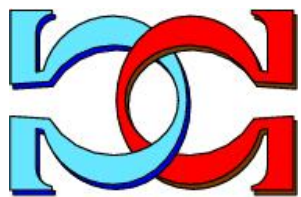
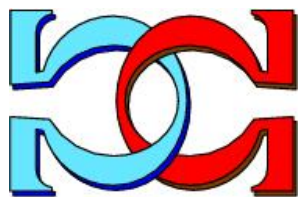
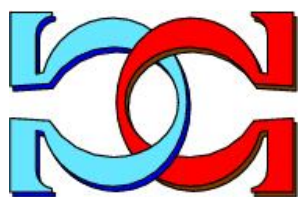
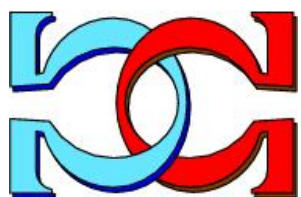


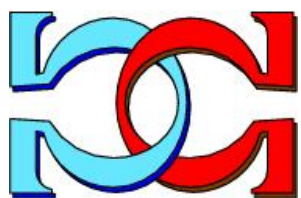
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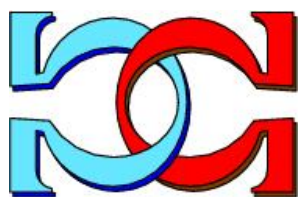
Analytic Continuation and
Randomness of a Generalized
Halting Probability



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Analytic Continuation, Natural Boundaries, and Randomness of a Generalized Halting Probability

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Abstract

Tadaki's generalization of Chaitin's halting probability associates with a self-delimiting Turing machine C the function

$$Z_C(s) = \sum_{p \in \text{dom}(C)} 2^{-s|p|}.$$

For a universal machine, $Z_C(1)$ is Chaitin's Omega number and, for every computable real $s > 1$, $Z_C(s)$ is partially random with compression rate $1/s$. For real $0 < s < 1$ the defining series diverges. We prove two complementary results showing that universality alone does not determine the complex-analytic behavior on the boundary $\Re(s) = 1$. First, starting effectively with an arbitrary universal self-delimiting Turing machine V , we construct a universal machine U for which Z_U can be analytically continued around $s = 1$ to the real interval $0 < s < 1$. The construction uses a positive bilateral Fourier kernel with a spectral gap, followed by a square-root branch carrying a scaled Omega number. For every computable real $s \in (0, 1)$, the absolute value of the imaginary part of either continued branch is a computably enumerable random real. Secondly, from the same arbitrary universal machine V we effectively construct a universal machine W with $\Omega_W = \Omega_V$ for which $\Re(s) = 1$ is a natural boundary. The second construction pairs every short simulation of a V -program with an equal-weight factorially padded block; a radial difference-quotient argument forces singularities at every root of unity. The two principal proofs were discovered by ChatGPT (OpenAI, GPT-5.6 Sol) in September 2026.

1 Introduction

Tadaki [5] introduced the one-parameter generalization of Chaitin's halting probability

$$Z_C(s) = \sum_{p \in \text{dom}(C)} 2^{-s|p|} \tag{1}$$

for a self-delimiting Turing machine C . At the critical value $s = 1$ we recover Chaitin's halting probability

$$\Omega_C = \sum_{p \in \text{dom}(C)} 2^{-|p|}.$$

When C is universal, Ω_C is computably enumerable (c.e.) and random; conversely, every c.e. random real is the halting probability of a universal self-delimiting machine [4, 2, 1]. Tadaki proved that if C is universal and $s > 1$ is computable, then $Z_C(s)$ is Chaitin $1/s$ -random. Calude and Stay [3] studied related generalized halting probabilities and zeta functions.

For a universal machine, the real series defining $Z_C(s)$ converges for $s \geq 1$ and diverges for $0 < s < 1$. It is therefore natural to ask whether universality determines the stronger complex-analytic behavior on the boundary line $\Re(s) = 1$. Must this line be a natural boundary, or can a universal machine have a generalized halting probability which continues through part of it?

We show that both behaviors occur, even when one starts with an arbitrary universal source machine and preserves it with constant overhead. In the first construction, a positive bilateral Fourier kernel produces a universal machine whose normalized length-generating function crosses an open arc of its circle of convergence. A second prefix branch then introduces a square-root singularity carrying a scaled Omega number. The continuation reaches the real interval $0 < s < 1$, and for every computable s there the imaginary part contains a c.e. random real up to a computable nonzero factor.

There is also an effective one-bit refinement of the first construction when the source language comes with a known free cylinder. This situation is natural for length-delimited programming languages: if a complete program x is explicitly known to diverge, and the framing of the language makes every proper prefix incomplete and every proper extension too long, then the cylinder $[x]$ contains no valid halting program. The cylinder gives an effective rational upper bound below one on Ω_V , allowing the Fourier gap to be narrowed until the entire auxiliary mass fits under the single complementary prefix bit. The same cylinder can then host the square-root branch, so the final machine satisfies $U(0p) = V(p)$. For language models whose halting probabilities are asymptotically random in the sense of Calude and Stay [3], the imaginary part of the continuation is asymptotically random as well.

The second construction goes in the opposite direction. Every halting program of an arbitrary universal machine is represented once at essentially its original length and again by a large equal-weight block at a factorially chosen length. At each root of unity the long blocks eventually have phase one. A term-by-term inequality shows that the long block dominates the worst possible cancellation from its short partner in a radial difference quotient. Hence every root of unity is singular. Since the roots of unity are dense, the unit circle is a natural boundary of the normalized generating function, and $\Re(s) = 1$ is a natural boundary of the corresponding Z -function.

The first main result is the following.

Theorem 1. *Let V be a universal self-delimiting Turing machine. We can effectively construct a universal self-delimiting Turing machine U with the following properties:*

1. U contains a fixed-prefix simulation of V ; in the construction below,

$$U(000p) = V(p), \quad p \in \text{dom}(V).$$

2. The function Z_U , initially defined by (1) in $\Re(s) > 1$, admits analytic continuation along paths around $s = 1$ to every real point $s \in (0, 1)$.
3. There are two conjugate continuations to each real $s \in (0, 1)$. For every computable real $s \in (0, 1)$,

$$|\Im(Z_U^{\text{cont}}(s))| = c_s \Omega_{U_0},$$

where U_0 is a universal machine effectively constructed from V and $c_s > 0$ is a computable real.

4. Consequently, for every computable real $s \in (0, 1)$, $|\Im(Z_U^{\text{cont}}(s))|$ is c.e. and random.

The complementary result is stronger than is needed for universality.

Theorem 2. *Let V be a self-delimiting Turing machine with infinitely many halting programs. We can effectively construct a self-delimiting Turing machine W such that*

$$\Omega_W = \Omega_V$$

and the unit circle is a natural boundary of A_W . Equivalently, $\Re(s) = 1$ is a natural boundary of Z_W . If V is universal, then W is universal and

$$H_W(x) = H_V(x) + 1$$

for every output x of V .

The next section places these results in the context of earlier work on generalized halting probabilities, algorithmic partition functions, fractal zeta functions, and natural boundaries. The remaining sections give the two constructions and their proofs, followed by some consequences and open questions.

2 Previous work

Tadaki's generalized halting probability was introduced in [5]. In later work he developed a statistical-mechanical interpretation of algorithmic information theory in which

$$Z(T) = \sum_{p \in \text{dom}(V)} 2^{-|p|/T}$$

plays the role of a partition function and the parameter T is identified with the compression rate of the resulting thermodynamic quantities [6]. Thus, with $T = 1/s$, Tadaki's partition function is the same one-parameter family as (1) on the positive real axis. Subsequent papers proved fixed-point results for partial randomness [7], showed that different optimal computers can give different thermodynamic behavior [8], and studied the change in computational behavior at the critical temperature $T = 1$ [9]. These results concern the real parameter and its algorithmic information content; the present paper asks instead how the same partition function behaves as a complex analytic function and what can occur on its line of convergence.

Calude and Stay [3] introduced natural halting probabilities and zeta numbers of machines, related them to Ω and program-size complexity, and developed notions of partial and asymptotic randomness. Baez and Stay [10] subsequently placed halting programs in a Gibbs-ensemble framework, treating program length (together with other observables) as an analogue of a thermodynamic observable and studying the domain in which the associated partition function converges. Their length-only specialization is, after the change of parameter $\gamma = s \log 2$, the same exponential sum considered here. In particular, this line of work already makes the boundary of convergence algorithmically significant, but it does not address analytic continuation through that boundary or the possibility of a natural boundary in the complex parameter plane.

There is a parallel analytic literature in which zeta functions encode geometric rather than computational data. Lapidus and van Frankenhuysen developed the theory of geometric zeta functions of fractal strings and their complex dimensions [11]. More recently, Lapidus, Radunović and Žubrinić studied meromorphic continuation of distance, tube,

and fractal-string zeta functions and constructed strongly and maximally hyperfractal examples for which the critical line is a natural boundary, or every point of that line is a nonremovable singularity [12]. The objects and coefficients are different from those considered here, but the distinction between a zeta function which crosses its critical line and one whose critical line is a natural boundary is closely analogous to the distinction proved below for generalized halting probabilities.

The natural-boundary construction also belongs to the classical circle of ideas surrounding lacunary power series. Fabry's gap theorem gives broad conditions under which a power series with increasingly sparse exponents has its circle of convergence as a natural boundary [13]. Our factorial padding is reminiscent of this mechanism, but Theorem 2 is not an application of the gap theorem: A_W contains the arbitrary short-program spectrum inherited from V as well as the factorially placed blocks. The roots-of-unity argument is used to prove directly that the long block paired with each short program prevents cancellation and forces a dense set of singularities.

3 Notation and preliminaries

Let $\Sigma = \{0, 1\}$ and let Σ^* be the set of finite binary strings. A self-delimiting Turing machine is a partial computable function $C : \Sigma^* \rightarrow \Sigma^*$ whose domain is prefix-free. Its program-size complexity is

$$H_C(x) = \min\{|p| \mid C(p) = x\}.$$

For such a machine put

$$d_C(n) = \#\{p \in \text{dom}(C) \mid |p| = n\}.$$

The Kraft inequality gives

$$\sum_{n \geq 0} d_C(n) 2^{-n} = \Omega_C \leq 1.$$

A self-delimiting machine U is universal if for every self-delimiting machine C there is a constant $c \geq 0$ such that for every x in the range of C ,

$$H_U(x) \leq H_C(x) + c.$$

Equivalently for our purposes, a universal machine can simulate an arbitrary fixed universal machine with an additive constant in program length. The constructions below preserve universality in the strongest possible way: they contain a literal fixed-prefix copy of the source machine.

For a self-delimiting machine C define the normalized length-generating function

$$A_C(w) = \sum_{n=0}^{\infty} d_C(n) 2^{-n} w^n. \quad (2)$$

Then

$$Z_C(s) = A_C(2^{1-s}). \quad (3)$$

Indeed,

$$A_C(2^{1-s}) = \sum_{n \geq 0} d_C(n) 2^{-n} 2^{(1-s)n} = \sum_{n \geq 0} d_C(n) 2^{-sn}.$$

Thus the line $\Re(s) = 1$ corresponds to the unit circle $|w| = 1$. The point

$$s = 1 + \frac{\pi i}{\log 2}$$

corresponds to $w = -1$.

We use the standard definition of a c.e. real and of a random real. If $\alpha = 0.x_1x_2\cdots$ is in $(0, 1)$, we write $\alpha[n] = x_1\cdots x_n$. A c.e. real α is random if, for one (equivalently, every) universal self-delimiting machine C , there is a constant c such that

$$H_C(\alpha[n]) \geq n - c$$

for every $n \geq 1$. This is equivalent to Martin-Löf randomness for c.e. reals [2, 1].

For a universal machine the unit circle is the actual circle of convergence of (2). Indeed, the density theorem for universal self-delimiting machines proved by Calude and Stay [3] implies

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log_2 \#\{p \in \text{dom}(C) \mid |p| \leq n\} = 1.$$

Consequently, for every real $0 < s < 1$, $Z_C(s)$ diverges, while Kraft's inequality gives convergence at $s = 1$. Hence the radius of convergence of A_C is exactly one.

We recall Tadaki's result in the notation used here.

Theorem 3 (Tadaki [5]). *Let C be a universal self-delimiting Turing machine and let $s > 1$ be computable. Then $Z_C(s)$ is Chaitin $1/s$ -random. In particular, for some constant c and every $n \geq 1$,*

$$H_C(Z_C(s)[n]) \geq \frac{n}{s} - c.$$

At $s = 1$, $Z_C(1) = \Omega_C$ and the randomness rate is one. The construction below shows that after analytic continuation around the critical point, the information rate can remain saturated at one.

4 A positive bilateral kernel

We begin with a simple Fourier-analytic observation. Define a 2π -periodic function by

$$\Theta(t) = \begin{cases} 4 \left(1 - \frac{2|t|}{\pi}\right), & |t| \leq \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < |t| \leq \pi. \end{cases} \quad (4)$$

Figure 1 shows one period of Θ .

Let

$$\kappa_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta(t) e^{-imt} dt, \quad m \in \mathbb{Z}.$$

The first several coefficients are shown in Figure 2.

Lemma 4. *The sequence $(\kappa_m)_{m \in \mathbb{Z}}$ has the following properties:*

1. $\kappa_m \geq 0$ for every $m \in \mathbb{Z}$;
2. $\kappa_0 = 1$;

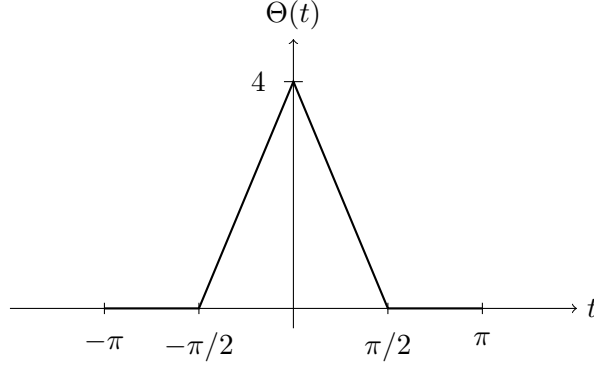


Figure 1: The function Θ on $[-\pi, \pi]$; it is extended periodically with period 2π .

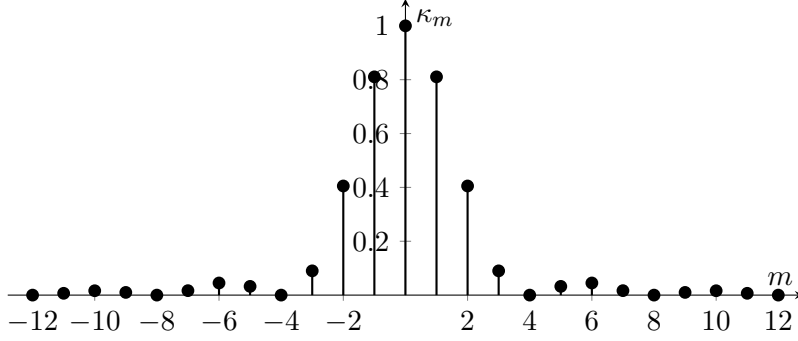


Figure 2: The Fourier coefficients κ_m for $-12 \leq m \leq 12$.

3. for $m \neq 0$,

$$\kappa_m = \frac{8(1 - \cos(\pi m/2))}{\pi^2 m^2};$$

4. $\sum_{m \in \mathbb{Z}} \kappa_m = 4$;

5. the absolutely convergent Fourier series

$$\sum_{m \in \mathbb{Z}} \kappa_m e^{imt} = \Theta(t)$$

vanishes on the open left semicircle, i.e. when $\pi/2 < |t| < \pi$.

Proof. The formula follows by an elementary integration. The numerator $1 - \cos(\pi m/2)$ is nonnegative, so all Fourier coefficients are nonnegative. The equality $\kappa_0 = 1$ follows from the area of the triangle in (4). Since $\kappa_m = O(m^{-2})$, the Fourier series converges absolutely and hence to Θ . Finally, its value at $t = 0$ gives $\sum_m \kappa_m = \Theta(0) = 4$. $\square$

The importance of this kernel is that positivity is compatible with an exact gap in the Fourier transform. Positivity will allow the coefficients to be realized by halting programs; the gap will supply analytic continuation.

5 A universal machine with a continuation arc

Fix a universal self-delimiting machine V . Put

$$d_j = d_V(j), \quad \nu_j = d_j 2^{-j},$$

and set $\nu_j = 0$ for $j < 0$. Hence

$$\sum_{j \geq 0} \nu_j = \Omega_V \leq 1.$$

For every $n \in \mathbb{Z}$, define

$$a_n = 2^{-2} \sum_{j \geq 0} \nu_j \kappa_{n-2-j}. \quad (5)$$

The constant 2 in the exponent and the subscript is the length of a prefix we will be adding to the programs of V . All terms are nonnegative. Moreover,

$$\sum_{n \in \mathbb{Z}} a_n = 2^{-2} \Omega_V \sum_{m \in \mathbb{Z}} \kappa_m = \Omega_V. \quad (6)$$

Thus $(a_n)_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$.

Its bilateral Fourier series factors exactly:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} a_n e^{int} &= 2^{-2} e^{i2t} \left(\sum_{j \geq 0} \nu_j e^{ijt} \right) \left(\sum_{m \in \mathbb{Z}} \kappa_m e^{imt} \right) \\ &= 2^{-2} e^{i2t} \left(\sum_{j \geq 0} \nu_j e^{ijt} \right) \Theta(t). \end{aligned} \quad (7)$$

Consequently, the bilateral Fourier series vanishes on the open left semicircle.

Define

$$A_+(w) = \sum_{n \geq 0} a_n w^n, \quad |w| < 1,$$

and

$$A_-(w) = - \sum_{n < 0} a_n w^n, \quad |w| > 1.$$

Both functions have continuous boundary values on $|w| = 1$ because $(a_n) \in \ell^1(\mathbb{Z})$. By (7), on the open left semicircle their boundary values agree:

$$A_+(e^{it}) = A_-(e^{it}).$$

Lemma 5. *The function A_+ admits a holomorphic continuation through every compact subarc of the open left semicircle. The continued function agrees with A_- on the exterior side.*

Proof. Glue A_+ and A_- along such a compact subarc. The resulting function is continuous across the arc and holomorphic on each side. Morera's theorem shows that it is holomorphic across the arc. $\square$

We next realize the positive-index coefficients by a self-delimiting machine without losing the continuation. The central coefficient $\kappa_0 = 1$ is the key. At $n = j + 2$, the j th source term in (5) is

$$2^{-2} \nu_j \kappa_0 = d_j 2^{-(j+2)}.$$

This is exactly the normalized weight of d_j programs obtained by prefixing the V -programs with 00.

Define

$$c_n = \begin{cases} d_{n-2}2^{-n}, & n \geq 2, \\ 0, & n < 2, \end{cases}$$

and put

$$r_n = a_n - c_n, \quad n \geq 0.$$

The numbers r_n are nonnegative and uniformly lower semicomputable. Also, using (6),

$$\sum_{n \geq 0} r_n \leq \Omega_V - \frac{\Omega_V}{4} = \frac{3\Omega_V}{4} < \frac{3}{4}. \quad (8)$$

For every $n \geq 0$ define the integer

$$E_n = \#\{h \in \mathbf{N} \setminus \{0\} \mid h < 2^n r_n\}. \quad (9)$$

The strict inequality in (9) is deliberate: since r_n is only lower semicomputable, it ensures that E_n is uniformly lower semicomputable even when $2^n r_n$ is an integer. With a non-strict inequality, equality with an integer could not in general be certified from an increasing approximation to r_n . Put

$$\varepsilon_n = r_n - E_n 2^{-n}.$$

Then

$$0 \leq \varepsilon_n \leq 2^{-n}. \quad (10)$$

By (8),

$$\sum_{n \geq 0} E_n 2^{-n} < \frac{3}{4}.$$

In particular $E_0 = 0$. We also have $E_1 = 0$: since $c_1 = 0$,

$$r_1 = a_1 = \frac{1}{4} \sum_{j \geq 0} \nu_j \kappa_{j+1} < \frac{1}{4},$$

because $\kappa_m < 1$ for every $m \neq 0$. Hence $2r_1 < 1$.

The complement of the cylinder $00\Sigma^*$ is the computable prefix-free forest with the three level-two roots 01, 10, and 11, of total Kraft measure $3/4$. Apply the usual Kraft–Chaitin construction with these three strings as the initial free roots. Since all requests have length at least two and their total Kraft weight is less than $3/4$, the construction gives a c.e. prefix-free set $X \subseteq \Sigma^* \setminus 00\Sigma^*$ containing exactly E_n strings of length n for every $n \geq 2$.

Define U_0 by

$$U_0(00p) = V(p), \quad p \in \text{dom}(V), \quad (11)$$

and

$$U_0(x) = \lambda, \quad x \in X,$$

where λ is the empty string. The two branches are prefix-disjoint, hence U_0 is self-delimiting. Equation (11) shows that U_0 is universal.

Let $D_n = d_{U_0}(n)$. By construction,

$$D_n 2^{-n} = a_n - \varepsilon_n.$$

Therefore

$$A_{U_0}(w) = A_+(w) - E(w), \quad E(w) = \sum_{n \geq 0} \varepsilon_n w^n. \quad (12)$$

By (10), E is holomorphic in $|w| < 2$.

Theorem 6. *The machine U_0 is universal and A_{U_0} admits analytic continuation through the open left semicircle. After crossing this arc, the continuation is given in $1 < |w| < 2$ by*

$$A_{U_0}^{\text{ext}}(w) = A_-(w) - E(w). \quad (13)$$

For real $1 < w < 2$, $A_{U_0}^{\text{ext}}(w)$ is real.

Proof. Universality was proved above. Analytic continuation follows from Lemma 5 and the fact that E is holomorphic in $|w| < 2$. Formula (13) follows by uniqueness of analytic continuation. Finally, both $A_-(w)$ and $E(w)$ are real for real w . $\square$

In the s -plane, the continuation can cross the line $\Re(s) = 1$ at the point corresponding to $w = -1$, namely $s = 1 + \pi i / \log 2$, and then move in the exterior $|w| > 1$ to a real point $1 < w < 2$. This reaches the interval $0 < s < 1$ without passing through the singular point $s = 1$.

6 A random square-root branch

The preceding construction gives analytic continuation, but the dyadic rounding error need not be algorithmically simple. We now add a separate branch in which the random information appears in an imaginary component and therefore cannot be cancelled by real rounding terms.

Consider

$$F(w) = 1 - \sqrt{1 - w}, \quad (14)$$

where the germ at $w = 0$ is chosen so that $\sqrt{1} = 1$. We have

$$F(w) = \sum_{n \geq 1} \rho_n w^n, \quad \rho_n = \frac{1}{2n-1} \binom{2n}{n} 4^{-n}. \quad (15)$$

The coefficients are positive and computable, and

$$\sum_{n \geq 1} \rho_n = F(1) = 1. \quad (16)$$

Fix an integer $b \geq 2$ and put

$$\alpha = 2^{-b} \Omega_{U_0}. \quad (17)$$

Since U_0 is universal, Ω_{U_0} is c.e. and random. Hence α is a positive c.e. random real.

For $n \geq 1$ define

$$M_n = \#\{k \in \mathbb{N} \setminus \{0\} \mid k < \alpha 2^n \rho_n\}. \quad (18)$$

The sequence (M_n) is uniformly lower semicomputable and

$$M_n 2^{-n} \leq \alpha \rho_n.$$

Consequently,

$$\sum_{n \geq 1} M_n 2^{-n} \leq \alpha < 1. \quad (19)$$

By the Kraft–Chaitin theorem there is a c.e. prefix-free set Y containing exactly M_n strings of length n .

Put

$$\eta_n = \alpha \rho_n - M_n 2^{-n}.$$

Then

$$0 \leq \eta_n \leq 2^{-n}. \quad (20)$$

If

$$B(w) = \sum_{n \geq 1} M_n 2^{-n} w^n, \quad H(w) = \sum_{n \geq 1} \eta_n w^n,$$

then, initially for $|w| < 1$,

$$B(w) = \alpha F(w) - H(w). \quad (21)$$

By (20), H is holomorphic in $|w| < 2$ and is real-valued on the real interval $(-2, 2)$.

We can now define the final machine U . Let

$$U(0q) = U_0(q), \quad q \in \text{dom}(U_0), \quad (22)$$

and

$$U(1y) = \lambda, \quad y \in Y.$$

Again the branches are prefix-disjoint. Since $U(0q) = U_0(q)$, U is universal; combining this with (11) gives

$$U(000p) = V(p), \quad p \in \text{dom}(V).$$

The normalized generating function of U is

$$A_U(w) = \frac{w}{2} (A_{U_0}(w) + B(w)) = \frac{w}{2} (A_{U_0}(w) + \alpha F(w) - H(w)). \quad (23)$$

7 Analytic continuation around the critical point

Formula (23) separates the three analytic ingredients. The function A_{U_0} crosses the unit circle through the left semicircle by Theorem 6; H is holomorphic throughout $|w| < 2$; and F is an algebraic function with its only finite branch point at $w = 1$.

Start with a real point $0 < w < 1$. Move inside the disk to the left semicircle, cross the unit circle there using the continuation of A_{U_0} , and then move through $1 < |w| < 2$ to a real point $1 < w < 2$, avoiding $w = 1$. The germ F can be continued along the same path. There are two choices, according as the path goes around $w = 1$ on one side or the other. At a real point $w > 1$ the two resulting germs satisfy

$$\sqrt{1-w} = \pm i \sqrt{w-1}, \quad (24)$$

and consequently

$$F^{\text{cont}}(w) = 1 \mp i \sqrt{w-1}. \quad (25)$$

By contrast, Theorem 6 gives

$$A_{U_0}^{\text{cont}}(w) \in \mathbf{R}$$

for real $1 < w < 2$, and $H(w) \in \mathbf{R}$ on the same interval. Therefore the entire imaginary part of the continued A_U comes from the square-root branch.

Theorem 7. For real $1 < w < 2$, either continuation described above satisfies

$$|\Im(A_U^{\text{cont}}(w))| = \frac{w}{2}\alpha\sqrt{w-1}. \quad (26)$$

Proof. Substitute (25) in (23). The other terms are real, so they do not contribute to the imaginary part. $\square$

Return now to the variable s by putting

$$w = 2^{1-s}.$$

For a real $s \in (0, 1)$ we have $1 < w < 2$. Equation (26) becomes

$$|\Im(Z_U^{\text{cont}}(s))| = 2^{-s}\alpha\sqrt{2^{1-s}-1} = c_s\Omega_{U_0}, \quad (27)$$

where

$$c_s = 2^{-b-s}\sqrt{2^{1-s}-1}. \quad (28)$$

For computable $s \in (0, 1)$, c_s is positive and computable.

This proves the analytic-continuation part of Theorem 1. Notice that the continuation is genuinely different from the divergent defining series in $0 < s < 1$. It is obtained by passing around the obstruction at $s = 1$. The two choices of path give conjugate values.

8 Randomness of the imaginary part

We finish the proof of Theorem 1 with the algorithmic information content of (27).

Lemma 8. Let $\beta \in (0, 1)$ be a c.e. random real and let $c > 0$ be computable, with $c\beta \in (0, 1)$. Then $c\beta$ is c.e. and random.

Proof. Since c is a positive computable real, it has an increasing computable sequence of positive rational approximations. Multiplying these by an increasing computable rational approximation to β shows that $c\beta$ is c.e.

For randomness, multiplication by a fixed positive computable constant, as well as multiplication by its reciprocal, changes prefix complexity by at most $O(1)$. Thus from the first $n + O(1)$ bits of $c\beta$ one can effectively recover the first n bits of β , and conversely. If $c\beta$ failed the Chaitin complexity criterion for randomness, then so would β . Hence $c\beta$ is random. $\square$

Theorem 9. For every computable real $s \in (0, 1)$,

$$|\Im(Z_U^{\text{cont}}(s))|$$

is a c.e. random real. In particular, there is a constant c (depending on s and U) such that for every $n \geq 1$,

$$H(|\Im(Z_U^{\text{cont}}(s))|[n]) \geq n - c. \quad (29)$$

Proof. By (27),

$$|\Im(Z_U^{\text{cont}}(s))| = c_s\Omega_{U_0}.$$

The machine U_0 is universal, so Ω_{U_0} is c.e. and random. For computable $s \in (0, 1)$, the number c_s in (28) is positive and computable. The result follows from Lemma 8. $\square$

Theorem 9 gives a simple information-theoretic interpretation of the continuation. Tadaki's theorem gives compression rate $1/s$ for computable $s > 1$. At $s = 1$ the rate

reaches one because $Z_U(1) = \Omega_U$. The formal continuation of the expression $1/s$ to $s < 1$ would exceed one, which is impossible for binary prefix complexity. In our construction the continued function instead contains a coordinate of full randomness: the information density saturates at the maximal value one.

There is a minor but useful distinction. On the original half-plane, for real $s > 1$, $Z_U(s)$ is real and positive because every summand in (1) is positive. The imaginary part in Theorem 9 is created by monodromy: after continuation around the branch point at $s = 1$, the real square-root term becomes imaginary. The two choices of continuation produce conjugate values.

9 A one-bit refinement from a free cylinder

The preceding construction is uniform in the source machine and therefore reserves the two-bit cylinder $00\Sigma^*$ for the copy of V . If the source machine is supplied with an effectively known free cylinder, the construction can be tightened so that the final machine simulates V with only one extra bit.

Definition 10. *A string $x \in \Sigma^*$ is a free cylinder for a self-delimiting machine V if x is prefix-incomparable with every member of $\text{dom}(V)$. Equivalently, the basic cylinder $[x]$ is disjoint from the union of the cylinders generated by $\text{dom}(V)$.*

If x is a free cylinder, then its measure is missing from the Kraft sum, and hence

$$\Omega_V \leq q := 1 - 2^{-|x|} < 1. \quad (30)$$

The point is that q is an effective rational upper bound, not merely the noneffective assertion $\Omega_V < 1$.

We also recall the weaker randomness notion introduced by Calude and Stay [3]. A real $\beta \in (0, 1)$ is *asymptotically random* if, for every computable real $r \in (0, 1)$, there is a constant c_r such that

$$H(\beta[n]) \geq rn - c_r \quad (31)$$

for all $n \geq 1$. Equivalently, its lower effective dimension is one. Every c.e. random real is asymptotically random, but the converse need not hold.

Theorem 11. *Let V be a self-delimiting machine and suppose an effectively known free cylinder x is given. Then one can effectively construct a self-delimiting machine U^* such that*

$$U^*(0p) = V(p), \quad p \in \text{dom}(V),$$

and Z_{U^} admits analytic continuation around $s = 1$ to every real point $s \in (0, 1)$. If V is universal, then U^* is universal. If Ω_V is asymptotically random, then for every computable real $s \in (0, 1)$ the absolute value of the imaginary part of either continued branch is asymptotically random. If Ω_V is random, then this imaginary part is random.*

Proof. Put $q = 1 - 2^{-|x|}$ as in (30). Choose a computable real τ satisfying

$$\frac{2\pi q}{1+q} < \tau < \pi. \quad (32)$$

Define a 2π -periodic tent by

$$\Theta_\tau(t) = \begin{cases} \frac{2\pi}{\tau} \left(1 - \frac{|t|}{\tau}\right), & |t| \leq \tau, \\ 0, & \tau < |t| \leq \pi. \end{cases}$$

Its Fourier coefficients satisfy

$$\kappa_0^{(\tau)} = 1, \quad \kappa_m^{(\tau)} = \frac{2(1 - \cos(m\tau))}{\tau^2 m^2} \geq 0 \quad (m \neq 0),$$

and

$$M_\tau := \sum_{m \in \mathbb{Z}} \kappa_m^{(\tau)} = \Theta_\tau(0) = \frac{2\pi}{\tau}. \quad (33)$$

The Fourier series vanishes on the nonempty open arc $\tau < |t| < \pi$. By (32),

$$(M_\tau - 1)q < 1. \quad (34)$$

With $\nu_j = d_V(j)2^{-j}$, define the bilateral coefficients

$$\tilde{a}_n = \frac{1}{2} \sum_{j \geq 0} \nu_j \kappa_{n-1-j}^{(\tau)}.$$

The central coefficient $\kappa_0^{(\tau)} = 1$ contributes exactly the normalized weight of the programs $0p$ with $p \in \text{dom}(V)$. The total off-center mass is at most

$$\frac{M_\tau - 1}{2} \Omega_V \leq \frac{M_\tau - 1}{2} q < \frac{1}{2}.$$

Round the remaining positive-index coefficients downward to dyadic program counts exactly as in (9). The rounding error again has coefficients bounded by 2^{-n} and therefore gives a function holomorphic in $|w| < 2$. Since the residual Kraft mass is less than $1/2$, all residual programs can be placed under the prefix 1 by the Kraft–Chaitin theorem. We obtain a machine $\tilde{U}_0$ with

$$\tilde{U}_0(0p) = V(p). \quad (35)$$

The same gluing argument as in Lemma 5 gives analytic continuation through the arc $\tau < |t| < \pi$ and hence, through the exterior annulus, around $w = 1$ to $1 < w < 2$.

Because every residual program begins with 1, the cylinder $[0x]$ remains free for $\tilde{U}_0$. Let

$$L = |0x| = |x| + 1.$$

Repeat the square-root construction (14)–(21), but now put

$$\alpha = 2^{-b} \Omega_V$$

for a fixed $b \geq 1$. Let Y be the resulting prefix-free set realizing

$$B(w) = \alpha F(w) - H(w),$$

with H holomorphic in $|w| < 2$ and real on the real interval $(-2, 2)$. Instead of prefixing the whole machine by another bit, place this auxiliary branch inside the already free cylinder $[0x]$: retain every computation of $\tilde{U}_0$ and add

$$U^*(0xy) = \lambda, \quad y \in Y.$$

Prefix-freeness follows from the fact that $[0x]$ was free. Equation (35) remains true for U^* , so if V is universal then U^* is universal with one bit of simulation overhead.

The normalized generating function is

$$A_{U^*}(w) = A_{\tilde{U}_0}(w) + \left(\frac{w}{2}\right)^L B(w). \quad (36)$$

For real $1 < w < 2$, the continued $A_{\tilde{U}_0}$ and H are real, while

$$F^{\text{cont}}(w) = 1 \mp i\sqrt{w-1}.$$

Consequently,

$$|\Im(A_{U^*}^{\text{cont}}(w))| = \left(\frac{w}{2}\right)^L \alpha \sqrt{w-1}. \quad (37)$$

Putting $w = 2^{1-s}$ gives

$$|\Im(Z_{U^*}^{\text{cont}}(s))| = 2^{-b-sL} \sqrt{2^{1-s}-1} \Omega_V. \quad (38)$$

For computable $s \in (0, 1)$, the factor multiplying Ω_V is positive and computable. Multiplication by such a constant preserves both c.e. randomness and asymptotic randomness: an $n + O(1)$ -bit approximation to one factor determines an n -bit approximation to the other, and hence the complexity inequalities differ by at most an additive constant. The final assertions follow. $\square$

The free-cylinder hypothesis in Theorem 11 is additional information when V is presented merely as an arbitrary partial computable prefix-free function. There is, however, an important class of programming languages for which such a cylinder can be found effectively from the language itself. Suppose that the length of a program is known before its execution begins, so that the parser knows exactly where a complete program ends. This is the usual situation for programs stored as files or other length-delimited objects: mathematically, one may regard the length or framing information as part of the program representation. If the language is universal, one can effectively write a syntactically valid program x which diverges, for example an unconditional infinite loop. Since x is already a complete program, every proper prefix of x is incomplete and every proper extension lies beyond the declared end of the program. Thus no halting program is prefix-comparable with x , and $[x]$ is an effectively known free cylinder. For this class of languages the cylinder required by Theorem 11 need not be supplied as separate non-computable information: it can be constructed directly from the syntax and operational semantics of the language.

This is also the reason that asymptotic randomness is relevant here. The halting probabilities arising from such concrete, length-delimited universal languages cannot be Chaitin random in the strongest sense. Nevertheless the resulting halting probabilities can be asymptotically random. Thus these languages provide a natural setting in which both ingredients needed for the one-bit construction occur together: their length-delimited syntax gives an effectively obtainable free cylinder, while their halting probabilities retain full asymptotic information density.

10 A universal machine with a natural boundary

We now prove Theorem 2. This construction is independent of the analytic-continuation construction above. Its purpose is to show that universality is also compatible with the opposite extreme: a dense set of unavoidable boundary singularities.

Let V be a self-delimiting machine with infinitely many halting programs, and put

$$d_j = \#\{p \in \text{dom}(V) \mid |p| = j\}, \quad \nu_j = d_j 2^{-j}.$$

For $j \geq 0$ define

$$N_j = (j + 2)!. \quad (39)$$

Whenever a computation $V(p)$ halts, with $|p| = j$ and output x , enumerate into the graph of W the pair

$$W(0p) = x$$

and also all pairs

$$W(1pz) = x, \quad z \in \Sigma^{N_j - j - 1}. \quad (40)$$

This is an effective definition of a partial computable function: when p enters $\text{dom}(V)$, only finitely many padded strings are enumerated. In particular, the construction does not require the machine to determine the total input length before execution.

Lemma 12. *The machine W is self-delimiting. For each $p \in \text{dom}(V)$ of length j , the short program $0p$ and the entire padded block in (40) have equal Kraft weight $2^{-(j+1)}$.*

Proof. The two branches are disjoint because they begin with different bits. Prefix-freeness of the short branch follows from prefix-freeness of $\text{dom}(V)$. Suppose two padded programs $1pz$ and $1p'z'$ are prefix-comparable. If $|p| < |p'|$, then p is a prefix of p' , contradicting prefix-freeness of $\text{dom}(V)$; similarly if $|p'| < |p|$. Hence $|p| = |p'|$, so $p = p'$, and the two padded programs have the same prescribed total length $N_{|p|}$; therefore neither is a proper prefix of the other.

There are $2^{N_j - j - 1}$ strings in the padded block belonging to p , each of length N_j . Their total Kraft weight is

$$2^{N_j - j - 1} 2^{-N_j} = 2^{-(j+1)},$$

which is the Kraft weight of the short program $0p$. $\square$

Summing the two equal contributions associated with each V -program gives immediately

$$\Omega_W = \Omega_V. \quad (41)$$

Moreover, every W -description of an output x arises from a V -description p of x and has length at least $|p| + 1$, while the short branch attains this bound. Hence

$$H_W(x) = H_V(x) + 1. \quad (42)$$

In particular, if V is universal then so is W .

The normalized generating function of W has the exact form

$$A_W(w) = \frac{1}{2} \sum_{j \geq 0} \nu_j (w^{j+1} + w^{N_j}). \quad (43)$$

Indeed, the short programs of source length j have total normalized weight $\nu_j/2$ at exponent $j + 1$, and the padded block has the same total normalized weight at exponent N_j . Since $\sum_j \nu_j = \Omega_V$, the series in (43) converges absolutely and uniformly on the closed unit disk. Thus A_W has a continuous boundary value at every point of $|w| = 1$.

We next show that every root of unity is nevertheless singular.

Lemma 13. *Every root of unity ζ is a singular point of A_W .*

Proof. Let ζ have order q . For all $j \geq q$, the factorial $N_j = (j+2)!$ is divisible by q , and hence

$$\zeta^{N_j} = 1.$$

Fix $0 < r < 1$. The contribution of an index $j \geq q$ to

$$A_W(\zeta) - A_W(r\zeta)$$

is

$$\frac{\nu_j}{2} [\zeta^{j+1}(1 - r^{j+1}) + (1 - r^{N_j})].$$

Its real part is bounded below by

$$\begin{aligned} \frac{\nu_j}{2} [-(1 - r^{j+1}) + (1 - r^{N_j})] &= \frac{\nu_j}{2} (r^{j+1} - r^{N_j}) \\ &\geq 0. \end{aligned}$$

After division by $1 - r$ this gives the lower bound

$$\frac{\nu_j}{2} \frac{r^{j+1} - r^{N_j}}{1 - r} = \frac{\nu_j}{2} \sum_{k=j+1}^{N_j-1} r^k. \quad (44)$$

For each fixed j , the right hand side increases as $r \uparrow 1$ to

$$\frac{\nu_j}{2} (N_j - j - 1).$$

We claim that the sum of these limiting quantities diverges. Since

$$(j+2)! \geq 2^{j+1} \quad \text{and} \quad 2^j \geq j+1,$$

we have

$$N_j - j - 1 \geq 2^j. \quad (45)$$

Therefore

$$\nu_j (N_j - j - 1) \geq d_j 2^{-j} 2^j = d_j.$$

Because V has infinitely many halting programs,

$$\sum_{j \geq q} d_j = \infty,$$

and hence

$$\sum_{j \geq q} \nu_j (N_j - j - 1) = \infty. \quad (46)$$

The finitely many indices $j < q$ contribute a polynomial in w , so their radial difference quotient remains bounded as $r \uparrow 1$. Thus for some finite constant B_q ,

$$\Re \left(\frac{A_W(\zeta) - A_W(r\zeta)}{1 - r} \right) \geq -B_q + \frac{1}{2} \sum_{j \geq q} \nu_j \sum_{k=j+1}^{N_j-1} r^k. \quad (47)$$

All terms in the double sum are nonnegative. By monotone convergence and (46), the right hand side tends to $+\infty$ as $r \uparrow 1$. Consequently,

$$\lim_{r \uparrow 1} \Re \left(\frac{A_W(\zeta) - A_W(r\zeta)}{1 - r} \right) = +\infty. \quad (48)$$

If A_W admitted a holomorphic continuation through ζ , continuity of the absolutely convergent boundary series would force the continued value to equal $A_W(\zeta)$. Taylor expansion at ζ would then give

$$\frac{A_W(\zeta) - A_W(r\zeta)}{1 - r} \longrightarrow \zeta A'_W(\zeta),$$

a finite complex number, contradicting (48). Hence ζ is singular. $\square$

Theorem 14. *The unit circle is a natural boundary for A_W , and $\Re(s) = 1$ is a natural boundary for Z_W .*

Proof. By Lemma 13, every root of unity is singular for A_W . The roots of unity are dense in the unit circle. If A_W admitted holomorphic continuation through any boundary point, it would continue through a small open arc, which would contain a root of unity. This is impossible. Hence the unit circle is a natural boundary.

Finally,

$$Z_W(s) = A_W(2^{1-s}).$$

The map $s \mapsto 2^{1-s}$ has nonzero derivative everywhere and maps the line $\Re(s) = 1$ onto the unit circle. A local continuation of Z_W through any point of that line would therefore induce, via a local inverse, a continuation of A_W through the corresponding point of the unit circle. Thus $\Re(s) = 1$ is a natural boundary for Z_W . $\square$

The singular points obtained explicitly in the s -plane include

$$1 + \frac{2\pi ir}{\log 2}, \quad r \in \mathbf{Q},$$

which form a dense subset of the boundary line. Notice also that (41) shows that the halting probability itself does not determine the continuation behavior: the construction changes the boundary behavior while preserving Ω exactly.

11 The author's use of AI

The author formulated the research problem and directed the mathematical investigation. In particular, he identified the complex-analytic behavior of generalized halting probabilities as the object of study; connected the problem with Tadaki's partition functions, Calude–Stay zeta functions, and earlier work on partial and asymptotic randomness; proposed examining periodic and length-delimited universal machines; introduced the practical motivation from programming languages whose program lengths are known before execution; emphasized the role of effectively known nonhalting cylinders; and repeatedly sharpened the constructions by asking whether universality overhead and Kraft-space requirements could be reduced. He also identified several technical issues in intermediate arguments, including the self-delimitation problem in the padded-program construction, and suggested the roots-of-unity and sign/cancellation questions that led to further results.

12 Conclusion and open questions

ChatGPT proved that universality alone does not determine whether $\Re(s) = 1$ is a natural boundary for Tadaki's generalized halting probability. Starting effectively from

an arbitrary universal machine V , the Fourier-kernel construction produces a universal machine U whose Z -function crosses an open part of the boundary and can be continued around $s = 1$ to the divergent real interval $0 < s < 1$. A square-root branch can additionally be used to protect a scaled Omega number in the imaginary part, so that every computable point of the continued real interval carries a c.e. random component.

Starting from the same arbitrary source machine, the factorial-padding construction produces a universal machine W with the opposite analytic behavior. The machine preserves the halting probability exactly, $\Omega_W = \Omega_V$, and changes program-size complexity only by one bit, yet every root of unity is singular for A_W . Thus the unit circle is a natural boundary and $\Re(s) = 1$ is a natural boundary for Z_W .

The two constructions show that neither optimality nor the value of Ω determines the analytic continuation behavior of the generalized halting probability. This suggests asking for a finer classification of the possible continuation domains of Z_U as U ranges over universal machines. It would also be interesting to determine which intermediate singular sets on $\Re(s) = 1$ can be realized, and to obtain analogous constructions for the exact-index zeta function and the prime product machines introduced by Calude and Stay [3]. In those cases the precise locations of programs, rather than only their lengths, enter the Dirichlet series, so the length-spectrum constructions used here do not transfer directly.

Competing interests

The author declares none.

References

- [1] Calude, C. S. (2002). *Information and Randomness: An Algorithmic Perspective*, 2nd edn, revised and extended. Berlin: Springer-Verlag.
- [2] Calude, C. S., Hertling, P., Khossainov, B. and Wang, Y. (2001). Recursively enumerable reals and Chaitin Ω numbers. *Theoret. Comput. Sci.* **255**, 125–149.
- [3] Calude, C. S. and Stay, M. A. (2006). Natural halting probabilities, partial randomness, and zeta functions. *Inf. Comput.* **204**, 1718–1739.
- [4] Chaitin, G. J. (1975). A theory of program size formally identical to information theory. *J. Assoc. Comput. Mach.* **22**, 329–340.
- [5] Tadaki, K. (2002). A generalization of Chaitin’s halting probability Ω and halting self-similar sets. *Hokkaido Math. J.* **31**, 219–253.
- [6] Tadaki, K. (2008). A statistical mechanical interpretation of algorithmic information theory. In: *Local Proceedings of Computability in Europe 2008 (CiE 2008)*, University of Athens, 425–434. Extended version: arXiv:0801.4194.
- [7] Tadaki, K. (2012). Fixed point theorems on partial randomness. *Ann. Pure Appl. Logic* **163**, 763–774.
- [8] Tadaki, K. (2012). A statistical mechanical interpretation of algorithmic information theory III: Composite systems and fixed points. *Math. Struct. Comput. Sci.* **22**, 752–770.

- [9] Tadaki, K. (2014). Phase transition and strong predictability. In: *Unconventional Computation and Natural Computation*, Lecture Notes in Computer Science **8553**, Springer, 340–352.
- [10] Baez, J. C. and Stay, M. (2012). Algorithmic thermodynamics. *Math. Struct. Comput. Sci.* **22**, 771–787.
- [11] Lapidus, M. L. and van Frankenhuijsen, M. (2013). *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*, 2nd edn. New York: Springer.
- [12] Lapidus, M. L., Radunović, G. and Žubrinić, D. (2017). Complex dimensions of fractals and meromorphic extensions of fractal zeta functions. *J. Math. Anal. Appl.* **453**, 458–484.
- [13] Fabry, E. (1896). Sur les points singuliers d’une fonction donnée par son développement en série et l’impossibilité du prolongement analytique dans des cas très généraux. *Ann. Sci. École Norm. Sup. (3)* **13**, 367–399.