1}[\mathcal{T}] = D^\alpha[\mathcal{T}]$. Given an infinite path P of $\mathcal{T}$, the following statements are equivalent:

- There is a node $x \prec P$ such that P is the only infinite path of $\mathcal{T}$ going through x;
- $P \notin D(\mathcal{T})$;
- There is a node $x \prec P$ with $x \notin d(\mathcal{T})$.

It follows that $D[\mathcal{T}]$ consists of exactly the infinite paths of $d(\mathcal{T})$. It can be proven by transfinite induction that also

$$D^\alpha[\mathcal{T}] = [d^\alpha(\mathcal{T})].$$

Assume now that $\alpha = \mathrm{CB}[\mathcal{T}]$. Then $d^\alpha(\mathcal{T})$ and $d^\beta(\mathcal{T})$ contain the same infinite paths for all $\beta > \alpha$, but $d^\alpha(\mathcal{T})$ might contain some nodes which are not on any infinite paths and therefore not contained in $d^{\alpha+1}(\mathcal{T})$. Thus the two CB-ranks might differ, but they differ at most by 1:

$$\mathrm{CB}[\mathcal{T}] \leq \mathrm{CB}(\mathcal{T}) \leq \mathrm{CB}[\mathcal{T}] + 1.$$

A witness $\mathcal{T}$ with $\mathrm{CB}[\mathcal{T}] \neq \mathrm{CB}(\mathcal{T})$ is the tree where the domain consists of the root 0 and, for every $n > 0$, the strings $01^{a_1}01^{a_2}0\ldots1^{a_n}0$ with $a_1 \geq a_2 \geq \ldots \geq a_n$; the ordering is the prefix-relation $\preceq$ restricted to this domain. One has for every node $01^{a_1}01^{a_2}0\ldots1^{a_n}0 \in T$ that $01^{a_1}01^{a_2}0\ldots1^{a_n}0 \in d^m(\mathcal{T}) \Leftrightarrow a_n \geq m$. So $d^\omega = \{0\}$. It follows that $\mathrm{CB}[\mathcal{T}] = \omega$ by $D^\omega(\mathcal{T}) = \emptyset$ while $\mathrm{CB}(\mathcal{T}) = \omega + 1$ by $d^{\omega+1}(\mathcal{T}) = \emptyset \neq d^\omega(\mathcal{T})$. This witness is also robust to small changes in the definition of d. If one, for example, takes $d(\mathcal{T})$ to contain exactly those nodes which are on infinitely many infinite paths of $\mathcal{T}$, then the resulting trees $d^\alpha(\mathcal{T})$ and derived CB-ranks are the same.

Here are some basic properties of CB–rank.

Property E.5.3 *If $\mathcal{T}$ is a countable tree with $\mathrm{CB}(\mathcal{T}) = \alpha$ then*

1. *α is a countable ordinal.*

2. *If $d^\alpha(\mathcal{T}) \neq \emptyset$ then $d^\alpha(\mathcal{T})$ and $\mathcal{T}$ contain uncountably many infinite paths.*

3. *If $d^\alpha(\mathcal{T}) = \emptyset$ then $\mathcal{T}$ contains only countably many infinite paths. Furthermore, α is either 0 or a successor ordinal.*

Proof For each β let $x_\beta \in d^\beta(\mathcal{T}) \setminus d^{\beta+1}(\mathcal{T})$. Since T is countable, and $\alpha \neq \beta$ implies that $x_\alpha \neq x_\beta$, the set of ordinals β such that $d^\beta(\mathcal{T}) \setminus d^{\beta+1}(\mathcal{T}) \neq \emptyset$ is also countable. Hence its least upper bound, a countable ordinal, say α, is $\mathrm{CB}(\mathcal{T})$. This proves (1).

If $d^\alpha(\mathcal{T})$ is not the empty tree, then for every $x \in d^\alpha(\mathcal{T})$ there exist $y, z \in d^\alpha(\mathcal{T})$ with $x \prec y, z$ and $y \| z$. In particular the full binary tree $(\{0,1\}^\star, \preceq_p)$ embeds in $d^\alpha(\mathcal{T})$. Since $d^\alpha(\mathcal{T})$ is a subset of $\mathcal{T}$, the full binary tree also embeds in $\mathcal{T}$. This proves (2).

If $d^\alpha(\mathcal{T})$ is the empty tree, then one shows that $\mathcal{T}$ has only countably many infinite paths as follows. For every infinite path P of $\mathcal{T}$ there is a minimum ordinal $\beta_p \leq \alpha$ such that $P \not\subset d^{\beta_P}(\mathcal{T})$. Furthermore, there is a node x_P in P such that $x_P \notin d^{\beta_P}(\mathcal{T})$. Since $x_P \in d^\gamma(\mathcal{T})$ for all $\gamma < \beta_P$, it follows that β_P is a successor ordinal $\delta+1$. Furthermore, P is the only infinite path of $d^\delta(\mathcal{T})$ which contains x_P. Thus the mapping $P \to (x_P, \beta_P)$ of the infinite paths of $\mathcal{T}$ to pairs of nodes and successor ordinals up to α is one-one. Since the range of this mapping is countable, so is its domain. Now for every $\gamma < \alpha$ the root of $\mathcal{T}$ is in $d^\gamma(\mathcal{T})$. So if $\alpha > 0$ then it is a successor ordinal. This proves (3). $\lhd$

As a matter of convenience we introduce a variation of CB–rank.

Definition E.5.4 *Suppose that $\mathcal{T}$ has countably many infinite paths. Define the* **CB_*–rank** *of $\mathcal{T}$, written $\mathrm{CB}_*(\mathcal{T})$, as the least ordinal α so that $d^\alpha(\mathcal{T})$ has finitely many nodes.*

This is well defined since $d^\alpha(\mathcal{T}) = \emptyset$ for some α. Note that CB_*–rank is non-increasing in the sense that if $x \preceq y$ then $\mathrm{CB}_*(\mathcal{T}(y)) \leq \mathrm{CB}_*(\mathcal{T}(x))$. Also since finite trees have no infinite paths, $\mathrm{CB}_*(\mathcal{T}) \leq \mathrm{CB}(\mathcal{T}) \leq \mathrm{CB}_*(\mathcal{T}) + 1$.

Lemma E.5.5 *Suppose $\mathcal{T}$ has countably many infinite paths and that $\mathcal{T}$ is finitely branching.*

1. *$\mathrm{CB}_*(\mathcal{T})$ is 0 or a successor ordinal.*

2. *If $\mathrm{CB}_*(\mathcal{T}) \geq \beta + 1$ then there is some $x \in T$ with $\mathrm{CB}_*(\mathcal{T}(x)) = \beta + 1$.*

Proof Say $\mathrm{CB}_*(\mathcal{T}) = \alpha$ and $\alpha > 0$ is a limit ordinal. Then $d^\alpha(\mathcal{T})$ contains an infinite path as follows. The root of $\mathcal{T}$, call it x_0, is in $d^\alpha(\mathcal{T})$ for otherwise $d^\gamma(\mathcal{T})$ is empty for some $\gamma < \alpha$. Since x_0 has finitely many immediate successors in $\mathcal{T}$, there must be one, call it x_1, with the property that $x_1 \in d^\alpha(\mathcal{T})$ for otherwise the maximum of the CB_*–ranks of the immediate successors of x_0 is $< \alpha$ and so $\mathrm{CB}_*(\mathcal{T}) < \alpha$. Proceed in this way to build an infinite path $x_0, x_1, x_2, \cdots$ of $d^\alpha(\mathcal{T})$. In particular then the CB_*–rank of $\mathcal{T}$ is not α. This proves (1).

Let $\mathrm{CB}_*(\mathcal{T}) \geq \beta + 1$. Then $d^\beta(\mathcal{T})$ is an infinite finitely branching tree and so contains some infinite path P. Moreover there must be some infinite path of $P \subset d^\beta(\mathcal{T})$ so that $P \not\subset d^{\beta+1}(\mathcal{T})$; for otherwise we could embed a copy of the infinite binary tree in $d^\beta(\mathcal{T})$ and so conclude that $\mathcal{T}$ has uncountably many infinite paths. Hence pick $x \in P$ with $x \notin d^{\beta+1}(\mathcal{T})$. Then $\mathrm{CB}_*(\mathcal{T}(x)) = \beta + 1$. This proves (2). $\lhd$

For the first result one associates the Kleene-Brouwer ordering $\mathcal{KB}_T = (T, <_{kb})$ with a tree $\mathcal{T}$ (see section E.4). In words, $x \leq_{kb} y$ if and only if either x is above y in the tree or x is to the left of y (with respect to $<_{llex}$ restricted to immediate successors). For example if $y_1 <_{llex} y_2 <_{llex} \cdots <_{llex} y_l$ are the immediate successors of the root r of $\mathcal{T}$ then $\mathcal{KB}_T = \mathcal{KB}_{T(y_1)} + \cdots + \mathcal{KB}_{T(y_l)} + \mathbf{1}$. Recall that $\mathcal{T}(x)$ denotes the subtree of $\mathcal{T}$ with root x and that its domain is written $T(x)$.

Theorem E.5.6 *The CB–rank of an automatic finitely branching tree with countably many infinite paths is finite.*

Proof Suppose $\mathcal{T}$ is finitely branching with countably many infinite paths. We now prove $(\dagger)$ that $\mathcal{KB}_T$ is scattered and $\mathrm{CB}_*(\mathcal{T}) = \mathrm{VD}_*(\mathcal{KB}_T)$. Consequently if $\mathcal{T}$ is automatic then so is $\mathcal{KB}_T$, which by Theorem E.2.7 has finite VD–rank and hence finite VD_*–rank. Then the CB_*–rank of $\mathcal{T}$ must be finite as required.

To prove $(\dagger)$ proceed by induction on $\mathrm{CB}_*(\mathcal{T})$. If $\mathcal{T}$ has CB_*–rank 0 then $\mathcal{KB}_T$ is finite and so has VD_*–rank 0. Before proceeding to the general case, we make some observations. Suppose $\mathcal{T}$ contains an infinite path $x_1, x_2, x_3, \cdots$. For a given i list $S(x_i)$ as follows: $y_1 <_{llex} \cdots <_{llex} y_k <_{llex} x_{i+1} <_{llex} z_1 <_{llex} \cdots <_{llex} z_l$. Define the set $L_i \subset T$ as $\cup T(y_j)$ and define $R_i \subset T$ as $\cup T(z_j)$. So $\mathcal{L}_i$ and $\mathcal{R}_i$ are forests of disjoint subtrees of $\mathcal{T}$. Abuse notation and define $\mathcal{KB}_{L_i}$ as the linear order $\mathcal{KB}_{T(y_1)} + \mathcal{KB}_{T(y_2)} + \cdots + \mathcal{KB}_{T(y_k)}$. Similarly define $\mathcal{KB}_{R_i}$ as the linear order $\mathcal{KB}_{T(z_1)} + \mathcal{KB}_{T(z_2)} + \cdots + \mathcal{KB}_{T(z_l)}$. Then by definition of $<_{kb}$,

$$\mathcal{KB}_T = (\mathcal{KB}_{L_1} + \mathcal{KB}_{L_2} + \mathcal{KB}_{L_3} + \cdots) + (\cdots + \mathcal{KB}_{R_3} + \mathbf{1}_3 + \mathcal{KB}_{R_2} + \mathbf{1}_2 + \mathcal{KB}_{R_1} + \mathbf{1}_1), \quad \text{(E.1)}$$

where $\mathbf{1}_i$ has order type $\mathbf{1}$ and represents the element x_i. In particular suppose $\mathcal{T}$ contains exactly one infinite path. Then every $\mathcal{KB}_{L_i}$ and $\mathcal{KB}_{R_i}$ is a finite linear order. So depending on whether there are infinitely many i such that $\mathcal{KB}_{L_i}$ (or $\mathcal{KB}_{R_i}$) is the empty linear order, $\mathcal{KB}_T$ has one of the following scattered order types: $\omega^\star$, $\mathbf{n} + \omega^\star$ for some $n \in \mathbb{N}$, or $\omega + \omega^\star$. Note that these orders have VD_*–rank 1.

For the general case, suppose the CB$_*$–rank of $\mathcal{T}$ is not 0. Then by Lemma E.5.5 (1) it is $\beta+1$ for some ordinal β. Let $X = \{x \in T \mid \mathrm{CB}_*(\mathcal{T}(x)) = \beta+1\}$. Then X is a downward closed subset of $\mathcal{T}$, and so $\mathcal{X}$ is a tree.

The tree $\mathcal{X}$ has infinitely many nodes. Indeed for every $x \in X$, the finitely branching tree $d^{\beta}(\mathcal{T}(x))$ is infinite and so contains an infinite path (w_i). For every i the tree $\mathcal{T}(w_i)$ has CB$_*$–rank $\beta+1$ and so w_i is in X. So $\mathcal{X}$, also being finitely branching, has at least one infinite path. Now if $\mathcal{X}$ has infinitely many infinite paths then since $\mathcal{X}$ is finitely branching we can construct an infinite path (z_i) of $\mathcal{X}$ such that for every i there are infinitely many infinite paths in $\mathcal{X}(z_i)$ (the subtree of $\mathcal{X}$ with root z_i). For infinitely many i there is a $y \in S(z_i) \setminus \{z_{i+1}\}$ with $y \in X$. So $\mathcal{T}$ contains the infinite path (z_i) with $\mathrm{CB}_*(\mathcal{T}(z_i)) = \beta+1$ and for infinitely many i there is $y \in T$ that is in $S(z_i) \setminus \{z_{i+1}\}$ with $\mathrm{CB}_*(\mathcal{T}(y)) = \beta+1$. So $d^{\beta+1}(\mathcal{T})$ contains the infinite path (z_i) contradicting that $\mathrm{CB}_*(\mathcal{T}) = \beta+1$. We conclude that $\mathcal{X}$ contains a non-zero finite number of infinite paths.

Let (x_i) be some infinite path of $\mathcal{X}$ and define $L_i \subset T$ and $R_i \subset T$ as above. The forest $\mathcal{L}_i$ (or $\mathcal{R}_i$) consists of finitely many disjoint subtrees of $\mathcal{T}$; list these as $\mathcal{T}(w_1), \cdots, \mathcal{T}(w_k)$, where $w_j \in S(x_i)$. Then $\mathrm{CB}_*(\mathcal{T}(w_j)) \leq \beta+1$. Moreover since $\mathcal{X}$ has only finitely many infinite paths, there exists $c \in \mathbb{N}$ such that for every $i \geq c$, every tree $\mathcal{T}(w_j)$ of $\mathcal{L}_i$ and $\mathcal{R}_i$ has CB$_*$–rank $\leq \beta$. By induction $\mathcal{KB}_{T(w_j)}$ is scattered and $\mathrm{CB}_*(\mathcal{T}(w_j)) = \mathrm{VD}_*(\mathcal{KB}_{T(w_j)})$. So for every $i \geq c$ the linear order $\mathcal{L}_i$ (and $\mathcal{R}_i$) being a finite sum of such $\mathcal{T}(w_j)$ is scattered and has VD$_*$–rank equal to the supremum of $\mathrm{VD}_*(\mathcal{T}(w_1)), \cdots, \mathrm{VD}_*(\mathcal{T}(w_k))$ which is at most β. Moreover by Lemma E.5.5 (2) there are infinitely many $m \geq c$ for which there exists a tree $\mathcal{T}(w_j)$ in $\mathcal{L}_m$ (or $\mathcal{R}_m$) that has CB$_*$–rank β. We conclude that there are infinitely many $\mathcal{KB}_{L_m}$ (or infinitely many $\mathcal{KB}_{R_m}$) with VD$_*$–rank exactly β. Hence using Equation E.1 and Lemma E.2.3 the linear order $\mathcal{KB}_{T(x_c)}$ has VD$_*$–rank $\beta+1$.

Pick $n < c$ so that $\mathcal{L}_n$ (or $\mathcal{R}_n$) contains a tree $\mathcal{T}(w_j)$ of CB$_*$–rank $\beta+1$. Argue as before with $\mathcal{T}(w_j)$ in place of $\mathcal{T}$. To this end define X' as $\{x \in T(w_j) \mid \mathrm{CB}_*(\mathcal{T}(x)) = \beta+1\}$. Then as before $\mathcal{X}'$ is a subtree of $\mathcal{T}(w_j)$ with finitely many infinite paths. However since $\mathcal{X}'$ does not contain the previous infinite path (x_i), the tree $\mathcal{X}'$ has fewer infinite paths than $\mathcal{X}$. This guarantees that after a finite number of iterations $c = 0$.

Since $\mathcal{X}$ has a finite (non-zero) number of infinite paths, we can write $\mathcal{KB}_T$ as a finite (non-zero) sum of linear orders of VD$_*$–rank $\beta+1$. This completes the induction. $\triangleleft$

Theorem E.5.7 *The CB–rank of every finitely branching automatic tree is finite.*

Proof Let $\mathcal{T}$ be a finitely branching automatic tree and $\mathcal{KB}_T$ the Kleene-Brouwer ordering of $\mathcal{T}$. Call a node $a \in T$ *scattered* if $\mathcal{T}(a)$ contains countably many infinite paths. By the proof Theorem E.5.6, $\mathcal{KB}_{T(a)}$ is a scattered linear ordering and $\mathrm{CB}_*(\mathcal{T}(a)) = \mathrm{VD}_*(\mathcal{KB}_{T(a)})$. But applying the proof of Theorem E.2.7 to the automatic linear order $\mathcal{KB}_T$, there exists $e \in \mathbb{N}$ such that $\mathrm{VD}([x,y]) \leq e$ for every scattered closed interval $[x,y]$ of $\mathcal{KB}_T$. In particular for every scattered $a \in T$, $\mathrm{VD}_*(\mathcal{KB}_{T(a)}) \leq \mathrm{VD}(\mathcal{KB}_{T(a)}) \leq e$. So $\mathrm{CB}(\mathcal{T}(a)) \leq e$ and $d^e(\mathcal{T})$ contains no scattered nodes.

We claim that $d^e(\mathcal{T}) = d^{e+1}(\mathcal{T})$. It is always the case that $d^{e+1}(\mathcal{T}) \subset d^e(\mathcal{T})$. If $\mathcal{T}$ has countably many infinite paths then $d^e(\mathcal{T})$ is empty. Otherwise suppose $x \in d^e(\mathcal{T})$. Then x is not a scattered node since all the scattered nodes have been removed, and so $\mathcal{T}(x)$ contains uncountably many infinite paths. In particular $x \in d^{e+1}(\mathcal{T})$. So $d^e(\mathcal{T}) \subset d^{e+1}(\mathcal{T})$, as required. $\lhd$

Next we remove the condition that the tree be finitely branching.

Definition E.5.8 *Given a tree $(T, \preceq)$, define a partial order $x \preceq' y$ on T by*

$$x \preceq y \vee (\exists v, w \in T)\,[x, w \in S(v) \wedge x \leq_{llex} w \wedge w \preceq y];$$

where $\leq_{llex}$ is the length lexicographic order and $S(v)$ the set of immediate successors of v with respect to $\preceq$.

Recall the set $S(x)$ is the set of $\prec$-immediate successors of $x \in T$. Then since $\leq_{llex}$ restricted to $S(x)$ has order type ω if $S(x)$ is infinite, $(T, \preceq')$ is indeed a tree which we denote by $\mathcal{T}'$. Let $s(x)$ be the length-lexicographically least element of $S(x)$ for the case $S(x) \neq \emptyset$ and let $s(x) = u$ for a default value $u \notin T$ if $S(x) = \emptyset$.

Note that $\preceq'$ extends $\preceq$. For $x \in T$ let $S'(x)$ be the set of successors with respect to $\preceq'$. Then $S'(x)$ contains $s(x)$ whenever $s(x) \neq u$ and the length-lexicographically next sibling y of x with respect to $\preceq$ whenever this y exists. Recall that y is a sibling of x with respect to $\preceq$ if there is a node z with $x, y \in S(z)$. Hence $\mathcal{T}' = (T, \preceq')$ is a finitely branching tree that is automatic if $\mathcal{T}$ is automatic.

Theorem E.5.9 *The CB–rank of an automatic tree $\mathcal{T} = (T, \preceq)$ is finite.*

Proof Let U and U' be the sets of infinite paths of $\mathcal{T} = (T, \preceq)$ and $\mathcal{T}' = (T, \preceq')$, respectively. Since every infinite path of $\mathcal{T}$ generates an infinite path of $\mathcal{T}'$, there is a one-one continuous mapping q from U to U'. This mapping satisfies for all $P \in U$ and all $x \in T$: $x \in P$ iff $s(x) \in q(P)$. Furthermore, U' contains besides the paths of the form $q(P)$ for some $P \in U$ also the paths generated by those sets $S(x)$ where $S(x)$ is infinite. Since there are countably many of these additional paths one has the following equivalence for all x: $\{P \in U : x \in P\}$ is uncountable iff $\{P' \in U' : s(x) \in P'\}$ is uncountable.

Now one shows by induction over n that the following implication holds for all $x \in T$ with $s(x) \neq u$ and $n \in \mathbb{N}$: $x \in d^n(\mathcal{T}) \Rightarrow s(x) \in d^n(\mathcal{T}')$. The property clearly holds for $n = 0$. Now assume the inductive hypothesis for n and consider any $x \in d^{n+1}(\mathcal{T})$. There are two distinct infinite paths $P, Q \in U$ such that $x \in P \cap Q$ and $P \cup Q \subset d^n(\mathcal{T})$. It follows that $s(x) \in q(P) \cap q(Q)$. By induction hypothesis and by q being one-one, $s(x)$ is a member of the two distinct infinite paths $q(P), q(Q)$ of $d^n(\mathcal{T}')$ and thus $s(x) \in d^{n+1}(\mathcal{T}')$. This completes the proof of this property.

By Theorem E.5.7, there is a natural number n such that $d^n(\mathcal{T}')$ contains exactly those nodes of the form $s(x)$ which are in uncountably many members of U'. Then all $x \in d^n(\mathcal{T})$ satisfy that x is in uncountably many members of U. On the other hand, every x being in uncountably many members of U is in $d^n(\mathcal{T})$. So $d^n(\mathcal{T})$ contains exactly the nodes x which are in uncountably many members of U and $d^{n+1}(\mathcal{T}) = d^n(\mathcal{T})$. The CB-rank of $\mathcal{T}$ is at most n. $\lhd$

E.6 Automatic versions of König's Lemma

König's Lemma says that every infinite finitely branching tree has at least one infinite path. This section consists of automatic versions of this and similar results, all of which will be referred to as automatic versions of König's Lemma. We briefly summarise the computable analogues. Here a tree is a computable set $T \subset \mathbb{N}^\star$ that is prefix-closed, partially ordered by prefix. Here are a sample of results that show the failure of computable versions of König's Lemma. There is an infinite computable binary branching tree without computable infinite paths, though this tree has uncountable many infinite paths, see [Odifreddi - 1989, Section 5]. There exists a computable finitely branching tree with exactly one infinite path, and that path is not computable. Finally there exists a computable infinitely branching tree with infinite paths, none of which is hyperarithmetical, see Rogers [1967]. Contrast these with the fact that König's Lemma and its variants have automatic analogues in the strongest possible sense. For instance, every automatic tree $(T, \preceq)$, not necessarily finitely branching, either has a regular infinite path or does not have an infinite path at all. Also if an automatic tree has countably many infinite paths then every infinite path is regular.

Proposition E.6.1 *It is decidable whether an automatic tree has an infinite path.*

Proof Let $(T, \preceq)$ be an automatic tree and recall that $(T, <_{kb})$ is an automatic linear order. By Proposition E.3.1 it is decidable whether this order is isomorphic to an ordinal. And this is the case if and only if $(T, \preceq)$ has no infinite path. To prove this last statement recall that a linear order is isomorphic to an ordinal if and only if it has no infinite decreasing chain. So suppose $(T, \preceq)$ has an infinite path $x_1 \prec x_2 \prec x_3 \ldots$. Then $x_1 >_{kb} x_2 >_{kb} x_3 \ldots$ is an infinite decreasing chain in $(T, \leq_{kb})$, and so $(T, \leq_{kb})$ is not isomorphic to an ordinal. Conversely, suppose $(T, <_{kb})$ is not isomorphic to an ordinal and let $x_1 >_{kb} x_2 >_{kb} x_3 \cdots$ be an infinite decreasing chain. We define an infinite path (p_i) of $(T, \prec)$ as follows.

1. Let $i = 1$ and $j = 1$.
2. **Repeat**
   (a) Define $p_i = x_j$.
   (b) Replace j with the smallest $k > j$ for which there is a $u \in S(p_i)$ with $u \preceq x_l$ for every $l \geq k$.
   (c) Replace i with $i + 1$.
3. **End Repeat**

If such a k exists in step $2(b)$ of every stage of the repeat loop, then the resulting sequence (p_i) is an infinite path in $(T, \preceq)$. So suppose that the algorithm has computed $p_1, p_2, \cdots, p_n$ with $p_1 \prec p_2 \prec \cdots \prec p_n$. So $i = n$ and $j \in \mathbb{N}$. For every $m > j$ define $u(x_m)$ as the immediate successor of p_i that is $\preceq x_m$. Then this sequence satisfies $u(x_m) \geq_{llex} u(x_{m+1}) \geq_{llex} u(x_{m+2}) \geq_{llex} \cdots$ since $x_m >_{kb} x_{m+1} >_{kb} x_{m+2} >_{kb} \cdots$. But since $\leq_{llex}$ is isomorphic to an ordinal (of type ω) it can not have an infinite decreasing sequence. Thus the sequence is eventually constant; that is, there is a (smallest) $k > j$ such that for every $l \geq k$ one has $u(x_k) = u(x_l) \preceq x_l$ as required. $\lhd$

Finitely branching automatic trees

An infinite tree is *pruned* if every element is on some infinite path. Note that for a tree $\mathcal{T}$ the set of elements $E(\mathcal{T})$ above which there are infinitely many elements is definable as $\{x \in T \mid (\exists^\infty y)\, x \preceq y\}$. So if $\mathcal{T}$ is finitely branching then $E(\mathcal{T})$ consists of those nodes of $\mathcal{T}$ that are on some infinite path. Indeed, if $x \in E(\mathcal{T})$ then by König's Lemma it is on an infinite path. Conversely if $x \notin E(\mathcal{T})$ then there are only finitely many elements above it (in $\mathcal{T}$) and so it is not on an infinite path. Hence the subtree $(E(\mathcal{T}), \preceq)$ is pruned and contains every infinite path of $\mathcal{T}$. Further if $\mathcal{T}$ is automatic then so is $E(\mathcal{T})$.

Theorem E.6.2 (Automatic König's Lemma 1) *If $\mathcal{T} = (T, \preceq)$ is an infinite finitely branching automatic tree then it has a regular infinite path. That is, there exists a regular set $P \subset T$ so that P is an infinite path of $\mathcal{T}$.*

Proof By the previous remark replace $\mathcal{T}$ with the pruned automatic tree $(E(\mathcal{T}), \preceq)$, and call the resulting tree $\mathcal{T}$. Recall that the length-lexicographic order $<_{llex}$ on Σ^* is automatic and therefore one can extend the presentation of $\mathcal{T}$ to include $<_{llex}$, namely $(T, \preceq, <_{llex})$ is an automatic structure. Now define the leftmost infinite path P with respect to the length-lexicographic order of the successors of any node. P contains those nodes x for which every $y \prec x$ satisfies that $\forall z, z' \in S(y)\, [z \preceq x \Rightarrow z <_{llex} z']$, and so by Proposition B.1.22 P is regular. This means, that the unique node $z \in S(y)$ which is below x is just the length-lexicographically least element of $S(y)$. Since the length-lexicographic ordering of Σ^* is a well-ordering (of type ω), this minimum always exists.

We briefly check that P is an infinite path. Firstly P is closed downward. Indeed, given $x \in P$, let $a \preceq x$. Then for every $y \prec a$, if $z, z' \in S(y)$ and $z \preceq y \preceq x$ so by hypothesis then $z \leq_{llex} z'$, as required. Secondly P is linearly ordered. For otherwise if $x, a \in P$ with $x \| a$, then let z be their $\prec$-maximal common ancestor. Consider two successors of z say v and w with $v \prec x$ and $w \prec a$. Without loss of generality suppose that $v <_{llex} w$. Then z, v and w form a counterexample to a's membership in P. Finally P is infinite (and hence maximal with these properties). Indeed if $x \in P$, then the $<_{llex}$-smallest element in $S(x)$ is also in P. Hence P is an infinite regular path in $\mathcal{T}$, as required. $\lhd$

If in the hypothesis above $\mathcal{T}$ contains finitely many infinite paths, then every infinite path is regular since after defining P, one considers the tree on domain $T \setminus P$ to find the next infinite path. The next theorem generalises this to the case when $\mathcal{T}$ contains countably many infinite paths.

Theorem E.6.3 (Automatic König's Lemma 2) *If $\mathcal{T} = (T, \preceq)$ is an automatic tree that is finitely branching and has countably many infinite paths, then every infinite path in it is regular.*

Proof As before replace $\mathcal{T}$ with the automatic pruned tree $(E(\mathcal{T}), \preceq)$. Then the derivative $d(\mathcal{T})$ is definable and so the elements of the tree $\mathcal{T} \setminus d(\mathcal{T})$ form a regular subset of T, call it R. Then R consists of countably many disjoint infinite paths, each definable as follows. For every $\prec$-minimal $a \in R$ define the infinite path P_a as $\{x \in T \mid x \preceq a \vee (a \prec x \wedge x \in R)\}$.

Now replace $\mathcal{T}$ by $d(\mathcal{T})$ and repeat the steps in the previous paragraph. Since $\mathrm{CB}(\mathcal{T})$ is finite, these steps can be iterated at most $\mathrm{CB}(\mathcal{T})$ times; after which time the resulting tree will be empty and every infinite path in the original $\mathcal{T}$ will have been generated at some stage. $\lhd$

The assumption that $\mathcal{T}$ has countably many infinite paths can not be dropped, since otherwise $\mathcal{T}$ necessarily has non-regular (indeed, uncountably many non-computable) infinite paths.

The general case

It turns out that automaticity allows one to remove the condition that $\mathcal{T}$ be finitely branching, under the assumption of course that $\mathcal{T}$ has at least one infinite path. This can be done if given an automatic tree $\mathcal{T}$, one can effectively construct an automatic copy of the pruned tree $E(\mathcal{T})$, the set of elements of $\mathcal{T}$ that are on an infinite path in $\mathcal{T}$. Then as in the finitely branching case, Theorem E.6.2, the $<_{llex}$-least path is definable and hence regular.

Theorem E.6.4 (Automatic König's Lemma 3) *If an automatic tree has an infinite path, then it has a regular infinite path.*

This follows immediately from the following construction.

Lemma E.6.5 *If $\mathcal{T}$ is an automatic tree then $E(\mathcal{T}) \subset T$ is a regular language.*

Proof Let $\mathcal{T} = (T, \preceq)$ be an automatic tree. Writing T' for $E(\mathcal{T})$, it is required that the set $T' \subset \Sigma^\star$ of all nodes in T that are on an infinite path is a regular language.

The idea is to construct a Büchi recognisable language[1] $\mathcal{B}$ over the alphabet $\Delta = \Sigma_\perp \times \Sigma$ so that its projection (onto the first co-ordinate) is of the form $T' \cdot \{\perp\} \cdot W^\omega$ for some regular $W \subset \Sigma_\perp^\star$. Then T' is regular since Büchi automata are closed under projection and an automaton for T' can be extracted from one for B.

Say that a word x *is on* $c_0 c_1 \ldots$, where each c_i is $(a_i, b_i) \in \Sigma_\perp \times \Sigma$, if and only if there exist $m, n \in \mathbb{N}$ such that

either $m = 0$, $x = a_0 a_1 \ldots a_n$ and $a_{n+1} = \perp$

[1] Recall that a (non-deterministic) Büchi automaton (S, ι, Δ, F) over Σ accepts an infinite string $\alpha \in \Sigma^\omega$ if it has a run $(q_i)_{i \in \mathbb{N}}$ such that there is some state $f \in F$ with $f = q_j$ for infinitely many $j \in \mathbb{N}$.

or $n \geq m > 0$, $x = b_0 b_1 \ldots b_{m-1} a_m a_{m+1} \ldots a_n$, $a_{m-1} = \perp$ and $a_{n+1} = \perp$.

In the first case we say that x is the *first word on* $c_0 c_1 \ldots$. Consider the set of all sequences $(a_0, b_0)(a_1, b_1) \ldots \in \Delta$ such that there are infinitely many words on the sequence and the words on the sequence generate an infinite path of T. More formally,

- $\exists^\infty n(a_n = \perp)$;
- if y, z are on $(a_0, b_0)\,(a_1, b_1)\,\ldots$ and $|y| \leq |z|$ then $y \preceq z$ and $y, z \in T$.

There is a Büchi automaton $\mathcal{B}$ accepting such sequences because the orderings $\preceq$ and length-comparison are automatic and T is regular. Further using that $\mathcal{T}$ is transitive, one need only check that adjacent words y, z on the sequence satisfy $y \preceq z$.

To complete the proof we prove that $x \in T'$ if and only if x is the first word on some sequence $c_0 c_1 \ldots$ satisfying the two conditions. The reverse implication is clear. For the forward implication let $x \in T$ be given and P be an infinite path witnessing that $x \in T'$. Define the sequences $a_0 a_1 \ldots$ and $b_0 b_1 \ldots$ described below.

1. **Choose** $n, a_0, a_1, \ldots, a_n$ such that $x = a_0 a_1 \ldots a_n$. **Let** $a_{n+1} = \perp$.
2. **Let** $m = 0$. **Let** $y = x$.
3. **Find** $b_m b_{m+1} \ldots b_{n+1}$ such that infinitely many nodes in P extend $b_0 b_1 \ldots b_{m-1}$ as strings.
4. **Update** $m = n + 2$.
5. **Find** a new value for n and $a_m a_{m+1} \ldots a_n$ such that $n \geq m$, the path P contains the node $z = b_0 b_1 \ldots b_{m-1} a_m a_{m+1} \ldots a_n$ and $y \preceq z$. **Let** $a_{n+1} = \perp$.
6. **Let** $y = z$. **Go to** 3.

Note that it is an invariant of the construction that whenever the algorithm comes to Step 3, either $m = 0$ or infinitely many nodes in P extend the string $b_0 b_1 \ldots b_{m-1}$. As there are only finitely many choices for the new part $b_m b_{m+1} \ldots b_{n+1}$, one can choose this part such that still infinitely many nodes in P extend $b_0 b_1 \ldots b_{n+1}$ as a string. In Step 4, m is chosen such that the precondition of Step 3 holds again and b_m is the first of the b-symbols not yet defined. For every $y \in P$ it holds that all but finitely many nodes z in P satisfy $y \preceq z$. Furthermore, for every finite length l, almost all nodes in P are represented by strings longer than l. Thus one can find a node z as specified in Step 5 and the algorithm runs forever defining the infinite sequence $(a_0, b_0)(a_1, b_1) \ldots$ in the limit. In particular, such a sequence exists. It is not required that the sequence can be constructed effectively since the path P might not even be computable. $\lhd$

From Theorem E.6.4, we see that if an automatic tree has *finitely* many infinite paths, then each is regular. The next theorem generalises this to trees with *countably* many infinite paths.

Theorem E.6.6 (Automatic König's Lemma 4) *If an automatic tree has countably many infinite paths then every infinite path in it is regular.*

Proof Let $\mathcal{T} = (T, \preceq)$ be an automatic tree with countably many infinite paths. Then the extendible part of $\mathcal{T}$, $E(\mathcal{T}) \subset T$, is regular by Lemma E.6.5. So the derivative $d(\mathcal{T})$ is automatic. Write $E^i(\mathcal{T}) \subset T$ for the extendible part of the domain of $d^i(\mathcal{T})$. Then since $\mathcal{T}$ is automatic $\mathrm{CB}(\mathcal{T})$ is finite, say n. And since $\mathcal{T}$ has countably many infinite paths, $d^n(\mathcal{T})$ is the empty tree. So the structure $(T, E^0(\mathcal{T}), E^1(\mathcal{T}), \ldots, E^n(\mathcal{T}), \preceq)$ is automatic.

Now for every $x \in \mathcal{T}$ there exists an $m < n$ such that x is in the domain the tree $d^m(\mathcal{T})$ and not in the domain of the tree $d^{m+1}(\mathcal{T})$. In particular if P is an infinite path of $\mathcal{T}$ then there is a largest $m < n$ such that $P \subset E^m(\mathcal{T})$. The path P is isolated on $(E^m(\mathcal{T}), \preceq)$ since otherwise P would also be an infinite path of $d^{m+1}(\mathcal{T})$ and a subset of $E^{m+1}(\mathcal{T})$. Define $x_P \in T$ to be the least, with respect to $\preceq$, element of P which is not in $E^{m+1}(\mathcal{T})$. Then P is the only infinite path of $E^m(\mathcal{T})$ containing x_P. So P is the set of all $y \in E^m(\mathcal{T})$ which are comparable to x_P with respect to $\preceq$. Hence P is regular. $\lhd$

If $\mathcal{T}$ is an automatic tree with countably many infinite paths, then there is a formula specifying all these paths which is built from the parameters $n, E^0(\mathcal{T}), E^1(\mathcal{T}), \ldots, E^n(\mathcal{T})$ defined in the previous proof. The formula is the following one:

$$\begin{aligned}\Phi(a,b) = \quad \bigvee_{i=0}^{n-1} \quad & [a \in E^i(\mathcal{T}) \wedge a \notin E^{i+1}(\mathcal{T}) \wedge b \in E^i(\mathcal{T})] \wedge \\ & [(b \preceq a) \vee (a \prec b \wedge (\forall c, d, e \in E^i(\mathcal{T})) \\ & [a \preceq c \wedge d, e \in S(c) \wedge d \preceq b \Rightarrow d \leq_{llex} e])].\end{aligned}$$

The formula Φ and each set P_a defined as $\{b \in T \mid \Phi(a,b)\}$ satisfy the following conditions:

- If $a \in E^0$ then P_a is an infinite path of $\mathcal{T}$;
- If $a \notin E^0$ then P_a is empty;
- For every infinite path P of $\mathcal{T}$ there is an a with $P = P_a$.

Most of the content of this chapter appears in Khoussainov et al. [2003a].

Chapter F

Classifying automatic structures

This chapter addresses two foundational problems in the theory of automatic structures. The first problem is that of classifying classes of automatic structures, namely determining each element of a class, up to isomorphism, in terms of relatively simple invariants. To this end we classify the automatic boolean algebras, and develop techniques to prove that some Fraïssé limits, such as the infinite random graph and the universal partial order, do not have automatic presentations. Typically this involves bounding the growth of some aspect of a possible presentation – for instance the lengths of strings encoding the elements of the structure.

The second problem concerns the isomorphism problem, namely deciding whether two automatic structures are isomorphic or not. We prove that the complexity of the isomorphism problem for the class of all automatic structures is Σ_1^1-complete.

Although the results in the first section imply that certain classes of automatic structures are limited, the Σ_1^1-completeness implies that the class of all automatic structures is rich, and we should not expect to be able to classify all automatic structures.

F.1 Growth level of string lengths

This method bounds the lengths of strings in an automatic presentation of a structure. A relation $R \subset A^{k+l}$ is called *locally finite* if for every tuple $\overline{a}$ (of size k) there are at most a finite number of tuples $\overline{b}$ (of size l) such that $(\overline{a}, \overline{b}) \in R$. If $\overline{b} = (b_1, \cdots, b_l)$ then write $b \in \overline{b}$ if $b = b_i$ for some $1 \leq i \leq l$. Similarly write $\overline{b} \in B^l$ if $b_i \in B$ for every $1 \leq i \leq l$. Note if $f : A^k \to A$ is a function then its graph is a locally finite $k+1$-ary relation. A structure is called *locally finite* if each of its atomic relations R_i is locally finite for some choice of k_i and l_i, written $R_i \subset A^{k_i+l_i}$.

Definition F.1.1 *Let $\mathcal{A} = (A, R_1, \cdots, R_s)$ be an automatic locally finite structure. Let G be a finite subset of A.*

1. *Define* $E_0(G) = G$ *and for* $n \in \mathbb{N}$ *let*

$$E_{n+1}(G) = \bigcup_{1 \le i \le s} \{b \in A | \, (\exists \overline{b} \in A^{l_i}, \exists \overline{a} \in E_n^{k_i}(G)) \, [b \in \overline{b} \wedge (\overline{a}, \overline{b}) \in R_i]\}.$$

2. *Define* $L_0(G) = E_0(G)$ *and for* $n \in \mathbb{N}$ *let* $L_{n+1}(G) = L_n(G) \cup E_{n+1}(G)$.

So $L_n(G)$ is the set of elements that can be formed by at most n applications of the atomic relations starting from G. A structure $\mathcal{A}$ is called *finitely generated* if there exists a finite set of *generators* $G \subset A$ such that $A = \bigcup_{i \in \mathbb{N}} L_i(G)$. The function $n \mapsto |L_n(G)|$ is called the *growth* of the structure $\mathcal{A}$ based at G.

The result then says that the growth (based at G) of a finitely generated locally finite automatic structure is bounded above by a single exponential. The proof rests on the following observation, which appears in a simpler case in Elgot and Mezei [1965].

Proposition F.1.2 *Suppose that* $R \subset A^{k+l}$ *is a locally finite FA recognisable relation. There exists a constant* p*, that depends only on the automaton for* R*, such that*

$$\max\{|y| \mid y \in \overline{y}\} - \max\{|x| \mid x \in \overline{x}\} \le p$$

for every $(\overline{x}, \overline{y}) \in R$.

Proof Let q be the number of states of the automaton recognising $\otimes(R)$. We will see that the required constant p may be taken as $q \times l$. Fix $(\overline{x}, \overline{y}) \in R$ and say $x' \in \Sigma^\star$ has length $\max\{|x| \mid x \in \overline{x}\}$. There is some largest number $m \le l$ so that $a_1, a_2, \cdots, a_m$ are distinct indices with the property that y_{a_i} has length at least $|x'|$ for every $1 \le i \le m$. Without loss of generality we may suppose that $|y_{a_1}| \le |y_{a_2}| \le \cdots \le |y_{a_m}|$. In particular $|y_{a_m}| = \max\{|y| \mid y \in \overline{y}\}$. Define d_1 as $|y_{a_1}| - |x'|$, and d_i as $|y_{a_i}| - |y_{a_{i-1}}|$ for $1 < i \le m$.

Suppose by way of contradiction that $|y_{a_m}| - |x'| > q \times l$. Then there is some i so that $d_i > q$ since $|y_{a_m}| - |x'| = d_1 + \cdots + d_m$ and $m \le l$. Let n be $|x'|$ if $i = 1$ and $|y_{a_{i-1}}|$ if $i > 1$. Now we can pump the string $\otimes(\overline{x}, \overline{y})$ between positions n and $n + d_i$. Here the essential property is that if $|y_j| \ge n$ then $|y_j| \ge n + d_i$. So for these y_j, write $y_j = v_j w_j$ where $|v_j| = n$. Then by the Pumping Lemma partition w_j into $a_j b_j c_j$ and conclude that for every $t \in \mathbb{N}$ the string $(x_1, \cdots, x_k, y'_1 \cdots, y'_l) \in \otimes(R)$ where y'_j is y_j if $|y_j| < n$ and $a_j b_j^t c_j$ if $|y_j| \ge n$. This contradicts that R is locally finite. $\lhd$

We introduce some notation.

Definition F.1.3 *Suppose* $\mathcal{D}$ *is a structure over alphabet* Σ*. Write* $\Sigma^{\le n}$ *for the strings of* Σ *of length at most* n*. Let* $\Sigma^{O(n)}$ *denote* $\Sigma^{\le kn}$ *for some* k*. Write* D_n *for* $D \cap \Sigma^{\le n}$*; that is the elements of the domain* D *of length at most* n*.*

The following essentially appeared in Khoussainov and Nerode [1995] and later in Blumensath [1999].

Proposition F.1.4 *Let $\mathcal{D}$ be an automatic structure containing locally finite atomic relations $R_1, \cdots, R_s$ and generating set $G = \{g_1, \cdots, g_j\} \subset D$. Then there is a linear function $t : \mathbb{N} \to \mathbb{N}$ such that $L_n(G)$ is a subset of all those words in D of length not exceeding $t(n)$. Hence, if $|\Sigma| = 1$ then $|L_n(G)| = O(n)$, and otherwise $|L_n(G)| = |\Sigma|^{O(n)}$.*

Proof Suppose $R_i \subset D^{k_i+l_i}$. Let $k = \max\{k_i\}$ and $l = \max\{l_i\}$. Let $\mathcal{A}$ be an automaton over Σ that recognises those $(x_1, \cdots, x_k, y_1, \cdots, y_l)$ such that

$$\vee_i[R_i(x_1, \cdots, x_{k_i}, y_1, \cdots, y_{l_i}) \wedge_{k_i<a\leq k, l_i<b\leq l} (x_a = \lambda \wedge y_b = \lambda)].$$

Note that $x \in L_n(G)$ with respect to $R_1, \cdots, R_s$ if and only if $x \in L_n(G)$ with respect to this $k + l$-ary relation.

Define $g = \max_i |g_i|$ and let p equal the number of states in $\mathcal{A}$. We show by induction on n that $E_n(G) \subset D_{g+np}$. For the base step, $E_0(G) \subset D_g$ by the definitions. For the induction step, assume that $E_n(G) \subset D_{g+np}$. Then $y \in E_{n+1}(G)$ if and only if there exists $\overline{x}$ and $\overline{y}$ so that $R_i(\overline{x}, \overline{y})$ where $\overline{x} = (x_1, \cdots, x_{k_i})$ has $x_j \in E_n(G)$ for every j and $y \in \overline{y}$. By the proof of Proposition F.1.2, $|y| \leq \max\{|x_j| \mid x_j \in \overline{x}\} + p$. So $E_{n+1}(G) \subset D_{(g+np)+p}$ which completes the induction. That $L_n(G) \subset D_{g+np}$ follows from the definition of $L_n(G)$.

For $|\Sigma| = 1$, since $|D_m| = m + 1$, $|L_n(G)| \leq g + np + 1$ as required. Otherwise, $|D_m| = |\Sigma|^m$ and so $|L_n(G)| \leq |\Sigma|^{g+np}$ as required. $\triangleleft$

Example F.1.5 *The free group on $k > 1$ generators is not automatically presentable.*
Indeed, for $G = \{g_1, \cdots, g_k\}$, $L_n(G)$ contains all words of length, in the generators, at most 2^n. So $|L_n(G)|$ is at least $k^{(2^n)}$. $\triangleleft$

Using this method Blumensath [1999] proves that the following structures do not have automatic presentations:

1. $(\mathbb{N}, \times)$ where $\times$ is usual multiplication, and

2. $(\mathbb{N}, p)$ where p is a pairing function (namely $p : \mathbb{N}^2 \to \mathbb{N}$ is a bijection).

The next section contains new proofs of these results.

We now apply Proposition F.1.2 to classify particular classes of algebraic structures, namely integral domains and Boolean algebras.

Recall that an integral domain is a commutative ring with identity $(D, +, \mathbf{0}, \cdot, \mathbf{1})$ with the property that if $x \cdot y = \mathbf{0}$ then $x = \mathbf{0}$ or $y = \mathbf{0}$.

Theorem F.1.6 *There is no infinite automatic integral domain.*

Proof Suppose by way of contradiction that $(D, +, \mathbf{0}, \cdot, \mathbf{1})$ is an infinite automatic integral domain over alphabet Σ. Suppose that $1 \in \Sigma$ and identify $(1^\star, \cdot)$ with $(\mathbb{N}, +)$. Recall that $D_n = D \cap \Sigma^{\leq n}$. Define a relation $M(n, x)$, for $n \in 1^\star$ and $x \in D$, by

$$(\forall a, b, a', b' \in D_n)\,[a \cdot x + b = a' \cdot x + b' \rightarrow (a = a' \wedge b = b')].$$

We show that for every n there exists an $x \in \Sigma^\star$ such that $M(n, x)$ holds. Suppose for some n that there were no x with $M(n, x)$. Then for every $x \in D$ there exist $a, b, a', b' \in D_n$ such that (a, b) and (a', b') are distinct pairs with the property that that $a \cdot x + b = a' \cdot x + b'$. Since D_n is finite and D is infinite there exist such distinct pairs (a, b) and (a', b') with $a \cdot y + b = a' \cdot y + b'$ for infinitely many y's. So $(a - a') \cdot y = (b' - b)$ for these y's. Now $a \neq a'$ for otherwise also $b = b'$, contradicting the assumption that $(a, b) \neq (a', b')$. Also $(a - a') \cdot y = (a - a') \cdot y' = b' - b$ for some $y \neq y'$. Rewriting one gets that $(a - a') \cdot (y - y') = \mathbf{0}$ and since $a \neq a'$ the property of being an integral domain implies that $y = y'$, a contradiction. This completes the proof of the existence of such an x.

Now define a relation $F(n, x)$ by

$$M(n, x) \wedge (\forall y)\,[M(n, y) \rightarrow x <_{llex} y].$$

Then $F(n, x)$ if x is the length-lexicographically least z such that $M(n, z)$. Now define a relation $E(n, x)$ by

$$(\exists a \in D)(\exists b \in D)[|a| \leq |n| \wedge |b| \leq |n| \wedge x = a \cdot F(n, x) + b].$$

But for every $(a, b) \neq (a', b')$ with (a, b) and (a', b') in $D_n \times D_n$, it is the case that $a \cdot F(n, x) + b \neq a' \cdot F(n, x) + b'$. So $\{x \in D \mid E(n, x)\}$ is at least $|D_n|^2$. Note that $E(n, x)$ is a locally finite relation and so by Proposition F.1.2 there is a $k \in \mathbb{N}$ such that $\{x \in D \mid E(n, x)\} \subset D_{n+k}$. Hence $|D_n|^2 \leq |\{x \in D \mid E(n, x)\}| \leq |D_{n+k}| = O(|D_n|)$ and so $|D_n| = O(1)$. Hence D is finite, contradicting that $\mathcal{D}$ presents an infinite structure. $\triangleleft$

Corollary F.1.7 *There is no infinite automatic field.*

Automatic Boolean Algebras

Recall that a Boolean algebra $\mathcal{B} = (B, \cup, \cap, \setminus)$ induces a partial order $x \subset y$ defined as $x \cap y = x$. Write $\mathbf{0}$ for the bottom element and $\mathbf{1}$ for the top B. Write $\bar{x}$ for $\mathbf{1} \setminus x$. If $S \subset B$ then write $\cup S$ for the element $\cup_{s \in S}$. Say that $y \in B$ *splits* $x \in B$ if $x \cap y \neq \mathbf{0}$ and $x \cap \bar{y} \neq \mathbf{0}$. A non-zero $x \in B$ is an *atom* if $y \subset x$ implies that $y = \mathbf{0}$ or $y = x$. Then $\mathcal{B}$ is called *atomless* if it has no atoms. Also $\mathcal{B}$ is called *atomic* if for every $x \in B$ there exists an atom $y \subset x$. Note that atomless implies not atomic, but the converse does not hold. Indeed if $\mathcal{B}$ is not atomic then there is an element $x \in B$ such that there is no atom below it. In this case $(\{y \in B \mid y \subset x\}, \cap, \cup, \setminus)$ is an atomless Boolean algebra, which is unique up to isomorphism. Note that this atomless Boolean algebra is first order definable in $\mathcal{B}$ with parameter x and hence is automatic if $\mathcal{B}$ is automatic.

However we will shortly prove that there is no automatic presentation of the atomless Boolean algebra and so conclude that if $\mathcal{B}$ is an automatic Boolean algebra, then it has no atomless subalgebra. Such an algebra is called *superatomic*.

The following lemma is useful.

Lemma F.1.8 *Suppose $\mathcal{B}$ is an automatic boolean algebra. There exists a constant $e \in \mathbb{N}$, that depends only on the automata for $\mathcal{B}$, such that for every finite set S of n elements of B, the length of $\cup S$ is at most*

$$\max_{s \in S}\{|s|\} + e \log n.$$

Proof We calculate a bound on $|\cup S|$ by considering a 'computation tree' computing $\cup S$. Let h be the least integer greater than or equal to $\log n$. Denote by B_h the full binary tree of height h. It has at least n leaves. Label the nodes of B_h as follows. The leaves are labelled by elements of S in such a way that every element of S is the label of at least one leaf. Suppose we have labelled all the elements on i-th level of the tree, $x_1, \cdots, x_k$, identifying the nodes of the trees with their labels. Label the $i-1$'st level as follows. For every odd $j < k$, label the parent of x_j and x_{j+1} by $x_j \cup x_{j+1}$. This completes the labelling of the tree B_h. Note that the label of the root of the tree is $\cup S$. By Proposition F.1.2 there is a constant d such that for every $a, b \in B$ it holds that $|a \cup b| \leq \max\{|a|, |b|\} + d$. Then by induction if $x \in B$ is a label of a node on the i-th level of the tree B_n then $|x| \leq \max_{s \in S}\{|s|\} + d(h-i)$. Setting $i = 0$ and noting that $h \leq 1 + \log n$ it holds that $|\cup S| \leq \max_{s \in S}\{|s|\} + d(1 + \log n)$, as required. $\triangleleft$

If $\mathcal{B}$ is a Boolean algebra and $G \subset B$ then write $\mathcal{B}(G)$ for the subalgebra generated by G, namely the smallest (with respect to set inclusion) subalgebra of $\mathcal{B}$ containing G. If G is a free set of generators of size n then $|\mathcal{B}(G)| = 2^n$. A property that distinguishes the atomless Boolean algebra $\mathcal{B}$ is the following: for every $d \in B$ (that is not the minimum element **0**) there exists $x \in B$ such that $x \cap d \neq \mathbf{0}$ and $\bar{x} \cap d \neq \mathbf{0}$.

Proposition F.1.9 *The countable atomless Boolean algebra has no automatic presentation.*

Proof Suppose by way of contradiction that $\mathcal{B}$ is an automatic presentation of the countable atomless Boolean algebra over alphabet Σ. For every binary string σ, define an element b_σ inductively as follows. Set $b_\lambda = \mathbf{1}$. Suppose b_σ has been defined and consider the length lexicographically least element $x \in B$ that splits b_σ. Define $b_{\sigma 0} = x \cap b_\sigma$ and $b_{\sigma 1} = \bar{x} \cap b_\sigma$. Note that for $\epsilon \in \{0, 1\}$, $b_{\sigma\epsilon}$ is first order definable from b_σ using the $<_{llex}$ ordering. Hence by Proposition F.1.2 there exists a constant k such that $|b_{\sigma\epsilon}| \leq |b_\sigma| + k$.

For every $n \in \mathbb{N}$ define $X_n = \{b_\sigma \mid |\sigma| = n\}$. Then by repeated application of Proposition F.1.2, $x \in X_n$ implies that $|x| \leq |\mathbf{1}| + kn$. However for every $x, y \in X_n$ it holds that $x \cap y = \mathbf{0}$ and so $\mathcal{B}(X_n)$, the Boolean algebra generated by X_n, has $2^{|X_n|}$ atoms. Let $Y \in \mathcal{B}(X_n)$ and note that $Y \in B$ is a union of at most 2^n elements of X_n. So by Lemma F.1.8, $|Y| \leq |\mathbf{1}| + kn + en$ for some $e \in \mathbb{N}$ independent of Y. Hence $\mathcal{B}(X_n) \subset \Sigma^{O(n)}$. However there are $2^{(2^n)}$ elements in $\mathcal{B}(X_n)$, which exceeds $|\Sigma^{O(n)}|$, a contradiction. $\triangleleft$

Hence every automatically presentable Boolean algebra is superatomic. We now classify the automatic superatomic Boolean algebras. Call two elements $a, b \in B$ *F-equivalent* if $(a \cap \bar{b}) \cup (b \cap \bar{a})$ is a finite union of atoms. This is a congruence relation on $\mathcal{B}$, and so write $\mathcal{B}/F$ for the quotient algebra. For $x \in B$, write x/F for the F-equivalence class of x. Call $x \in B$ *large* if there are infinitely many atoms of $\mathcal{B}$ below x. Call $x \in B$ *extra-large* if x/F is large in $\mathcal{B}/F$; that is if there are infinitely many atoms of $\mathcal{B}/F$ below x/F. Recall that $y \in B$ *splits* $x \in B$ if $x \cap y \neq \mathbf{0}$ and $x \cap \bar{y} \neq \mathbf{0}$. Note that if $l \in B$ is extra-large then there exists $y \in B$ that splits l such that $l \cap y$ is extra-large and $l \cap \bar{y}$ is large. Suppose that $\mathcal{B}$ is automatic. Then $\mathcal{B}/F$ is first order definable in $\mathcal{B}$, and so is automatic. Similarly the sets of large, and respectively extra-large, elements of $\mathcal{B}$ are regular.

Proposition F.1.10 *If $\mathcal{B}$ is a superatomic automatic Boolean algebra then $\mathcal{B}/F$ is finite.*

Proof Suppose that $\mathcal{B}$ is automatic and that $\mathcal{B}/F$ is infinite. For every $n \in \mathbb{N} \setminus \{0\}$ we will construct a finite binary $\mathcal{B}$-labelled tree T_n. The tree will be a prefix-closed subset of $\{0,1\}^\star$. Its labels will consists of $n+1$ extra-large elements, n large elements (that are not extra-large), and $n(n-1)/2$ atoms. Denote by L the set of large elements of $\mathcal{B}$, by XL the set of extra large elements of $\mathcal{B}$, and by A the set of atoms of $\mathcal{B}$. The following operations will be used. The function $f : XL \to L \times XL$ where $f(x) = (x \cap a, x \cap \bar{a})$ where a is the length-lexicographically least element of $\mathcal{B}$ that splits x such that $x \cap a$ is large and $x \cap \bar{a}$ is extra-large. The function $g : L \to A$ where $f(x)$ is the length-lexicographically least element of $\mathcal{B}$ that is an atom and is below x. The function $h : L \times A \to A$ where $h(x, y)$ is an atom below x and is the length-lexicographically least such that it is length-lexicographically greater than the element y. Since $\mathcal{B}$ is automatic, these functions are FA recognisable.

Define T_n as follows. Its domain is

$$\bigcup_{0 \leq i+j \leq n} 1^i 0^j.$$

Note that T_n is prefix-closed and each node has depth at most n. Now label the tree with elements of $\mathcal{B}$. The root λ is labelled with $\mathbf{1}$. Note that $\mathbf{1}$ is extra-large. Suppose we have labelled 1^i for some $i < n$ as y and let $f(y) = (l, x)$. Then label 1^{i+1} with the extra-large element x and $1^i 0$ with the large element l and $1^i 00$ (for $i < n-1$) with the atom $g(l)$. Now suppose we have labelled $1^i 0^j$ with the atom a for some $2 \leq j < n-i$. Then label $1^i 0^{j+1}$ with the atom $h(l, a)$. This completes the definition of T_n. Note that the $n+1$ nodes 1^i for $0 \leq i \leq n$ are labelled with extra-large elements; the n nodes $1^i 0$ for $0 \leq i < n$ are labelled with large elements (that are not extra-large); and the remaining nodes are labelled with atoms. The assumption that $\mathcal{B}/F$ is infinite and $\mathcal{B}$ superatomic implies that T_n exists for every n.

Recall that the projection function π_ϵ, that extracts the value at co-ordinate ϵ, is FA recognisable. Apply Proposition F.1.2 to the regular functions $\pi_\epsilon f$, f, g and h, yielding corresponding constants. Let k be the maximum of these constants. Now let σ be a node of the tree T_n, with label z say. Then z can be produced by exactly $|\sigma|$ applications of the functions, starting with $\mathbf{1}$. Hence for every n and every label z of T_n, it holds that $|z| \leq |\mathbf{1}| + kn$. In particular, let $X_n \subset B$

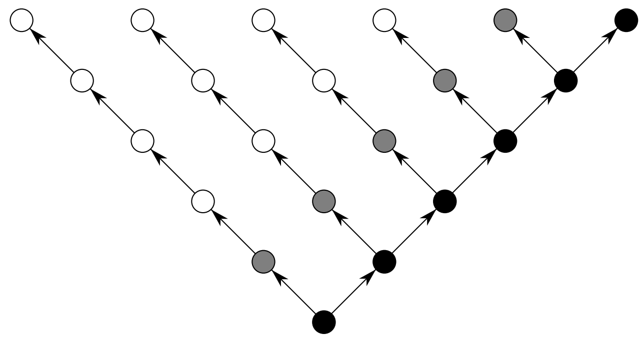

Figure F.1: The tree T_5. The black nodes are extra-large, the grey nodes are large, and the white nodes are atoms.

be the set of atoms labelling nodes in the tree T_n. Then $X_n \subset \Sigma^{O(n)}$. However since atoms are disjoint, the number of elements in $\mathcal{B}(X_n)$, the Boolean algebra generated by the elements in X_n, is $2^{|X_n|}$. Consider an element $Y = a_1 \cup \ldots \cup a_m$ of $\mathcal{B}(X_n)$. By Lemma F.1.8 there exists a constant k' such that $|Y| \leq |\mathbf{1}| + kn + k' \log m$. Since $m = O(n^2)$, it holds that $|Y| = O(n)$ and hence $\mathcal{B}(X_n) \subset \Sigma^{O(n)}$. However there are $2^{n(n-1)/2}$ elements in $\mathcal{B}(X_n)$ which exceeds $|\Sigma^{O(n)}|$, a contradiction. $\lhd$

Fact F.1.11 [see Goncharov - 1997, Corollary 1.6.1] *Every superatomic Boolean algebra is isomorphic to an interval algebra of some ordinal.*

Theorem F.1.12 [Khoussainov, Nies, Rubin, and Stephan - 2004] *A Boolean algebra is automatically presentable if and only if is isomorphic to the interval algebra of α for some ordinal $\alpha < \omega^2$.*

Proof The interval Boolean algebras of ordinals ωn are automatically presentable as in Example B.2.18. Conversely suppose $\mathcal{B}$ is an automatic Boolean algebra. Then by Proposition F.1.9 and the first paragraph of this section it is superatomic. So by Proposition F.1.10 the quotient $\mathcal{B}/F$ is finite. Note that ω^2 is the least ordinal α such that the quotient of the interval algebra of α is not a finite Boolean algebra. $\lhd$

Corollary F.1.13 *The isomorphism problem for automatic Boolean algebras is decidable.*

Proof If $\mathcal{A}$ is an automatic Boolean algebra then the congruence relation F is a regular predicate since it is definable from $\mathcal{A}$ (using $\exists^{\infty}$). Hence given two automatic Boolean algebras, $\mathcal{A}$ and $\mathcal{B}$, construct (uniformly in $\mathcal{A}$ and $\mathcal{B}$) the finite Boolean algebras $\mathcal{A}/F$ and $\mathcal{B}/F$. Then $\mathcal{A}$ is isomorphic to $\mathcal{B}$ if and only if $\mathcal{A}$ and $\mathcal{B}$ have the same number of elements. $\lhd$

F.2 Growth level of finite subsets of the domain

A theorem of Cantor states that every countable linear order is isomorphic to a subordering of the usual ordering of the rationals, η. This is expressed as saying that η is *universal* for the class of countable linear orders. The structure η also has the property that every isomorphism between finite substructures of η extends to an automorphism of η. This property is called *ultra-homogeneity*. Similarly, there exists a countable partial order, $\mathcal{P}$, which is universal for the class of countable partial orderings and is also ultra-homogeneous. These two properties can be used to prove that $\mathcal{P}$ is unique up to isomorphism. Moreover these are both examples of a more general phenomenon called Fraïssé limits. The aim of this section is to prove that unlike η, $\mathcal{P}$ and certain other Fraïssé limits are not automatically presentable.

Examples of Fraïssé limits

Here is a brief introduction to Fraïssé limits taken from Hodges [1993, Chapter 7.1].

Let $\mathcal{K}$ be a class of finite structures over a fixed signature, closed under isomorphism, with the following properties:

HP For every $\mathcal{A} \in \mathcal{K}$ all substructures of $\mathcal{A}$ are also in $\mathcal{K}$.

JEP For every $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ there exists $\mathcal{C} \in \mathcal{K}$ such that $\mathcal{A}$ and $\mathcal{B}$ are embeddable in $\mathcal{C}$.

AP For every $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{K}$ with embeddings $e : \mathcal{A} \to \mathcal{B}$ and $f : \mathcal{A} \to \mathcal{C}$ there exists $\mathcal{D} \in K$ and embeddings $g : \mathcal{B} \to \mathcal{D}$ and $h : \mathcal{C} \to \mathcal{D}$ such that $ge : \mathcal{A} \to \mathcal{D}$ equals $hf : \mathcal{A} \to \mathcal{D}$.

These are respectively called the *hereditary property*, *joint embedding property* and *amalgamation property*.

The *age* of a structure $\mathcal{D}$ is the set of all finite substructures of $\mathcal{D}$. The age of any structure $\mathcal{D}$ has HP and JEP. If a class of structures $\mathcal{K}$ satisfies all three conditions above then there exists a countable structure $\mathcal{D}_K$ such that the following holds.

1. $\mathcal{K}$ is the age of $\mathcal{D}_K$ and $\mathcal{D}_K$ is unique up to isomorphism with this property.
2. $\mathcal{D}_K$ is ultra-homogeneous.
3. The first order theory of $\mathcal{D}_K$ is ω-categorical. This means that every countable structure $\mathcal{B}$ with the same first order theory as $\mathcal{A}$ is isomorphic to $\mathcal{A}$.
4. $\mathcal{D}_K$ admits quantifier elimination.

Here are some examples of classes $\mathcal{K}$ that satisfy HP, JEP and AP, and the corresponding universal structures $\mathcal{D}_K$.

Example F.2.1 *The rational ordering η.*
Let $\mathcal{K}$ be the class of all finite linear orderings. Then $\mathcal{D}_K$ is isomorphic to the usual ordering of the rationals. $\lhd$

In the following two examples, all graphs are undirected.

Example F.2.2 *The random graph.*
Let $\mathcal{K}$ be the class of all finite graphs. Then $\mathcal{D}_K = (D, E)$ is called the *random graph* and is characterised by the following 'extension' property. For every finite subset $F \subset D$ and every partition X_1, X_2 of F, there exists $a \in D$ such that $(a, x_1) \in E$ and $(a, x_2) \notin E$ for every $x_1 \in X_1$ and $x_2 \in X_2$. It is called the random graph because it is with probability one isomorphic to the graph formed by placing an edge between two vertices with independent probability $\frac{1}{2}$. It is also isomorphic to the graph on vertices $\mathbb{N}$ and an edge between x and y if and only if there is a 1 in the x-th position of the binary expansion of y. $\lhd$

A graph is *complete* if there is an edge between every pair of vertices. Write C_p for the complete graph on p vertices.

Example F.2.3 *The random C_p-free graph.*
For $p \geq 3$ let $\mathcal{K}$ be the class of all finite graphs that do not contain a complete subgraph on p vertices. Then $\mathcal{D}_K = (D, E)$ is called the *random C_p-free graph*. It is characterised by the property that for every finite subset $F \subset D$, and every C_{p-1}-free subset $X \subset F$, there exists $a \in D$ such that $(a, x) \in E$ for every $x \in X$ and $(a, y) \notin E$ for every $y \in F \setminus X$. $\lhd$

Let $\mathcal{D} = (D, \preceq)$ be a partial order and let $X \subset D$. Then X is called a *chain* if the elements of X are pairwise comparable; that is for all $x, y \in X$ either $x \preceq y$ or $y \preceq x$. X is called an *anti-chain* if the elements of X are pairwise incomparable; that is for all $x, y \in X$ neither $x \preceq y$ nor $y \preceq x$, written $x \| y$.

Example F.2.4 *The random partial order.*
Let $\mathcal{K}$ be the class of all finite partial orders. Then $\mathcal{D}_K$ is called the *random partial order*. It has the following properties.

1. If F is a finite anti-chain of $\mathcal{D}_K$ and A and B partition F then there exists an element $x \in D_K$ such that for every $a \in A$, $x > a$ and for every $b \in B$, $x \| b$.

2. If F is a finite chain of $\mathcal{D}_K$ with least element a and largest element b then there exists an element $x \in D_K$ such that $a \prec x \prec b$ and for all $c \in F$ with $a \prec c \prec b$ it holds that $x \| c$. $\lhd$

Example F.2.5 *The atomless Boolean algebra.*
Let $\mathcal{K}$ be the class of all finite Boolean algebras. Then $\mathcal{D}_K$ is isomorphic to the countable atomless Boolean algebra. This Boolean algebra is characterised by the property that for every $x \neq \mathbf{0}$ there exists $y \neq \mathbf{0}$ with y strictly below x. Here $\mathbf{0}$ is the minimal element. $\lhd$

Non-automaticity of some Fraïssé limits

We now develop a technique that will be used to prove that the Fraïssé limits introduced above, with the exception of η, are not automatically presentable. It will also be used to prove the non automaticity of the pairing algebra $(\mathbb{N}, p)$ and the structure $(\mathbb{N}, \times)$.

Definition F.2.6 *Suppose $\mathcal{D}$ is a structure over alphabet Σ. Recall that D_n is the set $D \cap \Sigma^{\leq n}$; that is the elements of D of length at most n. Write S_n for $\{x \in \Sigma^n \mid \exists z \in \Sigma^\star \wedge xz \in D\}$; that is the set of strings of length n that are prefixes of words in D.*

Lemma F.2.7 *If $D \subset \Sigma^\star$ is a regular language then*

1. $|S_n| = O(|D_n|)$ *and*
2. $|D_{n+k}| = \Theta(|D_n|)$ *for every constant $k \in \mathbb{N}$.*

Proof Suppose the automaton recognising D has c states. Then for $x \in S_n$ there exists $z \in \Sigma^\star$ with $|z| \leq c$ such that $xz \in D$ $(\dagger)$. If $n \geq c$ then $|S_n| \leq |\Sigma|^c \times |S_{n-c}|$ since the map associating $x \in S_n$ with the word consisting of the first $n - c$ letters of x, is $|\Sigma|^c$-to-one. But by using $(\dagger)$ we see that $|S_{n-c}| \leq |D_n|$. So $|S_n| \leq |\Sigma|^c \times |D_n|$. This completes the first part.

Fix $k \in \mathbb{N}$. The mapping associating $x \in D_{n+k}$ to the prefix of x of length n is $|\Sigma|^k$-to-one. Hence $|D_{n+k}| \leq |\Sigma|^k \times |S_n| \leq |\Sigma|^k \times |\Sigma|^c \times |D_n|$. Since $D_n \subset D_{n+k}$ one has that $|D_n| \leq |D_{n+k}| = O(|D_n|)$. This completes the second part. $\triangleleft$

Definition F.2.8 *Suppose that the structure $\mathcal{D}$ contains an atomic binary relation E and $D \subset \Sigma^\star$. For $n \in \mathbb{N}$ and $y \in D$ define the function $c^E_{n,y} : D_n \to \{0, 1\}$ as*

$$c^E_{n,y}(x) = \begin{cases} 1 & if \quad \mathcal{D} \models E(x, y) \\ 0 & if \quad \mathcal{D} \models \neg E(x, y). \end{cases}$$

Write $\#c^E_n$ for the cardinality of the set $\{c^E_{n,y} \mid y \in D\}$. If there is no ambiguity we drop the superscript.

In general $\#c_n$ is at most $2^{|D_n|}$, the number of subsets of $|D_n|$. But in the case that $\mathcal{D}$ is automatic we can say more.

Proposition F.2.9 *If $\mathcal{D}$ is automatic then $\#c_n = O(|D_n|)$.*

Proof Suppose $\mathcal{D}$ is automatic over alphabet Σ. It is sufficient to prove that the cardinality of the set $C_n = \{c_{n,y} \mid y \in D \cap \Sigma^{>n}\}$ is $O(|D_n|)$ since the y's in D_n can supply at most $|D_n|$ many additional functions $c_{n,y}$. That is, $\#c_n \leq |C_n| + |D_n|$. Let (Q, ι, ρ, F) be a 2–tape automaton recognising $\otimes E$. Fix $n \in \mathbb{N}$. For $y \in D \cap \Sigma^{>n}$ we will associate with each $c_{n,y}$ two pieces of information, $J_y : D_n \to Q$ and $K_y \subset Q$ as follows. Let $\otimes(w, y) = \sigma_1\sigma_2 \cdots \sigma_k$, where $w \in D_n$ and $\sigma_i \in \Sigma^2_\perp$. Then define $J_y(w) := \rho(\iota, \sigma_1 \cdots \sigma_n)$. Define $K_y \subset Q$ as those states $s \in Q$ such that $\rho(s, \sigma_{n+1} \cdots \sigma_k) \in F$. Note that $\sigma_{n+1} \cdots \sigma_k = \otimes(\lambda, z)$ for some $z \in \Sigma^*$.

Consider the mapping sending $c_{n,y}$ to the pair (J_x, K_x). If $c_{n,y} \neq c_{n,y'}$ then there is some $w \in D_n$ such that without loss of generality $\mathcal{D} \models E(w, y)$ and $\mathcal{D} \models \neg E(w, y')$. So if $J_y = J_{y'}$ then there is a state q which is in K_y and not in $K_{y'}$. Hence the mapping is one-to-one. So the cardinality of the set C_n is less than or equal to $|\{J_y \mid y \in D \cap \Sigma^{>n}\}| \times 2^d$ for $d = |Q|$. Now J_y depends only on the first n letters of y in the sense that for $|v| = n$ and $w, w' \in \Sigma^\star$, $J_{vw} = J_{vw'}$. Hence $\{J_y \mid y \in D \cap \Sigma^{>n}\} = \{J_y \mid y \in S_n\}$, where S_n is the set $\{y \in \Sigma^n \mid (\exists w)\, [w \in \Sigma^\star \wedge yw \in D]\}$. By the first part of Lemma F.2.7, we get that

$$|C_n| = O(|S_n|) = O(|D_n|).$$

Hence $\#c_n = O(|D_n|)$ as required. $\lhd$

Corollary F.2.10 [Stephan - 2002] [also Delhommé - 2001b] *The random graph has no automatic presentation.*

Proof Suppose by way of contradiction that (D, E) is an automatic copy of the random graph. Applying the extension axiom to D_n we get that for every $K \subset D_n$ there exists $k \in D$ such that $c^E_{n,k}(x) = 1$ if and only if $x \in K$. Hence $\#c^E_n = 2^{|D_n|}$ contradicting Proposition F.2.9. $\lhd$

Proposition F.2.9 can be weakened to give a purely algebraic condition that is sufficient for proving non-automaticity. If $\Phi(x, y)$ is a $\mathcal{D}$-formula, $y \in D$ and $F \subset D$, then define $\Phi(F, y)$ as the set $\{x \in F \mid \mathcal{D} \models \Phi(x, y)\}$. Also write $\#c^\Phi(F)$ for $|\{\Phi(F, y) \mid y \in D\}|$. We drop the superscript if there is no ambiguity.

Suppose that $\mathcal{D}$ is automatic and let $\Phi(x, y)$ be a $\mathcal{D}$-formula. In what follows Φ may be a FO formula (with parameters) and possibly using the additional quantifiers $\exists^\infty$ and $\exists^{(k,m)}$. Recall one can expand every automatic presentation of $\mathcal{D}$ to one that includes Φ^D, by Theorem B.1.26.

So Proposition F.2.9 says that

$$|\{\Phi(D_n, y) \mid y \in D\}| = O(|D_n|).$$

In particular there exists $k \in \mathbb{N}$ such that for infinitely many $m \in \mathbb{N}$ (namely those $|D_n|$ for $n \in \mathbb{N}$), there exists an F (namely some D_n) with $|F| = m$ such that $\#c(F) < km$.

The contrapositive is summarised as follows.

Proposition F.2.11 *Let $\mathcal{D}$ be a structure with domain D. Suppose there exists a $\mathcal{D}$-formula $\Phi(x, y)$ so that for every $k \in \mathbb{N}$ it holds that*

$$\#c(F) \geq k|F|$$

for almost all finite subsets $F \subset D$. Then $\mathcal{D}$ is not automatically presentable.[1]

Example F.2.12 For the random graph choose $\Phi(x, y)$ as stating that there is an edge between x and y. Then $\#c(F) = 2^{|F|}$ for every finite subset F of the domain. $\triangleleft$

The following is part of an old combinatorial result known as Dilworth's Lemma.

Lemma F.2.13 *Let $\mathcal{D}$ be a* finite *partial order of cardinality n. Let a be the size of largest anti-chain in $\mathcal{D}$ and let c be the size of the largest chain in $\mathcal{D}$. Then $n \leq ac$.*

Proof Let c be the size of the largest chain in $\mathcal{D}$. For $1 \leq i \leq c$ define X_i as the set of all elements x such that the size of the largest chain in the subpartial order $(\uparrow x) = \{y \in D \mid x \preceq y\}$ is i. Then the X_i's partition D. Moreover if $a \prec b$ and $b \in X_i$ then the size of the largest chain in $(\uparrow a)$ is $> i$. Hence each X_i is an anti-chain. Thus $\mathcal{D}$ can be partitioned into exactly c many anti-chains. If a is the size of the largest anti-chain in $\mathcal{D}$ then $n \leq ac$ as required. $\triangleleft$

Corollary F.2.14 *$\mathcal{P} = (P, \leq)$ has no automatic presentation.*

Proof Let $\Phi(x, y)$ be the formula $x \leq y \vee y \leq x$ and fix a finite $F \subset P$.

Let A be an antichain of F. By property 1 in Example F.2.4, map each $X \subset A$ to an element $X' \in P$ such every $a \in X$ satisfies $X' > a$ and every $b \in A \setminus X$ satisfies $X' \| b$. Note that if $X \neq Y$ then $\Phi(F, X') \neq \Phi(F, Y')$. Hence there are at least $2^{|A|}$ distinct sets of the form $\Phi(F, x)$ where $x \in P$. That is, $\#c(F) \geq 2^{|A|}$.

Let C be a chain of F. Then by property 2 in Example F.2.4, map each pair (a, b) with $a < b \in C$ to an element $x_{(a,b)} \in P$ such that $x_{(a,b)} > a$ and $x_{(a,b)} < b$ and every $c \in C$ with $a < c < b$ satisfies $x_{(a,b)} \| c$. Then if $(a, b) \neq (e, f)$ then $\Phi(F, x_{(a,b)}) \neq \Phi(F, x_{(e,f)})$ and so $\#c(F) \geq |C|^2$.

Now if A is an anti-chain of F of maximal size, and C is a chain of F of maximal size, then by Lemma F.2.13, $(\#c(F))log^2(\#c(F)) \geq |F|^2$. So by Proposition F.2.11, $\mathcal{D}$ is not automatically presentable. $\triangleleft$

In what follows, let $\mathcal{F}$ be a finite graph. For a vertex v, write $E(v)$ for the set of vertices adjacent to v. The *degree* of a vertex is the cardinality of $E(v)$. Write $\Delta(\mathcal{F})$ for the maximum degree over all the vertices of F. Call a subgraph $\mathcal{G}$ with no edges an *independent* graph. Let $\alpha(\mathcal{F})$ be the number of vertices of a largest independent subgraph of $\mathcal{F}$.

Recall that C_p denotes the complete graph on p vertices; that is, there is an edge between every pair of vertices. A graph is called *C_p-free* if it has no subgraph isomorphic to C_p.

[1]This observation arose in conversation with Leonid Libkin and Michael Benedikt.

Lemma F.2.15 *For every $p \geq 3$, there is a polynomial $Q_p(x)$ of degree $p-1$ so that if $\mathcal{F}$ is a finite C_p-free graph then $Q_p(\alpha(\mathcal{F})) \geq |F|$.*

Proof We first prove that for every finite graph $\mathcal{F}$,

$$\alpha(\mathcal{F}) \geq |F|/(\Delta(\mathcal{F})+1). \tag{F.1}$$

Let $\mathcal{G}$ be an independent subgraph of $\mathcal{F}$ with a maximal number of vertices. That is, $\alpha(\mathcal{F}) = |\mathcal{G}|$. For every $d \in G$ let $N(d) = E(d) \cup \{d\}$, where $E(d)$ is the set of vertices in $\mathcal{F}$ adjacent to d. Then since $\mathcal{G}$ is maximal, for every $x \in F$ there is some (not necessarily unique) $d \in G$ such that $x \in N(d)$. Hence $\mathcal{F} = \cup_{d \in G} N(d)$. But $|N(d)|$ equals the degree (in $\mathcal{F}$) of d plus one, and so the largest cardinality amongst the $N(d)$'s is at most $\Delta(\mathcal{F})+1$. Hence $|F| \leq |G| \times (\Delta(\mathcal{F})+1)$ as required.

The lemma is proved by induction on p. We will show that $Q_p(x) = \Sigma_{i=1}^{p-1} x^i$. For the case $p=3$ note that for every vertex v, the subgraph on domain $E(v)$ is independent. For otherwise if $x, y \in E(v)$ were joined by an edge then the subgraph of $\mathcal{F}$ on $\{x, y, v\}$ is C_3. In particular then $\alpha(\mathcal{F}) \geq \Delta(\mathcal{F})$. Combining this with Inequality F.1, we get $\alpha(\mathcal{F})[\alpha(\mathcal{F})+1] \geq |F|$ as required.

For the inductive step, let $\mathcal{F}$ be a C_p-free graph with $p > 3$. For every vertex v, the set $E(v)$ is C_{p-1}-free for otherwise the subgraph of $\mathcal{F}$ on $E(v) \cup \{v\}$ has a copy of C_p. Applying the induction hypothesis to $E(v)$ we get that $E(v)$ must have an independent set X so that $Q_{p-1}(|X|) \geq |E(v)|$. But X is also independent in $\mathcal{F}$ so $Q_{p-1}(\alpha(\mathcal{F})) \geq \Delta(\mathcal{F})$. Combining this with Inequality F.1, we get that $\alpha(\mathcal{F})[Q_{p-1}(\alpha(\mathcal{F}))+1] \geq |F|$. Hence $Q_p(\alpha(\mathcal{F})) \geq |F|$ as required. $\lhd$

Corollary F.2.16 *For $p \geq 3$, the random C_p-free graph is not automatically presentable.*

Proof Fix $p \geq 3$ and let (D, E) be a copy of the random C_p-free graph. Let F be a finite subset of D. Then for every C_{p-1}-free subset $K \subset F$ there exists an $x \in D$ that is connected to every vertex in K and none in $F \setminus K$. So let $\mathcal{G}$ be an independent subgraph of $\mathcal{F}$ so that $Q_p(|G|) \geq |F|$ as in the lemma. Then $\mathcal{G}$ has $2^{|G|}$ subsets and so $\#c(F)$ is at least $2^{|G|}$ which is not linear in $|F|$. Hence by Proposition F.2.11, $\mathcal{D}$ is not automatically presentable. $\lhd$

The next result was observed by Leonid Libkin, though originally proven in Blumensath [1999] using the method of growth levels of string lengths. Recall that a pairing function is a bijection of the form $p : D^2 \to D$. The resulting structure (D, p) is called a *pairing algebra*.

Corollary F.2.17 [Blumensath - 1999] *No pairing algebra $(\mathbb{N}, p)$ has an automatic presentation.*

Proof Let $\Phi(x, y)$ be the formula $(\exists z)[p(x, z) = y \vee p(z, x) = y]$. Note that for every pair $(a, b) \in F \times F$, there is a $y \in D$ so that $p(a, b) = y$. That is, $\Phi(a, y)$ holds and $\Phi(b, y)$ holds and since p is one-to-one, there is no other c for which $\Phi(c, y)$ holds. So in general $\Phi(F, y)$ defines every two (and one) element subset of F as y varies over $\mathbb{N}$. Hence $\#c(F) \geq \binom{|F|}{2}$ for every finite subset $F \subset \mathbb{N}$ and so by Proposition F.2.11, $(\mathbb{N}, p)$ is not automatically presentable. $\triangleleft$

The following was also originally proven in Blumensath [1999] using the method of growth levels of string lengths. In fact, there the divisibility structure $(\mathbb{N}, |)$ was proven to have no automatic presentation.

Corollary F.2.18 [Blumensath - 1999] *The structure* $(\mathbb{N}, \times)$ *has no automatic presentation.*

Proof Let $(D, \times)$ be an automatic presentation of $(\mathbb{N}, \times)$ over Σ. Expand this presentation to one that includes the following.

1. A function $sqr : D \rightarrow D$ where $sqr(a) = a \times a$.
2. The divisibility predicate $x \div y$ defined as $(\exists z)\ [y = x \times z]$.
3. The locally finite relation $vid(x, y)$ defined as $y \div x$.
4. The identity **1** definable as $\{x \mid (\forall y)\ x \div y\}$.
5. The set of primes P definable as $\{x \in D \mid x \neq \mathbf{1} \wedge (\forall y \div x)\ [y = \mathbf{1} \vee y = x]\}$.
6. The set of prime powers R is definable as $\{x \in D \mid (\exists p \in P)(\forall y \div x)\ [p \div y \vee y = \mathbf{1}]\}$.
7. A function $next : P \rightarrow P$ where $next(p)$ is the length-lexicographical least prime length-lexicographically greater than p; that is, the graph of $next$ is the immediate successor relation of $<_{llex}$ restricted to the primes P.

By Proposition F.1.2 applied to function $next$, $|p_{i+1}| \leq |p_i| + e$ for some constant $e \in \mathbb{N}$. So $|p_{i+1}| \leq |p_1| + ei$ or in other words $D_{|p_1|+ei}$ contains $\{p_1, \cdots, p_{i+1}\}$. So the number of primes in D_n, denoted by $p(n)$, is at least n/d for some constant $d \in \mathbb{N}$ that depends only on $|p_1|$ and e. Similarly by repeated application of Proposition F.1.2 applied to the function sqr, there is a constant $k' \in \mathbb{N}$ such that if $a \in D_n$ then for every $x \in \mathbb{N}$, $a^{(2^x)}$ is in $D_{n+k'x}$. Also by Proposition F.1.2 applied to the relation vid there is a constant $k'' \in \mathbb{N}$ such that if $a \in D_n$ then every divisor of a is in $D_{n+k''}$. Let k be the larger of k' and k''. In particular then for every $x \in \mathbb{N}$, the set D_{n+kx} contains every number of the form $p^{(2^y)}$ for every prime p in D_n and $y \leq x$.

For every prime p, let $r_n(p)$ be the largest exponent such that the number $p^{r_n(p)}$ is in D_n. Then applying Proposition F.2.9 to the relation $\div$, we have that for every $x, n \in \mathbb{N}$,

$$2^{O(n+kx)} \geq \#c^{\div}_{n+kx} \geq \prod_{p \text{ prime in } D_n} r_{n+kx}(p) \geq \prod_{i=1}^{p(n)} 2^x = 2^{xp(n)} \geq 2^{xn/d},$$

which is a contradiction since $d, k \in \mathbb{N}$ are independent of x and n. $\lhd$

Finally, here is an alternative proof that the atomless Boolean algebra is not automatically presentable (see Proposition F.1.9). Recall that if $\mathcal{B}$ is a boolean algebra and $G \subset B$ then $\mathcal{B}(G)$ denotes the subalgebra of $\mathcal{B}$ generated by G.

Corollary F.2.19 *The atomless Boolean algebra has no automatic presentation.*

Proof Let $(D, \cap, \cup, \setminus)$ be an automatic copy over Σ of the countable atomless Boolean algebra. Recall that D_n is defined as $D \cap \Sigma^{\leq n}$ so that $|D_n|$ as a function from $\mathbb{N}$ to $\mathbb{N}$ is non-decreasing. Suppose that $1 \in \Sigma$. Expand this presentation to an automatic one that includes the following relations where $n \in \mathbf{1}^\star$ and $x \in D$.

- the minimum element $\mathbf{0}$ satisfies $(\forall x \in D)\,[x \cup \mathbf{0} = x]$.
- the partial order $x \subset y$ defined as $x \cap y = x$.
- $Atom(n, x)$ defined as $(\exists y)[|y| \leq |n| \wedge x \subset y] \wedge (\forall y)[|y| \leq |n| \rightarrow (x \subset y \vee x \subset \bar{y})]$.
- $MaxAtom(n, x)$ defined as $Atom(n, x) \wedge (\forall y)[(Atom(n, y) \wedge y \subset x) \rightarrow x = y]$. So $MaxAtom(n, x)$ holds if x is an atom of the finite Boolean algebra generated by the set $\{y \in D \mid |y| \leq |n|\}$.
- $Gen(n, x)$ defined as $(\forall a)[MaxAtom(n, a) \rightarrow (a \subset x \vee a \subset \bar{x})]$.
- $MinGen(n, x)$ defined as

$$\begin{aligned} Gen(n, x) \quad \wedge \quad & (\forall y)(\forall x)[Gen(n, y) \wedge MaxAtom(n, a)] \rightarrow \\ & [(a \subset x \iff a \subset y) \rightarrow (x \subset y)]. \end{aligned}$$

- $Split(n, x)$ defined as $(\forall a)[MaxAtom(n, a) \rightarrow (a \cap x \neq \mathbf{0} \wedge a \cap \bar{x} \neq \mathbf{0})]$.
- $MinSplit(n, x)$ defined as $Split(n, x) \wedge (\forall y)[Split(n, y) \rightarrow x \leq_{llex} y]$.

Identifying $(1^\star, \cdot)$ with $(\mathbb{N}, +)$ we may write $n \in \mathbb{N}$. Then $MaxAtom(n, x)$ holds if and only if x is a free generator of $\mathcal{B}(D_n)$. So $MinGen(n, x)$ holds if and only if x is in $\mathcal{B}(D_n)$. Note that $MinGen(n, x)$ is a locally finite relation and so by Proposition F.1.2 there exists a constant k independent of n such that $\mathcal{B}(D_n) \subset D_{n+k}$ for every $n \in \mathbb{N}$.

The property of being atomless implies that for every n there exists a unique $x \in D$ with $MinSplit(n, x)$. Since $x \notin \mathcal{B}(D_n)$ and using Proposition F.1.2 one has that there exists a constant k' independent of n such that $n \leq |x| \leq n + k'$. Let $c = k' + k$. Then D_{n+c} contains $\mathcal{B}(D_{n+k'})$ which in turn contains at least twice as many free generators as $\mathcal{B}(D_n)$. So $|D_{n+c}| \geq |\mathcal{B}(D_{n+k'})| \geq |\mathcal{B}(D_n)|^2 \geq |D_n|^2$. However using the second part of Lemma F.2.7 one has that $|D_{n+c}| = O(|D_n|)$. So $D_n = O(1)$ which implies that D is finite which contradicts that $\mathcal{D}$ presents an infinite structure. $\lhd$

F.3 The isomorphism problem

The *isomorphism problem* for a class of automatic structures $\mathcal{C}$ is as follows: Given automatic presentations of two structures $\mathcal{A}$ and $\mathcal{B}$ from $\mathcal{C}$, is $\mathcal{A}$ isomorphic to $\mathcal{B}$? Recall the isomorphism problem for automatic ordinals or for automatic Boolean algebras is decidable (Corollaries E.3.3 and F.1.13). Recall also that it is undecidable whether two automatic permutation structures are isomorphic or not (Theorem D.2.11). In measuring the complexity of the isomorphism problem, we follow the standard notation, see Rogers [1967]. The proof of Theorem D.2.11 says that the isomorphism problem for automatic permutation structures is at least Π^0_1.

Call a directed graph *locally finite* if the degree of every vertex in the underlying undirected graph is finite. A (*connected*) *component* of a directed graph is a maximal set of vertices that are connected in the underlying undirected graph.

Proposition F.3.1 *The complexity of the isomorphism problem for automatic locally finite directed graphs is Π^0_3-complete.*

Proof Let $\mathcal{A}$ and $\mathcal{B}$ be automatic locally finite directed graphs. If $x \in A$, and $n \in \mathbb{N}$, write $L(x, n)$ for the set of $x' \in A$ such that there is a path of length at most n between x and x'. Note that since $\mathcal{A}$ is locally finite, $L(x, n)$ is finite. And moreover since $\mathcal{A}$ is automatic, the set $L(x, n)$ is computable from x and n. So for $x \in A$ and $y \in B$ the property that there is an isomorphism from $L(x, n)$ to $L(y, n)$ that extends $x \mapsto y$ is computable since there are only finitely many maps to test for isomorphism.

Let $x \in A$ and $y \in B$ and consider the following sentence Φ. 'For every $n \in \mathbb{N}$, $L(x, n)$ is isomorphic to $L(y, n)$'. Whether this sentence holds is Π^0_1. Write C for the component containing x and D for the component containing y; so $\mathcal{C}$ and $\mathcal{D}$ are directed graphs. We claim that $\mathcal{C}$ is isomorphic to $\mathcal{D}$ if and only if this sentence holds. Indeed, if $\mathcal{C}$ is isomorphic to $\mathcal{D}$, then the isomorphism witnesses the truth of the sentence. Conversely, suppose the sentence holds. Define an isomorphism $f : C \to D$ inductively. At stage k we will define an isomorphism f_k from $L(x, k)$ to $L(y, k)$ extending f_{k-1}. By assumption choose $x \in C$ and $y \in D$ and define $f_0(x) = y$. Denote the isomorphism from $L(x, n)$ to $L(y, n)$ by g_n and let G_0 be the set of all g_n for $n \in \mathbb{N}$. This completes stage 0. The inductive hypothesis is that in stage k, f_k has domain $L(x, k)$ and range $L(y, k)$, and G_k is an infinite subset of G_{k-1} such that every isomorphism g_n in G_k extends f_k. Now extend f_k to $L(x, k + 1)$ as follows. Denote by g'_n the restriction of $g_n \in G_k$ to $L(x, k + 1)$. Then since the graphs are locally finite, there are only finitely many distinct such g'_n, and so at least one, say g'_j, is equal to g'_n for infinitely many $g_n \in G_k$. Define G_{k+1} to be the set of all $g_n \in G_k$ such that $g'_n = g'_j$, and let $f_{k+1} = g'_j$. This completes stage $k + 1$.

The predicate 'a and b are in different components' is Π^0_1 since checking whether there is a path (in the underlying graph) of length n from a to b is computable. Consider the following sentence Ψ: 'For every $k \in \mathbb{N}$, for every $x_1, \cdots, x_k$ elements of $\mathcal{A}$ there exist $y_1, \cdots, y_k$ elements of $\mathcal{B}$ with the following properties:

- For each $i \leq k$, x_i and x_j are in different components of $\mathcal{A}$ if and only if y_i and y_j are in different components of $\mathcal{B}$.

- For every $n \in \mathbb{N}$ and $i \leq k$ there is an isomorphism from $L(x_i, n)$ to $L(y_i, n)$.'

Denote by Ψ' the same sentence but with $\mathcal{A}$ and $\mathcal{B}$ interchanged. Then whether the sentence $\Psi \wedge \Psi'$ holds is in Π^0_3 and expresses whether $\mathcal{A}$ is isomorphic to $\mathcal{B}$. Indeed that the sentence holds follows immediately from the existence of an isomorphism. Conversely, suppose the sentence holds. If $\mathcal{A}$ were not isomorphic to $\mathcal{B}$ then there exists a connected locally finite directed graph $\mathcal{C}$, such that the number $a \leq \omega$ of components of $\mathcal{A}$ isomorphic to $\mathcal{C}$ is not equal to the number $b \leq \omega$ of components of $\mathcal{B}$ isomorphic to $\mathcal{C}$. Say $b < a < \omega$ and let x_i be an element of $\mathcal{A}$ from the ith component isomorphic to $\mathcal{C}$, for $i \leq a$. Then apply Ψ to get $y_1, \cdots, y_a$ in B, all in different components, each component isomorphic to $\mathcal{C}$, contradicting that $b < a$. The other cases $a < b < \omega$, $b < a = \omega$ and $a < b = \omega$ are similar. This completes the proof that the isomorphism problem is Π^0_3.

For hardness, recall that the set of Turing Machines that diverge on infinitely many inputs is Π^0_3-complete. We reduce this set to the isomorphism problem for locally finite graphs. In fact the graphs will be injection structures. Define an automatic injection structure $\mathcal{F}$ as consisting of infinitely many $\mathbb{Z}$-orbits, infinitely many $\mathbb{N}$-orbits, and infinitely many orbits of every finite size and type, see Example B.2.3. For a deterministic Turing machine $\mathcal{M}$, construct an equivalent reversible machine $\mathcal{R}$. Let the configuration graph of $\mathcal{R}$ be $\mathcal{C}(R) = (C, E)$ over Σ. Make the following two alterations to $\mathcal{C}(R)$ that preserve automaticity.

For the first change, for every configuration c with indegree 0 that is not an initial configuration, define a graph $\mathcal{J}_c$ with domain $\{c\} \times 1^\star$, here 1 is a new symbol not in Σ, and an edge from 1^n to 1^{n-1} for every $n > 0$. The union of all such $\mathcal{J}_c$'s is automatic. Now identify (c, λ) with c. Similarly for every configuration d with outdegree 0 that is not a final configuration, define a graph $\mathcal{J}_d$ with domain $\{d\} \times 2^\star$, here 2 is a new symbol, and an edge from 2^n to 2^{n+1} for every $n \geq 0$. Now identify (d, λ) with d. These changes transform non-valid computations into $\mathbb{Z}$-orbits.

Take the resulting automatic structure and form its union with the automatic structure consisting of infinitely many $\mathbb{Z}$-orbits and infinitely many orbits of every finite size. Call the resulting automatic structure $\mathcal{I}(\mathcal{R})$. Then $\mathcal{R}$ diverges on infinitely many words if and only if $\mathcal{I}(\mathcal{R})$ contains infinitely many $\mathbb{N}$-orbits if and only if $\mathcal{I}(\mathcal{R})$ is isomorphic to $\mathcal{F}$. Hence the isomorphism problem for automatic injection structures, and hence automatic locally finite graphs, is Π^0_3-complete. $\triangleleft$

We now make some naming conventions regarding trees. Recall that $(T, \preceq)$ is a *tree* if $\preceq$ is a partial order on T with a minimum element such that every set $\{x \in T \mid x \prec y\}$ for $y \in T$ is a finite linear order. In this case write $S_\preceq$ for the immediate successor relation induced by $\preceq$; namely $S_\preceq(x, y)$ holds if $x \prec y$ and there is no z in T such that $x \prec z \prec y$.

Definition F.3.2 *Let $E \subset T \times T$ and $\preceq$ be the transitive closure of E. Then call the directed graph (T, E) a* **successor tree** *if $(T, \preceq)$ is a tree and $E = S_\preceq$.*

Note that if $(T, \preceq)$ is a tree then $(T, S_{\preceq})$ is a successor tree.

Denote the set of finite strings from $\mathbb{N}$ by $\mathbb{N}^\star$. Then $(\mathbb{N}^\star, \preceq_p)$ is a tree while

$$(\mathbb{N}, \{(w, wn) \mid w \in \mathbb{N}^\star, m \in \mathbb{N}\})$$

is a successor tree that equals $(\mathbb{N}^\star, S_{\preceq_p})$.

The successor tree $(\mathbb{N}^\star, S_{\preceq_p})$ is computably isomorphic to the automatic successor tree with domain $N = \{0,1\}^\star 1 \cup \{\lambda\}$ and edge relation

$$E_N = \{(x, y) \mid x \prec_p y \wedge (\neg\exists z)\,[x \prec_p z \prec_p y]\}.$$

The computable mapping sending $n_1 \dots n_k$ to $0^{n_1}1 \dots 0^{n_k}1$ and the root to λ establishes the isomorphism.

Definition F.3.3 *Let $T \subset N$ and let E_T denote E_N restricted to T. Call (T, E_T) a* **downward closed subtree** *of (N, E_N) if for every $x, y \in N$ if $x \cdot y \in T$ then $x \in T$. Note that (T, E_T) is a successor tree. Call (T, E_T)* **computable** *if there is a Turing machine that decides on input $x \in N$, whether or not $x \in T$.*

Lemma F.3.4 *The complexity of the isomorphism problem for computable downward closed subtrees of (N, E_N) is Σ_1^1-complete.*

Proof From recursion theory a subset T of $\mathbb{N}^\star$ is called a tree if it is downward closed, namely for every $x, y \in \mathbb{N}$, if $x \cdot y \in T$ then $x \in T$, and is called *computable* if T is computable. Recall that $(\mathbb{N}^\star, \preceq_p)$ and (N, E_N) are computably isomorphic. This isomorphism shows that if $X \subset \mathbb{N}^\star$ is a computable tree in the sense just defined, then it is computably isomorphic to a downward closed successor subtree of (N, E_N), and vice versa.

Hence there is a computable reduction from the isomorphism problem for computable subtrees of $\mathbb{N}^\star$ to the computable downward closed subtrees of (N, E_N).

Now apply the following result whose proof can be found in Goncharov and Knight [2002, Theorem $4.4\,b$.]: The complexity of the isomorphism problem for computable subtrees of $\mathbb{N}^\star$ is Σ_1^1-complete. $\triangleleft$

The following theorem is joint with A. Nies and B. Khoussainov.

Theorem F.3.5 *The complexity of the isomorphism problem for automatic structures is Σ_1^1-complete.*

Proof First note that the problem can be expressed as 'there exists a function f from A to B such that for every atomic symbol R, for all $\overline{a}$, it holds that $\mathcal{A} \models \overline{a} \in R^A$ if and only if $\mathcal{B} \models f(\overline{a}) \in R^B$'. Since automata for R^A and R^B are computable from (an index for) R, this sentence is in Σ_1^1.

We now establish that the isomorphism problem for computable downward closed subtrees of (N, E_N) is Turing reducible to the isomorphism problem for automatic structures. It is sufficient to prove that for every computable downward closed subtree (T, E_T) of (N, E_N) there exists an automatic graph $\mathcal{A}_T$ with the following property: T_1 and T_2 are isomorphic (as successor trees) if and only $\mathcal{A}_{T_1}$ and $\mathcal{A}_{T_2}$ are isomorphic (as graphs).

So given $\mathcal{T}$ let $\mathcal{M}_T$ be a reversible Turing machine that computes T modified so that it only halts in an accepting state; that is, instead of halting in a rejecting state, it loops forever. Let $\mathcal{C}(\mathcal{M}_T)$ be its configuration space over alphabet Σ. Assume that the input alphabet is $\{0, 1\}$. Recall that configurations are of the form $(q, (x_1, x_2), (y_1, y_2))$ where x_1x_2 is the content of the first tape, y_1y_2 of the history tape and q is the state of $\mathcal{M}_T$. For $w \in N$, the initial configuration of $\mathcal{M}_T$ with w on the first tape is $(\iota, (w, \lambda), (\lambda, \lambda))$. If $w \in T$, then the corresponding computation is finite, otherwise by the modification it is infinite.

We now define some graphs ending with the definition of $\mathcal{A}_T$. A *chain* is a graph isomorphic to an initial segment of $(\mathbb{N}, S)$ and may be of finite or infinite length. Note that a chain is a successor tree. The *base* of a chain is its unique element with no predecessor. An *isolated chain with base* w in a successor tree T is a chain $P \subset T$ starting with $w \in T$ such that for every $x \in N$, if $wx \in T$ then $wx \in P$. Note that since $\mathcal{M}_T$ is reversible, the configuration space consists of chains.

The graph $\mathcal{F}$. Let $\mathcal{F}$ be the graph consisting of infinitely many chains of every finite length. Then $\mathcal{F}$ is automatically presentable as follows. The set consisting of one chain of every finite length is automatically presentable with domain $0^\star 01^\star$ and an edge from x to y if and only if $|x| = |y|$ and y is the least string lexicographically greater than x. Take the ω-fold disjoint union of this structure to get $\mathcal{F}$.

The graph $\mathcal{I}$. Let $\mathcal{I}$ be the graph consisting of infinitely many chains of every finite length and exactly one infinite chain, all joined at a single vertex. Then $\mathcal{I}$ is automatically presentable as it may be formed as the disjoint union of $\mathcal{F}$ and a single infinite chain, and then identifying the bases of the chains. Note that the resulting graph is a successor tree.

The graphs $\mathcal{F}_w$. For each $w \in N$, define an automatic graph as follows. Assume the graph $\mathcal{F}$ is automatic over a new alphabet. The domain is $\{wf \mid f \in F\}$ and there is an edge from wf to wg in case there is one from f to g in $\mathcal{F}$. Finally identify the bases of each chain and call the resulting graph $\mathcal{F}_w$. Note that $\mathcal{F}_w$ is a successor tree; and we may suppose that its root is w.

The graph $\mathcal{O}$. Let $\mathcal{V}$ be an automatic copy of the successor tree $(\mathbb{N}^\star, \preceq_p)$ over a new alphabet and let $\mathcal{I}$ be automatic over a new alphabet. Assume that the root of $\mathcal{I}$ is λ. For every $v \in V$ define the automatic successor tree $\mathcal{I}_v$ as consisting of domain $\{vf \mid f \in I\}$ and an edge from vf to vg in case there is one from f to g in $\mathcal{I}$. Now form the union (*not disjoint union*) of $\mathcal{V}$ and all the $\mathcal{I}_v$'s for $v \in V$. Call the resulting automatically presentable graph $\mathcal{O}$. Note that $\mathcal{O}$ is a successor tree.

The graphs $\mathcal{O}_w$. For each $w \in N$, define the automatic graph $\mathcal{O}_w$ as follows. Assume that $\mathcal{O}$ is automatic over a new alphabet. The domain of $\mathcal{O}_w$ is $\{wf \mid f \in O\}$ and there is an edge from

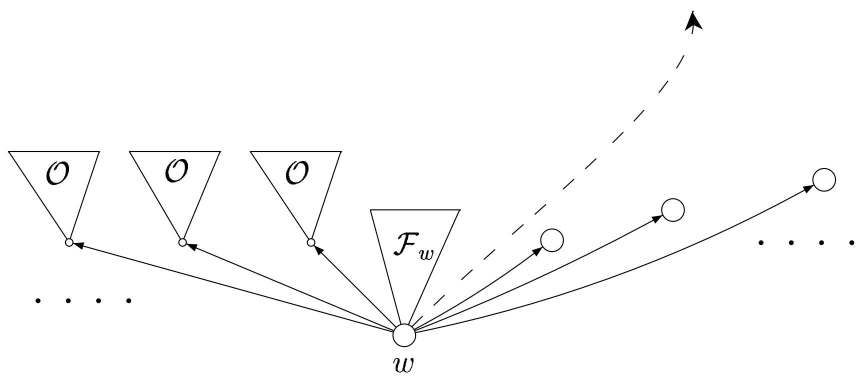

Figure F.2: Partial schematic of the successor tree $\mathcal{B}_T$ above a node w. The dotted path represents the computation of $\mathcal{M}_T$ on input w.

wf to wg in case there is one from f to g in $\mathcal{O}$. Note that $\mathcal{O}_w$ is a successor tree; and we may suppose that its root is w.

The graph $\mathcal{C}'$. Let $\mathcal{C}$ be an automatic copy of the configuration space $\mathcal{C}(\mathcal{M}_T)$ with inputs $w \in N$ identified with their corresponding initial configuration $(\iota, (w, \lambda), (\lambda, \lambda))$. Define $\mathcal{C}'$ as the union (*not disjoint union*) of $\mathcal{C}$ with the $\mathcal{F}_w$'s for $w \in N$. Then $\mathcal{C}'$ consists of the configuration space and attached to every initial configuration w is a copy of $\mathcal{F}$.

The graph $\mathcal{B}_T$. Define $\mathcal{B}_T$ as the union (*not disjoint union*) of the automatic graphs $\mathcal{C}'$ and each $\mathcal{O}_w$ for $w \in N$. The resulting graph is automatic and is a successor tree.

The graph $\mathcal{A}_T$. Now take the disjoint union of $\mathcal{B}_T$ and an automatic copy of infinitely many chains of every length, finite and infinite. The resulting graph is automatic.

This completes the definition of $\mathcal{A}_T$.

Property F.3.6 *For every $w \in N$, w is rejected by $\mathcal{M}_T$ if and only if there is an infinite isolated chain in $\mathcal{A}_T$ with base w. Hence T is exactly those elements of $\mathcal{A}_T$ that are not the base of an infinite isolated chain but are the base of infinitely many finite chains of every length.*

Indeed, let $w \in N$ be a node in the successor tree $\mathcal{T}$. Then w is accepted by $\mathcal{M}_T$ and so there is a finite computation, say of length n, in $\mathcal{C}(\mathcal{M}_T)$ starting at the initial state $c = (\iota, (w, \lambda), (\lambda, \lambda))$. Hence $\mathcal{B}_T$ contains a finite chain of length n with base w. But there are infinitely many finite isolated chains of length n with base w by construction, namely those in $\mathcal{F}_w$, and in particular there is no infinite isolated chain with base w. Conversely suppose x is an element of $\mathcal{A}_T$ that is not the base of an infinite chain, but is the base of infinitely many finite chains of every length. Then by construction x must be in N. Moreover, x is in T since the computation of $\mathcal{M}_T$ on x is finite. This completes the proof of the property.

Property F.3.7 *Say $w \in N$ is not in T. Then the successor tree above w is isomorphic to $\mathcal{O}$.*

Indeed suppose $w \in N$ is not in T. Then since T is downward closed, for every $v \in N$ above w it holds that v is not in T. Hence for every $v \in N$ in the successor tree above w, by the previous property, v is the base for infinitely many finite isolated chains of every finite length, and exactly one infinite isolated chain. The property now follows by the construction of $\mathcal{B}_T$ and $\mathcal{O}$.

It remains to prove the following property.

Property F.3.8 *$\mathcal{T}_1$ is isomorphic to $\mathcal{T}_2$ if and only if $\mathcal{A}_{T_1}$ is isomorphic to $\mathcal{A}_{T_2}$.*

Suppose ϕ is an isomorphism between $\mathcal{A}_{T_1}$ and $\mathcal{A}_{T_2}$. Note that for every $k \leq \omega$, w is the base of an isolated chain of $\mathcal{B}_{T_1}$ of length k if and only if $\phi(w)$ is the base of an isolated chain of $\mathcal{B}_{T_2}$ of length k. Hence by the first property ϕ is an isomorphism between successor trees $\mathcal{T}_1$ and $\mathcal{T}_2$.

Conversely suppose ϕ is an isomorphism between $\mathcal{T}_1$ and $\mathcal{T}_2$, considered as subgraphs of $\mathcal{A}_{T_1}$ and $\mathcal{A}_{T_2}$ respectively. We extend ϕ to an isomorphism from $\mathcal{A}_{T_1}$ to $\mathcal{A}_{T_2}$ as follows. First extend ϕ to the elements of $A_{T_i} \setminus B_{T_i}$. Indeed, each such set consists of the disjoint union of infinitely many isolated chains of every length (finite and infinite), independently of the chains of the configuration graphs that do not correspond to initial configurations. Now we extend ϕ to an isomorphism from the successor tree $\mathcal{B}_{T_1}$ to the successor tree $\mathcal{B}_{T_2}$ as follows.

If $w \in T_1$ then there is an isomorphism between all the isolated chains with base w and all the isolated chains with base $\phi(w)$. Indeed each consists of finitely many chains of every finite length and no infinite chain, independently of the possibly different lengths of the computation of w in $\mathcal{M}_{T_1}$ and $\phi(w)$ in $\mathcal{M}_{T_2}$. Hence extend ϕ to the isolated chains defined by elements of T_1 to the isolated chains defined by elements of T_2.

Now suppose $w \in T_1$ and let S_1 be the set of immediate successors of w that are initial configurations (namely in N) but not in T_1. Similarly write S_2 for the set of immediate successors of $\phi(w)$ that are initial configurations but not in T_2. It may be the case that the cardinalities of S_1 and S_2 are different. However both w and $\phi(w)$ each have infinitely many immediate successors that are not in S_1 and S_2 respectively. Moreover each such successor is the root of a copy of the successor tree $\mathcal{O}$. Hence, by the second property, extend ϕ from the immediate successors of w that are not in T_1 (and the successor trees above them) to the immediate successors of $\phi(w)$ that are not in T_2 (and the successor trees above them). This completes the proof of the property and the theorem is proved. $\lhd$

Corollary F.3.9 *The complexity of the isomorphism problem for automatic successor trees is Σ_1^1-complete.*

Proof Adapt the construction of the previous theorem as follows. Consider the graph $\mathcal{A}_T$ and recall that it a forest of successor trees; namely $\mathcal{B}_T$, with root λ, and infinitely many isolated chains of every length. The roots of the isolated chains are definable in $\mathcal{A}_T$. So identify them and form a successor tree; call the root r. Now place an edge from r to λ. Call the resulting structure $\mathcal{D}_T$ and note that it is an automatic successor tree. Finally note that $\mathcal{A}_{T_1}$ is isomorphic to $\mathcal{A}_{T_2}$ if and only if $\mathcal{D}_{T_1}$ is isomorphic to $\mathcal{D}_{T_2}$. $\triangleleft$

Most of the content of this chapter appears in Khoussainov et al. [2004].

Chapter G

Intrinsic regularity

This chapter investigates the relationship between regularity and definability in automatic structures. A relation is called *intrinsically regular* in an automatically presentable structure $\mathcal{A}$ if it is regular in every automatic presentation of $\mathcal{A}$. Although the definition of intrinsic regularity is new, the topic previously received much attention in some specific cases. Of particular note is the work of Cobham, Semenov, Muchnik, Bruyére et al. that investigates the relationship between regular relations of (coded) natural numbers and definability in certain fragments of arithmetic; see Bruyère et al. [1994] for a good exposition.

The definition of intrinsic regularity mimics that of intrinsically computable relations from Ash and Nerode [1981]. This is concerned with understanding the relationship between definability and computability (see Ershov et al. [1998, Chapter 3] for the current state of the area). For a computable structure $\mathcal{A}$, that is one whose atomic diagram is a computable set, a relation R is *intrinsically computably enumerable* if in all computable isomorphic copies of $\mathcal{A}$ the relation R is computably enumerable. Ash and Nerode [1981] show that under some natural conditions on $\mathcal{A}$, the relation R is intrinsically computably enumerable if and only if it is definable as an effective disjunction of existential formulae. One may therefore regard the topic of this chapter as a refined version of the Ash-Nerode program in which the class of automatic structures is considered rather than the class of all computable structures.

G.1 Preliminaries

Definition G.1.1 *A relation R on the domain of an automatically presentable structure $\mathcal{A}$ is called* **intrinsically regular in $\mathcal{A}$** *if the image of R under every isomorphism from $\mathcal{A}$ to an automatic structure $\mathcal{B}$ is finite automaton recognisable. Write* $\mathrm{IR}(\mathcal{A})$ *for the class of intrinsically regular relations in $\mathcal{A}$.*

In order to show that a particular relation is intrinsically regular in a given automatically presentable structure, one needs to provide a mechanism for extracting an automaton recognising the relation from automatic presentations of the structure. A perfect illustration of this is the subset like construction of Theorem B.1.26 which is restated in the present terminology in the next theorem.

Proposition G.1.2 *If $R \subset A^n$ is first order definable with the additional quantifiers $\exists^\infty$ and $\exists^{(k,m)}$ in an automatically presentable structure $\mathcal{A}$, then R is intrinsically regular in $\mathcal{A}$.*

Proof Let $\mathcal{B}$ be an automatic presentation of $\mathcal{A}$. Then use the definition of R to extract an automaton, from the automata in the presentation $\mathcal{B}$, that recognises R. $\lhd$

For instance every set of the form $\{x \in \mathbb{N} \mid x \equiv k \pmod{m}\}$ is intrinsically regular in $(\mathbb{N}, \leq)$ since each is definable as $(\exists^{(k,m)} y)\,[y < x]$.

Write $\mathrm{FO}^{\infty,\mathrm{mod}}(\mathcal{A})$ for the class of relations that are first order definable with $\exists^\infty$ and $\exists^{(k,m)}$ in $\mathcal{A}$. Then the previous proposition can be summarised as

$$\mathrm{FO}^{\infty,\mathrm{mod}}(\mathcal{A}) \subset \mathrm{IR}(\mathcal{A}).$$

Proposition G.1.3 *If $\mathcal{A}$ is automatically presentable then* $\mathrm{IR}(\mathcal{A})$ *is closed under union, complementation and projection.*

Proof Suppose $R_0, R_1 \subset A^n$ are n–ary intrinsically regular relations in $\mathcal{A}$. Let $\mathcal{B}$ be an automatic presentation of $\mathcal{A}$, and let ν be the isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Then the structure $(\mathcal{B}, \nu(R_0), \nu(R_1))$ is automatic. Since regular relations are closed under finite union, complementation and projection, we have that the images $\nu(R_0 \cup R_1), \nu(A^n \setminus R)$ and the projection of the i-th co-ordinate $\nu(\pi_i(R))$ are also regular in $\mathcal{B}$. $\lhd$

If $\mathcal{B}$ is a reduct of $\mathcal{A}$ and every relation in $\mathcal{A}$ not in $\mathcal{B}$ is $\mathrm{FO}^{\infty,\mathrm{mod}}$ definable in $\mathcal{B}$, then call $\mathcal{A}$ a $\mathrm{FO}^{\infty,\mathrm{mod}}$ *definitional expansion* of $\mathcal{B}$.

Proposition G.1.4 *Suppose $\mathcal{A}$ is automatically presentable.*

1. *If $\mathcal{B}$ is a reduct of $\mathcal{A}$ then* $\mathrm{IR}(\mathcal{B}) \subset \mathrm{IR}(\mathcal{A})$.
2. *If $\mathcal{A}$ is a* $\mathrm{FO}^{\infty,\mathrm{mod}}$ *definitional expansion of $\mathcal{B}$ then* $\mathrm{IR}(\mathcal{A}) = \mathrm{IR}(\mathcal{B})$.

Proof First note that the property of being automatically presentable is preserved under taking reducts and $\mathrm{FO}^{\infty,\mathrm{mod}}$ definitional expansions.

item 1. Let R be a relation on $B (= A)$ and suppose $R \in \mathrm{IR}(\mathcal{B})$. Let $\mathcal{C}$ be an automatic presentation of $\mathcal{A}$, with corresponding isomorphism $\nu : \mathcal{A} \to \mathcal{C}$. By forgetting the automata that recognise those atomic relations that are in $\mathcal{A}$ and not in $\mathcal{B}$, produce an automatic presentation $\mathcal{C}'$ of $\mathcal{B}$, and note that the corresponding isomorphism is the restriction of ν to $\mathcal{B}$. Since $R \in \mathrm{IR}(\mathcal{B})$, the relation $\nu(R)$ on $C' (= C)$ is regular. But since $\mathcal{C}$ was arbitrary, $R \in \mathrm{IR}(\mathcal{A})$.

item 2. By the previous item it is sufficient to establish that

$$\mathrm{IR}(\mathcal{A}) \subset \mathrm{IR}(\mathcal{B}).$$

Let R be a relation on $A(= B)$ and suppose $R \in \mathrm{IR}(\mathcal{A})$. Let $\mathcal{C}$ be an automatic presentation of $\mathcal{B}$, with corresponding isomorphism $\nu : \mathcal{B} \to \mathcal{C}$. Extend $\mathcal{C}$ to an automatic presentation $\mathcal{C}'$ of $\mathcal{A}$ by producing automata for the missing atomic relations using the $\mathrm{FO}^{\infty,\mathrm{mod}}$ definitions as in Theorem B.1.26. Note that the corresponding isomorphism from $\mathcal{A}$ to $\mathcal{C}'$ is the one induced by $\nu : B \to C(= C')$. Since $R \in \mathrm{IR}(\mathcal{A})$, the relation $\nu(R)$ on C is regular. But since $\mathcal{C}$ was arbitrary, $R \in \mathrm{IR}(\mathcal{B})$. $\lhd$

For instance we have the following:

$$\mathrm{IR}(\mathbb{N}) \subset \mathrm{IR}(\mathbb{N}, S) \subset \mathrm{IR}(\mathbb{N}, \leq) = \mathrm{IR}(\mathbb{N}, \leq, (\equiv_n)_{n\in\mathbb{N}}) \subset \mathrm{IR}(\mathbb{N}, +) \subset \mathrm{IR}(\mathbb{N}, +, |_2).$$

The equality holds since $\equiv_n \in \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N}, \leq)$ as follows:

$$x \equiv_n x' \iff \bigvee_{0\leq i<n} \left[(\exists^{(i,n)} y)\,[y < x] \wedge (\exists^{(i,n)} y')\,[y' < x']\right].$$

G.2 Some characterisations of IR

In this section we characterise the intrinsically regular relations of certain fragments of arithmetic. Each result can be summarised as saying that the intrinsically regular relations in $\mathcal{A}$ are exactly those that are first order definable with $\exists^\infty$ and $\exists^{(k,m)}$ in $\mathcal{A}$. Recall that $\mathrm{FO}^{\infty,\mathrm{mod}}(\mathcal{A}) \subset \mathrm{IR}(\mathcal{A})$ for $\mathcal{A}$ automatically presentable.

For each $m > 1$ consider the presentation $\mathcal{A}_m$ of $\mathbb{N}$ over the alphabet $\Sigma_m = \{0, \ldots, m-1\}$. Here the natural number $n \in \mathbb{N}$ is represented in A_m as the shortest least significant digit first base m-representation. The structure $\mathcal{A}_m = (A_m, +_m, |_m)$ is automatic and is isomorphic to $(\mathbb{N}, +, |_m)$.

Denote the unary relation $\{x \in \mathbb{N} \mid x \equiv 0 \pmod{m}\}$ by mult_m.

Proposition G.2.1 *For every $m > 1$,*

$$\mathrm{IR}(\mathbb{N}, +, |_m) = \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N}, +, |_m).$$

Proof Fix $m > 1$. Let $R \in \mathrm{IR}(\mathbb{N}, +, |_m)$ and $R^{(m)}$ be the image of R in $\mathcal{A}_m$. Then by the definition of intrinsic regularity $R^{(m)}$ is FA recognisable over Σ_m. Then since $(\mathbb{N}, +, |_m)$ is complete for the regular relations (Theorem C.2.6), $R^{(m)}$ is first order definable in $\mathcal{A}_m$. Hence $R \in \mathrm{FO}(\mathbb{N}, +, |_m)$. $\lhd$

Proposition G.2.2

$$\mathrm{IR}(\mathbb{N}, +) = \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N}, +).$$

Proof Let $R \subset \mathbb{N}^n$ with $R \in \mathrm{IR}(\mathbb{N},+)$. Then the image $R^{(m)}$ of R in $(A_m, +_m)$ is regular. The Cobham-Semenov Theorem (see Bruyère et al. [1994]) states that if both $R^{(k)}$ and $R^{(l)}$ are regular for multiplicatively independent k and l, then R is definable in $(\mathbb{N},+)$. Here k and l are multiplicatively independent if there are no integers $m, n \geq 1$ such that $k^m = l^n$. Hence $R \in \mathrm{FO}(\mathbb{N},+)$ and so $\mathrm{IR}(\mathbb{N},+) = \mathrm{FO}(\mathbb{N},+)$. $\lhd$

Note that in both of the previous cases, $\mathrm{IR}(\mathcal{A}) = \mathrm{FO}(\mathcal{A})$. Here is an example where the $\exists^{(m,k)}$ quantifiers are essential.

Proposition G.2.3

$$\mathrm{IR}(\mathbb{N},\leq) = \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N},\leq).$$

Proof It is sufficient to establish that

$$\mathrm{IR}(\mathbb{N},\leq) \subset \mathrm{FO}(\mathbb{N},\leq,(\equiv_n)_{n\in\mathbb{N}}) \subset \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N},\leq).$$

For the first containment, write $\mathcal{U}$ for the presentation of $(\mathbb{N},\leq,(\equiv_n)_{n\in\mathbb{N}})$ over the alphabet $\Sigma = \{1\}$, namely the one induced by coding $n \in \mathbb{N}$ by 1^n. Recall Theorem C.2.9 that says that $\mathcal{U}$ is complete for the unary automatic structures. Let $R \in \mathrm{IR}(\mathbb{N},\leq)$ and write R' for the image of R in $\mathcal{U}$. By intrinsic regularity, $(\mathcal{U}, R')$ is automatic over $\{1\}$. So R is first order definable in $(\mathbb{N},\leq,(\equiv_n)_{n\in\mathbb{N}})$.

For the second containment note that $\equiv_n \in \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N},\leq)$. $\lhd$

Although we conjecture that $\mathrm{IR}(\mathbb{N},S) = \mathrm{FO}^{\infty,\mathrm{mod}}(\mathbb{N},S)$, we only have the following results.

Theorem G.2.4 *For every $k \geq 2$, there is an automatic presentation of $(\mathbb{N},S)$ in which the image of the set* mult_k *is not regular.*

Proof Fix $k \geq 2$ and let $\Sigma = \{0,1,\ldots,k-1\}$. We construct an automatic structure $(\Sigma^\star, f)$ isomorphic to $(\mathbb{N},S)$. To do this, for any given string $x \in \Sigma^\star$, we introduce the following auxiliary notations: $ep(x)$ is the string represented by bits of x at even positions; $op(x)$ is the string represented by bits of x at odd positions; n and m are the lengths of strings $ep(x)$ and $op(x)$, respectively. We may also treat $ep(x)$ and $op(x)$ as natural numbers written in least-significant-digit-first base k, and in particular perform addition on them. For example, if $x = 0111001$ then $ep(x) = 0101$, $op(x) = 110$, $n = 4$ and $m = 3$; note that $m \leq n \leq m+1$ and $|x| = m+n$. We may regard the string x as the ordered pair of strings, written $\langle ep(x), op(x)\rangle$, and think of $op(x)$ as a parameter. Call strings x for which $ep(x) = k^{n-1}$ *midpoints* and strings for which $ep(x) = 0$ modulo k^n *startpoints*. Now we describe rules defining the function f. In brackets [[like this]] we explain the meaning of each rule if needed. We note in advance that all arithmetic is performed modulo k^n. Define an auxiliary function $\mathrm{next}(x) = ep(x) + (k \times op(x)) + k - 1$ modulo k^n.

1. If $n \leq 2$ then $f(x)$ is the successor of x with respect to length-lexicographic ordering.

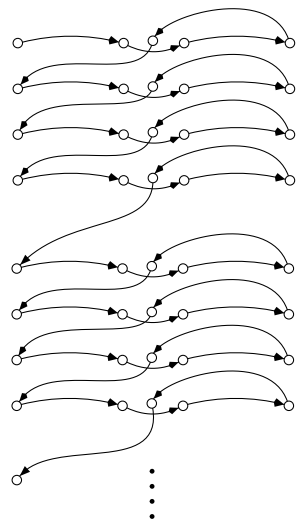

Figure G.1: Schematic of the function f.

2. If $\langle \mathrm{next}(x), op(y)\rangle$ is neither a midpoint nor a startpoint then $f(x) = y$, where $ep(y) = \mathrm{next}(x)$ and $op(y) = op(x)$. [[This is the generic case according to which the successor of the string x, regarded as the pair $\langle ep(x), op(x)\rangle$, is $\langle \mathrm{next}(x), op(x)\rangle$.]]

3. If $\langle \mathrm{next}(x), op(y)\rangle$ is a midpoint then $f(x) = y$, where $|y| = |x|$, $ep(y) = ep(x) + \mathrm{next}(\mathrm{next}(x))$ modulo k^n and $op(y) = op(x)$. [[This case says that if adding $\mathrm{next}(x)$ to $ep(x)$ produces a midpoint then the midpoint should be skipped. Note that $ep(y) = ep(x) + 2\mathrm{next}(x)$.]]

4. If $\langle \mathrm{next}(x), op(x)\rangle$ is a startpoint then $f(x) = y$, where $|y| = |x|$, $ep(y) = k^{n-1}$ and $op(y) = op(x)$. [[The successor of the endpoint is the midpoint.]]

5. If $\langle ep(x), op(x)\rangle$ is a midpoint and $op(x) < k^m - 1$ then $f(x) = y$, where $|y| = |x|$, $ep(y) = 0$ and $op(y) = op(x) + 1$ modulo k^n. [[This is the case when the parameter $op(x)$ is incremented by 1, and the string $ep(x)$ is initialised to the string consisting of n zeros.]]

6. If $\langle ep(x), op(x)\rangle$ is a midpoint and $op(x) = k^m - 1$ then $f(x) = 0^{n+m+1}$. [[This is the only case when the length of string x increases by one.]]

Now we explain how f acts. Fix $b \in \mathbb{N}$ congruent to $k-1$ modulo k. For every $a \in \mathbb{N}$ there is a unique number $c \in \{0, 1, \ldots, k^n - 1\}$ such that $a = b \cdot c$ modulo k^n. In other words, every element $c \in \{0, 1, \ldots, k^n - 1\}$ appears exactly once in the sequence $0, b, 2b, 3b, \ldots, (k^n - 1)b$, where elements are taken modulo k^n. Moreover, $k^{n-1}b$ equals k^{n-1} modulo k^n. Hence, $k^{n-1}b$ appears in the middle of this sequence. Let us assume that x is such that $ep(x) = 0$ and let $b = kop(x) + k - 1$. Then by rules 2, 5 and 6, the function f consecutively applied $k^n - 1$ times to $\langle 0, op(x)\rangle$ produces the following sequence:

$$\begin{aligned}\langle 0, op(x)\rangle\, \langle b, op(x)\rangle \quad &\ldots \quad \langle k^{n-1} - b, op(x)\rangle\, \langle k^{n-1} + b, op(x)\rangle\, \ldots \\ &\ldots \quad \langle k^n - b, op(x)\rangle\, \langle k^{n-1}, op(x)\rangle.\end{aligned}$$

Note that the midpoint $\langle k^{n-1}, op(x)\rangle$ has been removed from the middle of the sequence

$$\langle 0, op(x)\rangle\,\langle b, op(x)\rangle\ \ldots\ \langle k^n - b, op(x)\rangle,$$

and placed at the end. Finally rules 3 and 4 imply that f applied to the last string v in the sequence produces the string $\langle 0, op(x)+1\rangle$ if $op(x) \neq k^m - 1$; otherwise $f(v) = 0^{n+m+1}$. This completes the description of f.

The function f is FA recognisable because all the rules used in the definition of f be can tested by finite automata. It can be checked that $(\Sigma^\star, f)$ is isomorphic to $(\mathbb{N}, S)$, say via mapping $\pi : \Sigma^\star \to \mathbb{N}$.

Our goal is to show that the image of the set $\text{mult}_k = \{x \mid x \text{ is a multiple of } k\}$ is not regular in the described automatic presentation of $(\mathbb{N}, S)$. For this we need to have a finer analysis of the isomorphism π from $(\Sigma^\star, f)$ to $(\mathbb{N}, S)$. Denote by x' the string $\langle 0, op(x)\rangle$. One can inductively check the following for the case that $n \geq 3$.

1. The number $\pi(x')$ is congruent to 0 modulo k for all non-empty strings x.
2. There is a unique $u \leq k^n - 1$ such that $ep(x) = u \cdot (kop(x) + k - 1)$ modulo k^n. Moreover:
   (a) If $u < k^{n-1}$ then $\pi(x) = \pi(x') + u$.
   (b) If $u > k^{n-1}$ then $\pi(x) = \pi(x') + u - 1$.
   (c) If $u = k^{n-1}$ then $\pi(x) = \pi(x') + k^n - 1$.
3. If $ep(y) = 0$ and $op(y) = op(x') + 1 \leq k^m - 1$ then $\pi(y) = \pi(x') + k^n$.

Thus, from the above it is easy to see that x is in the image of mult_k if and only if either $u < k^{n-1}$ and u is congruent to 0 modulo k or $u \geq k^{n-1}$ and u is congruent to 1 modulo k. In order to show that the image of mult_k is not regular, consider all the strings x such that n is odd, $ep(x) = 1^n$ (its numerical value is $k^{n+1} - 1$), $op(x) = 0^{m-r}1^r$ (its numerical value is $k^{r+1} - 1$, so that $kop(x) + k - 1 = k^{r+2} - 1$), and $n > r + 4$. Then under these premises for every $r \in \mathbb{N}$ the minimal $n \in \mathbb{N}$ for which $x \in \pi(\text{mult}_k)$ is when $n = 2r + 5$:

Indeed, $(k^{n-1} + k^{r+2} + 1) \cdot (k^{r+2} - 1) = k^{2r+4} - k^{n-1} - 1$ modulo k^n. So under the assumption that $n = 2r + 5$, this is equal to $-1 = ep(x)$ modulo k^n. Hence $u = k^{n-1} + k^{r+2} + 1 > k^{n-1}$ and so by item 2b above conclude that $\pi(x) = \pi(x') + k^{n-1} + k^{r+1}$ and so $x \in \pi(\text{mult}_k)$.

For the converse, $(k^{r+2} + 1) \cdot (k^{r+2} - 1) = k^{2r+4} - 1$ modulo k^n. Hence under the assumption that $n < 2r + 5$, this is equal to $-1 = ep(x)$ modulo k^n. Now if further $r + 3 < n - 1$, then $u = k^{r+2} + 1 < k^{n-1}$, and so by item 2a above conclude that $\pi(x) = \pi(x') + k^{r+2} + 1$ and so $x \notin \pi(\text{mult}_k)$.

Now we can check that $\pi(\text{mult}_k)$ is not regular. Note that in the presence of $n = 2r + 5$ the assumption that $n > r + 4$ is redundant since $n \leq r + 4$ implies that $r \leq -1$ which contradicts that $r \in \mathbb{N}$. So consider the non regular set

$$Y = \{x \in \Sigma^\star \mid ep(x) = 1^n, op(x) = 0^{m-r}1^r, n = 2r + 5\}.$$

It can be defined from $\pi(\text{mult}_k)$ as the set of all $x \in \Sigma^\star$ such that $ep(x) = 1^n$, for some odd n, $op(x) = 0^{m-r}1^r$ for some $m, r \in \mathbb{N}$, $n > r + 4$, $x \in \pi(\text{mult}_k)$ and if $r + 4 < s < n$ then $(1^s, op(x)) \notin \pi(\text{mult}_k)$. But since Y is not regular, neither is $\pi(\text{mult}_k)$, as required. $\lhd$

So we have established that the sets mult_k are not intrinsically regular relations in $(\mathbb{N}, S)$. Since mult_k is in $\text{FO}^{\infty,\text{mod}}(\mathbb{N}, \leq)$, it is the case that $\leq$ is also not intrinsically regular in $(\mathbb{N}, S)$.

The theory $\text{FO}(\mathbb{N}, S)$ admits quantifier elimination, see for instance Enderton [1972, Theorem 32A]. Hence $R \subset \mathbb{N}^n$ is first order definable in $(\mathbb{N}, S)$ if and only if R can be formed using the operations of union, intersection and complementation from sets of the form

$$\{(x_1, \cdots, x_n) \mid x_i + k = x_j\},\ k \in \mathbb{N}.$$

In particular note that if $R \subset \mathbb{N}$ then R is either finite or co-finite.

A set $R \subset \mathbb{N}$ is *eventually periodic* if there exists $p \in \mathbb{N}$ (the period), and $t \in \mathbb{N}$ (the threshold) such that for all $m \geq t$, $m \in R \iff m + p \in R$. In case that $t = 0$, R is called *periodic*. And if R is periodic with period p then R is periodic with period pk for every integer $k \geq 1$.

Corollary G.2.5 *A unary relation $R \subset \mathbb{N}$ is intrinsically regular in $(\mathbb{N}, S)$ if and only if it is in* $\text{FO}^{\infty,\text{mod}}(\mathbb{N}, S)$.

Proof The reverse direction is immediate. For the forward direction it is sufficient to prove that if $R \subset \mathbb{N}$ is intrinsically regular in $(\mathbb{N}, S)$ then it is finite or co-finite; in this case it is in $\text{FO}(\mathbb{N}, S)$ and so certainly in $\text{FO}^{\infty,\text{mod}}(\mathbb{N}, S)$. We will shortly prove if R is an eventually periodic set, and if it is infinite and co-infinite, then there is some period p of R such that mult_p is first order definable $(\mathbb{N}, S, R)$. Assuming this for now proceed as follows. Let $R \subset \mathbb{N}$ be intrinsically regular in $(\mathbb{N}, S)$. Since $(1^\star, \otimes\{(1^n, 1^{n+1}) \mid n \in \mathbb{N}\})$ is an automatic presentation of (N, S), R must be eventually periodic. If R is finite or co-finite we are done. Otherwise R is regular in every presentation of $(\mathbb{N}, S)$ and using the fact there exists a period p of R such that mult_p is first order definable in $(\mathbb{N}, S, R)$ we get that mult_p is also intrinsically regular in $(\mathbb{N}, S)$ contradicting the previous theorem.

So all that is left is to prove the required property. Let $R \subset \mathbb{N}$ be an eventually periodic set with period p and threshold t. Let $\{n_1, \cdots, n_k\}$ be $R \cap [t, t + p - 1]$. Consider the periodic set $R' = \cup_i [n_i]_p \subset \mathbb{N}$ where $[n_i]_p := \{x \in \mathbb{N} \mid x \equiv n_i \text{ modulo } p\}$. Note that R' is equal to R except possibly on finitely many elements (namely those less than or equal t). Hence R is first order definable in $(\mathbb{N}, S, R')$ if and only if R' is first order definable in $(\mathbb{N}, S, R)$. Also R is infinite and co-infinite if and only if R' is.

Hence we may assume without loss of generality that $R \subset \mathbb{N}$ is an infinite and co-infinite *periodic* set with period p. We aim to establish the existence of a period q of R such that mult_q is first order definable in $(\mathbb{N}, S, R)$.

First we find the period. Write $R = [n_1]_p \cup \cdots \cup [n_k]_p$ with $1 < n_1 < n_2 < \cdots < n_k \leq p$ and $1 \leq k < p$. Given $1 \leq x \leq p$ define $R + x$ as $[n_1 + x]_p \cup \cdots \cup [n_k + x]_p$. Consider the equation $R+x=R$. Note that a is a solution if and only if every element in $[a]_p$ is a solution. It has $x=p$ as a solution. Suppose that $1 \leq q < p$ is also a solution. Then $y \in R$ if and only if $y+q \in R$. So R is periodic with period $q<p$. Rewrite $R=[m_1]_q \cup \cdots \cup [m_l]_q$ with $1 \leq m_1 < \cdots < m_l \leq q$ and $1 \leq l < q$, and repeat this procedure until the only solution to $R+x=R$ in the range $[1,q]$ is $x=q$. The resulting q is the required period. Note that since R is infinite and co-infinite $1 \leq l < q$.

Now $\mathrm{mult}_q = [0]_q = \{x \in \mathbb{N} \mid R+x=R\}$. Indeed 0 and hence every element in $[0]_q$ is a solution of the equation. Conversely, if $R+x=R$ then every $y \in [x]_q$ is also a solution and in particular so is y satisfying $1 \leq y \leq q$; then by minimality of q we have $y=q$ and hence $x \in \mathrm{mult}_q$.

We claim that mult_q is first order definable in $(\mathbb{N}, S, R)$ as

$$\{x \in \mathbb{N} \mid \bigwedge_i S^{m_i}(x) \in R \wedge \bigwedge_j S^{m'_j}(x) \notin R\}$$

where $\{m'_1, \cdots, m'_{l'}\} = [1,q] \cap (\mathbb{N} \setminus R)$. (Note that the m_i's and m'_j's in this formula are fixed and there are only finitely many of them). Indeed if $R+x=R$ then $m+x \in R \iff m \in R$ for every $m \in \mathbb{N}$, and in particular for every $m \leq q$. Conversely if x is such that $m_i + x \in R$ for every $1 \leq i \leq l$ then $[m_i]_q + x \subset R$ for every i and so $R+x \subset R$. Similarly if x is also such that $m'_j + x \notin R$ for every $1 \leq j \leq l'$ then $[m'_j]_q + x \cap R = \emptyset$ for every j and so $(\mathbb{N} \setminus R) + x \subset (\mathbb{N} \setminus R)$. This completes the proof. $\triangleleft$

Finally we provide an automatic presentation of $(\mathbb{N}, S)$ in which $\leq$ is not regular but all the unary relations $\mathrm{mult}_2, \mathrm{mult}_3, \ldots$ are regular. This shows that regularity of each of the sets mult_i and the successor relation S do not imply that the relation $\leq$ is regular.

Theorem G.2.6 *There is an automatic presentation of the structure*

$$(\mathbb{N}, S, \mathrm{mult}_2, \mathrm{mult}_3, \ldots)$$

in which the relation $\leq$ is not regular.

Proof We construct an automatic presentation (D, S_D) of $(\mathbb{N}, S)$, where $D = 0^\star 1^\star$. Define the function S_D in the following manner:

$$S_D(0^n 1^m) = \begin{cases} 0^n 1 & \text{if } m = 0; \\ 0^{n-1} 1^{m+2} & \text{if } m \text{ is odd and } n > 0; \\ 1^{m+1} & \text{if } m \text{ is odd and } n = 0; \\ 0^{n+1} 1^{m-2} & \text{if } m \text{ is even and } m > 0. \end{cases}$$

The function S_D is automatic as all the four conditions defining S_D can be tested by finite automata. To understand the action of S_D consider the input 0^n. Applying S_D to the input $n+2$ times successively we produce the sequence:

$$0^n, 0^n 1, 0^{n-1} 1^3, 0^{n-2} 1^5, 0^{n-3} 1^7, \ldots, 1^{2n+1}, 1^{2n+2}.$$

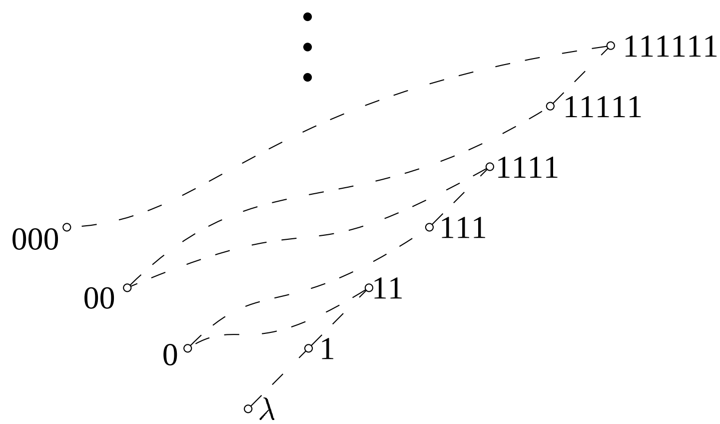

Figure G.2: Schematic of the function S_D.

Applying S_D to 1^{2n} exactly $n+1$ times we produce the next sequence:

$$1^{2n+2}, 0^1 1^{2n}, 0^2 1^{2n-2}, 0^3 1^{2n-4}, \ldots, 0^{n+1}.$$

Note that the function S_D defines the order $\leq_D$ on $0^\star 1^\star$ as follows: $0^n 1^m \leq_D 0^{n'} 1^{m'}$ if and only if there exists a t such that $S_D^t(0^n 1^m) = 0^{n'} 1^{m'}$. This linear order, which is the image of the order $\leq$ on $\mathbb{N}$, is not regular. For otherwise let $\mathcal{A}$ be an automaton recognising $\otimes(\leq_D)$. Then $\otimes(1^{2n}, 0^n)$ is accepted by $\mathcal{A}$ for every n. In particular for n greater than the number of states of $\mathcal{A}$, there exists j such that $\otimes(1^{2n+jk}, 0^n)$ is accepted by $\mathcal{A}$ for every $k \in \mathbb{N}$. This contradicts that there are only finitely many strings x such that $x \leq_D 0^n$.

We need to show that each set mult$_i$ is a regular relation under this presentation of $(\mathbb{N}, S)$. Let π be the isomorphism from (D, S_D) to $(\mathbb{N}, S)$. The lengths of x and $S_D(x)$ differ by 1. Thus, $\pi(0^n 1^m)$ is odd if and only if the length $n+m$ of the string is odd. So, mult$_2$ is a regular language in this presentation of $(\mathbb{N}, S)$.

To show that mult$_k$ is a regular language in this presentation of $(\mathbb{N}, S)$ we argue as follows. Consider the table below that lists all the elements of the domain D ordered by the relation $\leq_D$:

Strings ordered by $\leq_D$
λ,
1,
11, 0
01, 111
1111, 011, 00
001, 0111, 11111
111111, 01111, 0011, 000,
...

This table shows that the equalities $\pi(0^n) = n \cdot (n+2)$ and $\pi(1^{2m}) = m \cdot (m+1)$ for all $n, m \in \mathbb{N}$ hold true. Hence for $0 \leq j < k$, the sets $A_j^k = \{0^n \mid \pi(0^n) \equiv j \pmod{k}\}$ and $B_j^k = \{1^{2m} \mid \pi(1^{2m}) \equiv j \pmod{k}\}$ are regular. Indeed for fixed j and k, A_j^k is definable as the disjunction over the finite set $\{a \mid 0 \leq a < k, \ a(a+2) \equiv j \pmod{k}\}$ of the regular predicates $|0^n| \equiv a \pmod{k}$. By similar reasoning B_j^k is regular. This table also shows the following. For any string $w \in 0^\star 1^\star$, if $w = 0^x 1 1^{2m}$ then $S_D^{m+1}(0^{x+m}) = w$; and if $w = 0^x 1^{2m}$ then $S_D^x(1^{2m+2x}) = w$.

Here is a procedure that decides the set mult_k for any given fixed k. The correctness of the procedure can be proved by using the facts in the preceding paragraph. So, given an input w do the following:

Case 1: Assume that $w = 0^x 1 1^{2m}$ (in this case $S_D^{m+1}(0^{x+m}) = w$.)

(a) Calculate $x + m$ modulo k.

(b) Calculate $\pi(0^{x+m}) + m + 1$ modulo k.

(c) If the last value equals 0 modulo k then $w \in \text{mult}_k$; otherwise $w \notin \text{mult}_k$.

Case 2: Assume that $w = 0^x 1^{2m}$ (in this case $S_D^x(1^{2m+2x}) = w$.)

(a) Calculate $2m + 2x$ modulo k.

(b) Calculate $\pi(1^{2m+2x}) + x$ modulo k.

(c) If the last value equals 0 modulo k then $w \in \text{mult}_k$; otherwise $w \notin \text{mult}_k$.

As the sets A_j^k and B_j^k are regular, the procedure described can be carried out by a finite automaton while processing the string w. We conclude that mult_k is a regular language. Finally note that automata for mult_k can be constructed uniformly in k. Hence $(\mathbb{N}, S, \text{mult}_2, \text{mult}_3, \cdots)$ is automatically presentable. $\triangleleft$

Each of these characterisations seem to require more or less *ad hoc* proof techniques, and so it would be interesting to know whether there is a more general principle. The general problem is to characterise the intrinsically regular relations in $\mathcal{A}$ as those that are definable in a suitable logic in $\mathcal{A}$. A natural conjecture at this point is that the logic is $\text{FO}^{\infty,\text{mod}}$.

Some applications

The proof of Theorem G.2.4 can be used to construct automatic structures with pathological properties. The first application of the results concerns the reachability relation in automatic graphs. The reachability problem for automatic graphs is undecidable (Theorem D.2.11). Such automatic graphs necessarily have infinitely many components. In fact for automatic graphs with finitely many components the reachability problem is decidable. A natural question is whether or not the reachability relation for automatic graphs with finitely many components can be recognised by finite automata. To answer this question, consider the following graph $\mathcal{G} = (\{0,1\}^\star, Edge)$, where $Edge(x,y)$ if and only if $f^2(x) = y$ and f is the function defined in the proof of Theorem G.2.4 for $k = 2$. The graph $\mathcal{G}$ is automatic with exactly two infinite components each being isomorphic to $(\mathbb{N}, S)$. One of the components coincides with mult_2. Hence, we have the following:

Corollary G.2.7 *There is an automatic graph with exactly two components each of which is not regular.*

The second application of this theorem is on the structure $(\mathbb{Z}, S)$. A *cut* is a set of the form

$$\{x \in \mathbb{Z} \mid x \geq n\},$$

where $n \in \mathbb{Z}$ is fixed.

Corollary G.2.8 *There is an automatic presentation of* $(\mathbb{Z}, S)$ *in which no cut is regular.*

Proof It is sufficient to find a presentation of $(\mathbb{Z}, S, 0)$ in which $\{x \in \mathbb{Z} \mid x \geq 0\}$ is not regular since this cut is first order definable from every other cut. We modify the presentation $(\{0,1\}^2, f)$ of $(\mathbb{N}, S)$ in the proof of Theorem G.2.4 for $k = 2$, and construct a function $g : \{0,1\}^\star \to \{0,1\}^\star$ where g is defined using the same notation as before. All arithmetic below is performed modulo 2^n.

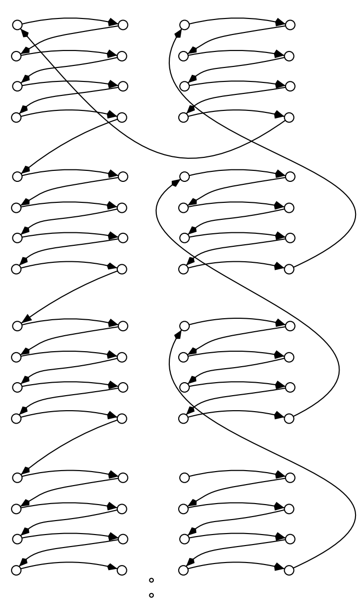

Figure G.3: Schematic of the function g.

1. If $n \leq 2$ then $g(x)$ is the length-lexicographic successor of x.

2. If $(ep(x) + 2op(x) + 1, op(x))$ is neither a midpoint nor a startpoint then $g(x) = y$ with $|x| = |y|$ and $ep(y) = ep(x) + 2op(x) + 1$ and $op(y) = op(x)$.

3. If $(ep(x) + 2op(x) + 1, op(x))$ is a midpoint, then

   (a) if $op(x) < 2^m - 1$ then $g(x) = y$ with $|x| = |y|$ and $ep(y) = 0$ and $op(y) = op(x) + 1$.

   (b) if $op(x) = 2^m - 1$ then $g(x) = y$ with $|y| = |x| + 1$ and $ep(y) = 0$ and $op(x) = 0$.

4. If $(ep(x) + 2op(x) + 1, op(x))$ is a startpoint, then

   (a) if $op(x) < 2^m - 1$ then $g(x) = y$ with $|x| = |y|$ and $ep(y) = 2^{n-1}$ and $op(y) = op(x) + 1$.

   (b) if $op(x) = 2^m - 1$ then

      i. if $n = 3$ and $m = 2$ then $g(x) = \epsilon$. Otherwise,

      ii. if $n = m+1$ then $g(x) = y$ with $|ep(y)| = n-1$, $|op(y)| = m$ and $ep(y) = 2^{n-2}$ and $op(y) = 0$.

      iii. if $n = m$ then $g(x) = y$ with $|ep(y)| = n$ and $|op(y)| = m-1$ and $ep(y) = 2^{n-1}$ and $op(y) = 0$.

Thus, $(\{0,1\}^\star, g, \epsilon)$ is an automatic presentation of $(\mathbb{Z}, S, 0)$ in which the cut above 0 consists of exactly those strings $x \in \{0,1\}^\star$ such that $u < 2^n - 1$ (recall u is the unique number less than 2^n that satisfies $ep(x) = u \cdot (2op(x) + 1)$). But this set of strings is not regular since it can be used to give a first order definition of the image of mult_2 in $(\{0,1\}^\star, f)$. $\triangleleft$

Most of the content of this chapter appears in Khoussainov et al. [2003b].

Chapter H

Open problems

This thesis has only dealt with automata operating on finite words. One may replace finite words with finite trees, for instance, and get the larger class of tree automatic structures. Although some work has been done, a similar investigation for this class is wanting. However there are still many problems in the word case. Here is a sample.

Problem 1 *Which Abelian groups have automatic presentations ? For instance does the group of rationals* $(\mathbb{Q}, +)$ *have an automatic presentation ?*

Problem 2 *Investigate the complexity in the arithmetic hierarchy of the isomorphism problem for classes of automatic structures. For instance, what is the complexity of the isomorphism problem for automatic linear orders ?*

Problem 3 *Characterise the intrinsically regular relations of* $\mathcal{A}$ *in a suitable logic. For instance does it hold that*

$$\mathrm{IR}(\mathcal{A}) = \mathrm{FO}^{\infty,\mathrm{mod}}(\mathcal{A}),$$

for every automatically presentable structure $\mathcal{A}$ *? Or even, is there an automatic presentation of* $(\mathbb{Z}, +)$ *in which* $\leq$ *is not regular ?*

Problem 4 *Which* ω*-categorical structures have automatic presentations ? In particular which Fraïssé limits have automatic presentations ?*

Bibliography

C.J. Ash and A. Nerode. Intrinsically recursive relations. In *Aspects of effective algebra* (*Clayton*, 1979), pages 26–41. Upside Down A Book Co. Yarra Glen, 1981.

M. Benedikt, L. Libkin, T. Schwentick, and L. Segoufin. A Model-Theoretic approach to regular string relations. In *Proceedings of the 16th Annual IEEE Symposium on Logic in Computer Science* (LICS), pages 431–440, 2001.

C. H. Bennett. Logical reversibility of computation. *IBM Journal of Research and Development*, 17(6):525–532, 1973.

A. Blumensath. *Automatic Structures*. Diploma thesis, RWTH Aachen, 1999.

A. Blumensath. Prefix-recognisable graphs and monadic second-order logic, 2001. Tech. Rep. AIB-2001-06, RWTH Aachen.

A. Blumensath and E. Grädel. Automatic structures. In *15th Symposium on Logic in Computer Science* (LICS), pages 51–62, 2000.

A. Blumensath and E. Grädel. Finite presentations of infinite structures : Automata and interpretations. In *Proc. 2nd Int. Workshop on Complexity in Automated Deduction* (CIAD), 2002.

V. Bruyère, G. Hansel, C. Michaux Christian, and R. Villemaire. Logic and p-recognizable sets of integers. *Bull. Belg. Math. Soc. Simon Stevin*, 1(2):191–238, 1994. Journées Montoises (Mons, 1992).

J.R. Büchi. Weak second-order arithmetic and finite automata. *Z. Math. Logik Grundlagen Math.*, 6:66–92, 1960.

J.R. Büchi. On a decision method in restricted second order arithmetic. *Logic, Methodology and Philosophy of Science* (*Proc.* 1960 *Internat. Congr*), pages 1–11, 1962.

J. W. Cannon, D. B. H. Epstein, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston. *Word processing in groups*. Jones and Bartlett, 1992.

D. Cenzer and J.B. Remmel. Complexity-theoretic model theory and algebra. In *Handbook of recursive mathematics, Vol. 1*, volume 138 of *Stud. Logic Found. Math.*, pages 381–513. North-Holland, Amsterdam, 1998.

A. Church. Logic, arithmetic, and automata. In *Proc. Internat. Congr. Mathematicians* (*Stockholm*, 1962), pages 23–35. Inst. Mittag-Leffler, Djursholm, 1963.

C. Delhommé. Non-automaticity of ω^ω, 2001a. manuscript.

C. Delhommé. The rado graph is not automatic, 2001b. manuscript.

C. Delhommé, V. Goranko, and T. Knapik. Automatic linear orderings, 2003.

S. Eilenberg, C.C. Elgot, and J.C. Shepherdson. Sets recognised by n–tape automata. *Journal of Algebra*, 13(4):447–464, 1969.

C.C. Elgot. Decision problems of finite automata design and related arithmetics. *Trans. Amer. Math. Soc.*, 98:21–51, 1961.

C.C Elgot and J.E. Mezei. On relations defined by generalised finite automata. *IBM J. Res. Develop.*, 9:47–68, 1965.

C.C. Elgot and M.O. Rabin. Decidability of extensions of theory of successor. *Journal of Symbolic Logic*, 31(2):169–181, 1966.

H.B. Enderton. *A mathematical introduction to logic*. Academic Press, 1972.

Yu.L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, editors. *Handbook of Recursive Mathematics Volume 1*. Studies in Logic and the foundations of Mathematics. Elsevier, 1998.

M. Fischer and M.0. Rabin. Super-exponential complexity of Presburger arithmetic. In *Complexity of computation (Proc. SIAM-AMS Sympos., New York, 1973)*, pages 27–41. SIAM–AMS Proc., Vol. VII. Amer. Math. Soc., 1974.

S.S. Goncharov. *Countable Boolean algebras and decidability*. Siberian School of Algebra and Logic, 1997.

S.S. Goncharov and J.F. Knight. Computable structure and non-structure theorems. *Algebra and Logic*, 41(6):351–373, 2002.

E. Grädel. Simple interpretations among complicated theories. *Information Processing Letters*, 35(5):235–238, 1990.

F. Hausdorff. Grundzüge einer theorie der geordnete mengen. *Math. Ann*, 65:435–505, 1908.

W. A. Hodges. *Model Theory*. Cambridge University Press, 1993.

B.R. Hodgson. *Théories décidables par automate fini*. Phd thesis, University of Montréal, 1976.

B.R. Hodgson. Decidabilite par automate fini. *Ann. Sci. Math.*, 7:39–57, 1983.

H. Ishihara, B. Khoussainov, and S. Rubin. Some results on automatic structures. In *17th Symposium on Logic in Computer Science* (LICS), pages 235–242, 2002.

J.H. Johnson. Rational equivalence relations. *Theoretical Computer Science*, 47(1):39–60, 1986.

B. Khoussainov and A. Nerode. Automatic presentations of structures. *Lecture Notes in Computer Science*, 960:367–392, 1995.

B. Khoussainov and A. Nerode. *Automata theory and its applications*, volume 21 of *Progress in Computer Science and Applied Logic*. Birkhäuser Boston Inc., Boston, MA, 2001.

B. Khoussainov, A. Nies, S. Rubin, and F. Stephan. Automatic structures: Richness and limitations. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science* (LICS), pages 44–53, 2004.

B. Khoussainov and S. Rubin. Graphs with automatic presentations over a unary alphabet. *Journal of Automata, Languages and Combinatorics*, 6(4):467–480, 2001.

B. Khoussainov and S. Rubin. Automatic structures – overview and future directions. *Journal of Automata, Languages and Combinatorics*, 8(2):287–301, 2003.

B. Khoussainov, S. Rubin, and F. Stephan. Automatic linear orders and trees, 2003a. CDMTCS technical report 208, Department of Computer Science, University of Auckland.

B. Khoussainov, S. Rubin, and F. Stephan. Definability and regularity in automatic presentations of subsystems of arithmetic, 2003b. CDMTCS technical report 209, Department of Computer Science, University of Auckland.

F. Klaedtke. On the automata size for presburger arithmetic. *Technical report No.* 186 *of the Institute of Computer Science at Freiburg University*, 2003.

S. C. Kleene. Representation of events in nerve nets and finite automata. In *Automata studies*, Annals of mathematics studies, no. 34, pages 3–41. Princeton University Press, Princeton, N. J., 1956.

D. Kuske. Is Cantor's Theorem automatic ? *Proceedings of the 10th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning* (LPAR), 2850:332–345, 2003.

L. Libkin and F. Neven. Logical definability and query languages over unranked trees. *Proceedings of the 18th Annual IEEE Symposium on Logic in Computer Science* (LICS), pages 178–187, 2003.

M. Lohrey. Automatic structures of bounded degree. *Proceedings of the 10th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning* (LPAR), 2850: 344–358, 2003.

O. Ly. Automatic graphs and D0L-sequences of finite graphs. *Journal of Computer and System Sciences*, 67(3):497–545, 2003.

A.R. Meyer. Weak monadic second order theory of successor is not elementary-recursive. In *Logic Colloquium* (*Boston, Mass,* $1972 - -1973$), volume 453 of *Lecture Notes in Mathematics*, pages 132–154. Springer, Berlin, 1975.

C. Michaux and F. Point. Les ensembles k-reconnaissables sont définissables dans $\langle \mathbf{N}, +, V_k \rangle$. *C. R. Acad. Sci. Paris Sér. I Math.*, 303(19):939–942, 1986.

A.A. Nabebin. Multitape automata in a unary alphabet. *Trudy Moskov. Orden. Lenin. Ènerget. Inst.*, 292:7–11, 1976.

P. Odifreddi. *Classical recursion theory. The theory of functions and sets of natural numbers*. Studies in Logic and the Foundations of Mathematics, 125. North-Holland, 1989.

P. Odifreddi. *Classical recursion theory. Volume II*. Studies in Logic and the Foundations of Mathematics, 143. North-Holland, 1999.

D.C. Oppen. A superexponential upper bound on the complexity of Presburger arithmetic. *J. Comput. System Sci.*, 16(3):323–332, 1978.

H. Rogers. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967.

J.G. Rosenstein. *Linear Orderings*. Academic Press, 1982.

S. Rubin. Finite automata and well ordered sets. *New Zealand Journal of Computing*, 7(2): 39–46, 1999.

F. Stephan. The random graph is not automatically presentable, 2002. manuscript.

A. Szilard, S. Yu, K. Zhang, and J. Shallit. Characterizing regular languages with polynomial densities. In I.M. Havel and V. Koubek, editors, *Mathematical Foundations of Computer Science 1992, 17th International Symposium*, volume 629 of *Lecture Notes in Computer Science*, pages 494–503, 1992.

J.W. Thatcher. Decision problems for multiple successor arithmetics. *The Journal of Symbolic Logic*, 31:182–190, 1966.

B.A. Trahtenbrot. Finite automata and the logic of one-place predicates. (Russian). *Siberian Mathematical Journal*, 3:103–131, 1962. English translation: American Mathematical Society Translations, Series 2, 59 (1966), 23-55.

Index