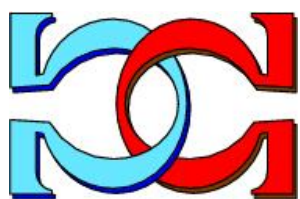
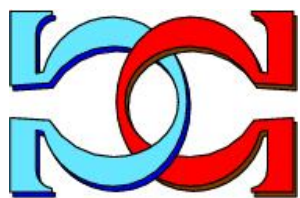
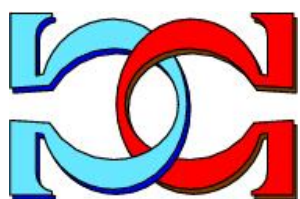


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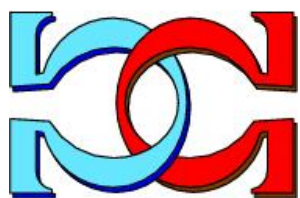
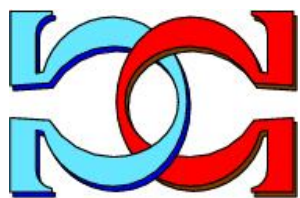


**A note on Automatic BAIRE  
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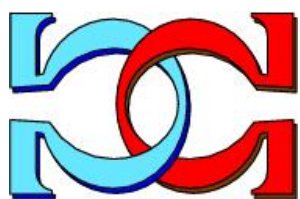


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# A note on Automatic BAIRE property

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## Abstract

Automatic Baire property is a variant of the usual Baire property which is fulfilled for subsets of the Cantor space accepted by finite automata. We consider the family  $\mathcal{A}$  of subsets of the Cantor space having the Automatic Baire property. In particular we show that not all finite subsets have the Automatic Baire property, and that already a slight increase of the computational power of the accepting device may lead beyond the class  $\mathcal{A}$ .

In [Fin20, Fin21] Finkel introduced an automata-theoretic variant of the topological Baire property for subsets of the Cantor space. He showed that this Automatic Baire property is valid for regular  $\omega$ -languages, that is, for subsets of the Cantor space definable by finite automata.

In this note we investigate which  $\omega$ -languages beyond regular ones have the Automatic Baire property. We get a full characterisation of  $\omega$ -languages of first Baire category as well as of finite  $\omega$ -languages having

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the Automatic Baire property. In this respect, disjunctive  $\omega$ -words, that is,  $\omega$ -words random w.r.t. to finite automata in the measure-theoretic approach (cf. [Sta18]) play a major rôle. Here, as a tool, we use the measure-category coincidence for regular  $\omega$ -languages (see [Sta76], Theorem 3 of [Sta98], [VV06], or Section 9.4 of [VV12]).

Moreover, we show that, besides definability by finite automata, other computational constraints do not imply Automatic Baire property. To this end we derive  $\omega$ -languages closed or open in the topology of the Cantor space definable by simple one-counter automata not having the Automatic Baire property.

## 1 Preliminaries

### 1.1 Notation

We introduce the notation used throughout the paper. By  $\mathbb{N} = \{0, 1, 2, \dots\}$  we denote the set of natural numbers. Its elements will be usually denoted by letters  $i, \dots, n$ . Let  $X$  be an alphabet of cardinality  $|X| \geq 2$ . Then  $X^*$  is the set of finite words on  $X$ , including the *empty word*  $e$ , and  $X^\omega$  is the set of infinite strings ( $\omega$ -words) over  $X$ . Subsets of  $X^*$  will be referred to as *languages* and subsets of  $X^\omega$  as  *$\omega$ -languages*.

For  $w \in X^*$  and  $\eta \in X^* \cup X^\omega$  let  $w \cdot \eta$  be their *concatenation*. This concatenation product extends in an obvious way to subsets  $W \subseteq X^*$  and  $B \subseteq X^* \cup X^\omega$ . For a language  $W$  let  $W^* := \bigcup_{i \in \mathbb{N}} W^i$ , and  $W^\omega := \{w_1 \cdots w_i \cdots : w_i \in W \setminus \{e\}\}$  be the set of infinite strings formed by concatenating non-empty words in  $W$ . Furthermore,  $|w|$  is the *length* of the word  $w \in X^*$  and  $\text{pref}(B)$  is the set of all finite prefixes of strings in  $B \subseteq X^* \cup X^\omega$ . We shall abbreviate  $w \in \text{pref}(\{\eta\})$  ( $\eta \in X^* \cup X^\omega$ ) by  $w \sqsubseteq \eta$ .

An  $\omega$ -word  $\xi \in X^\omega$  is *ultimately periodic* if there are words  $w, v \in X^*$  such that  $\xi = w \cdot v^\omega = w \cdot v \cdot v \cdots$ , and an  $\omega$ -word  $\zeta \in X^\omega$  is *disjunctive* (or *rich*, [Sta98]) if every  $w \in X^*$  is an infix of  $\zeta$ , that is,  $\zeta \in \bigcap_{w \in X^*} X^* \cdot w \cdot X^*$ .

### 1.2 Regular $\omega$ -languages

As usual, a language  $W \subseteq X^*$  is *regular* if it is obtained from finite languages via the operations union, concatenation and star. An  $\omega$ -language  $F \subseteq X^\omega$  is *regular* if it is of the form  $F = \bigcup_{i=1}^n W_i \cdot V_i^\omega$  where  $W_i, V_i \subseteq X^*$  are regular languages.

We assume the reader to be familiar with the basic facts of the theory of regular languages and finite automata. For more details on  $\omega$ -languages

and regular  $\omega$ -languages see the books [PP04, TB73] or the survey papers [Sta97, Tho90].

The following is well-known.

**Theorem 1** *The family of regular  $\omega$ -languages is a Boolean algebra, and every non-empty regular  $\omega$ -language contains an ultimately periodic  $\omega$ -word.*

### 1.3 The Cantor space

We consider  $X^\omega$  as a topological space (Cantor space). The *closure* (smallest closed set containing  $F$ )  $\mathcal{C}(F)$  of a subset  $F \subseteq X^\omega$  is described as  $\mathcal{C}(F) := \{\xi : \mathbf{pref}(\{\xi\}) \subseteq \mathbf{pref}(F)\}$ . The *open sets* in Cantor space are the  $\omega$ -languages of the form  $W \cdot X^\omega$ .

Next we recall some topological notions, see [Kur66, Oxt80]. As usual, an  $\omega$ -language  $F \subseteq X^\omega$  is *dense in  $X^\omega$*  if  $\mathcal{C}(F) = X^\omega$ . This is equivalent to  $\mathbf{pref}(F) = X^*$ . An  $\omega$ -language  $F \subseteq X^\omega$  is *nowhere dense in  $X^\omega$*  if its closure  $\mathcal{C}(F)$  does not contain a non-empty open subset. This property is equivalent to the fact that for all  $v \in \mathbf{pref}(F)$  there is a  $w \in X^*$  such that  $v \cdot w \notin \mathbf{pref}(F)$ . If a regular  $\omega$ -language  $F \subseteq X^\omega$  is nowhere dense then there is a word  $w \in X^*$  such that  $F \subseteq X^* \cdot w \cdot X^\omega$  [Sta76].

Moreover, a subset  $F \subseteq X^\omega$  is *meagre* or of *first Baire category* if it is a countable union of nowhere dense sets.

Any subset of a nowhere dense set is nowhere dense, hence, every subset of a meagre set is again meagre. A finite union of nowhere dense sets is nowhere dense, and a countable union of meagre sets is meagre.

The following property is a consequence of the fact that in Cantor space no non-empty open subset is of first Baire category.

**Property 2** *Let  $F \subseteq X^\omega$  be of first Baire category and  $E \subseteq X^\omega$  be open. If  $F \Delta E$  is of first Baire category then  $E = \emptyset$ .*

## 2 Measure and Category

In this section we consider the relation between measures on Cantor space and topological density.

For every  $w \in X^*$  the ball  $w \cdot X^\omega = \bigcup_{x \in X} wx \cdot X^\omega$  is a disjoint union of its sub-balls. Thus  $\mu(w \cdot X^\omega) = \sum_{x \in X} \mu(wx \cdot X^\omega)$  for every measure  $\mu$  on  $X^\omega$ . The *support* of a measure  $\mu$  on  $X^\omega$ ,  $\mathbf{supp}(\mu)$ , is the smallest closed subset of  $X^\omega$  such that  $\mu(\mathbf{supp}(\mu)) = \mu(X^\omega)$ .

As measures  $\mu$  on  $X^\omega$  we consider finite non-null measures ( $0 < \mu(X^\omega) < \infty$ ) having the following property that the measure of a non-null sub-ball  $wx \cdot X^\omega$  does not deviate too much from  $\mu(w \cdot X^\omega)$  (cf. [Sta98, VV12]).

**Definition 1 (Balance condition)** A measure  $\mu$  on  $X^\omega$  is referred to as *balanced* (or *bounded away from zero* [VV12]) provided there is a constant  $c_\mu > 0$  depending only on  $\mu$  such that for all words  $w \in X^*$  and every  $x \in X$  we have  $\mu(wx \cdot X^\omega) = 0$  or  $c_\mu \cdot \mu(w \cdot X^\omega) \leq \mu(wx \cdot X^\omega)$ .

In the book by Oxtoby [Oxt80] analogies between topological density and measure, in particular, the “duality” between measure and category are discussed. The papers [Sta76, Sta98, VV06] and [VV12] show that for regular  $\omega$ -languages in Cantor space measure and category coincide.

**Theorem 3 (Theorem 3 of [Sta98])** Let  $F \subseteq X^\omega$  be a regular  $\omega$ -language. Then the following conditions are equivalent:

1. No  $\zeta \in F$  is a disjunctive  $\omega$ -word.
2.  $F$  is of first Baire category.
3. For all measures  $\mu$  with  $\text{supp}(\mu) = X^\omega$  satisfying the balance condition it holds  $\mu(F) = 0$ .
4. There is a measure  $\mu$  with  $\text{supp}(\mu) = X^\omega$  satisfying the balance condition such that  $\mu(F) = 0$ .

Theorem 3.1 shows that the union of all regular  $\omega$ -languages of first Baire category  $R_0$  can be characterised as follows (see e.g. [Sta76, Korollar 8]).

$$R_0 = \bigcup_{w \in X^*} (X^\omega \setminus X^* \cdot w \cdot X^\omega) \quad (1)$$

### 3 Baire property and Automatic Baire property

Automatic Baire property was introduced by Finkel [Fin20, Fin21]. Here we define this variant of the usual Baire property and derive several of its properties. First we recall the following (see e.g. [Kur66, Oxt80]).

**Definition 2** A subset  $F \subseteq X^\omega$  has the *Baire property* if there is an open set  $E \subseteq X^\omega$  such that their symmetric difference  $F \Delta E$  is of first Baire category.

**Theorem 4** Every Borel set of the Cantor space has the Baire property.

The Automatic Baire property requires the sets  $E$  and  $F \Delta E$  to be restricted in some sense to regular  $\omega$ -languages.

**Definition 3 (Automatic Baire property)** A subset  $F \subseteq X^\omega$  has the *Automatic Baire property* if

$$F \Delta E \subseteq F', \quad (2)$$

where  $E$  is a regular and open  $\omega$ -language and  $F'$  a regular  $\omega$ -language of first Baire category.

Then it holds the following.

**Theorem 5 ([Fin20, Fin21])** Every regular  $\omega$ -language has the Automatic Baire property.

We derive some properties of the class  $\mathcal{A}$  of all  $\omega$ -languages having the Automatic Baire property. It is obvious that every  $\omega$ -language which has the Automatic Baire property has also the Baire property.

**Lemma 6**  $\mathcal{A}$  is a Boolean algebra.

**Proof.** This follows from  $(F_1 \cup E_1) \Delta (F_2 \cup E_2) \subseteq (F_1 \Delta E_1) \cup (F_2 \Delta E_2)$  and  $(X^\omega \setminus F) \Delta (X^\omega \setminus E) = F \Delta E$  and the fact that the union of two regular  $\omega$ -languages of first Baire category is also regular and of first Baire category.  $\square$

We derive a necessary condition for sets to be of first Baire category.

**Lemma 7** Let  $F \Delta E \subseteq F'$  where  $E \subseteq X^\omega$  is open and  $F' \subseteq X^\omega$  a regular  $\omega$ -language of first Baire category. Then for every measure  $\mu$  with support  $\text{supp}(\mu) = X^\omega$  satisfying the balance condition it holds  $\mu(F) = 0$  if and only if  $F$  is of first Baire category.

**Proof.** Let  $F \Delta E \subseteq F'$  where  $E$  is open and  $F'$  is regular and of first Baire category. According to Theorem 3 we have  $\mu(F') = 0$ .

If  $\mu(F) = 0$  then  $\mu(E) = \mu(E) - \mu(F) \leq \mu(E \setminus F) \leq \mu(E \Delta F) \leq \mu(F') = 0$  implies  $E = \emptyset$ . Thus  $F = E \Delta F$  is of first Baire category.

If  $F$  and  $E \Delta F$  are of first Baire category then  $E \subseteq (E \Delta F) \cup F$  is also of first Baire category. Thus  $E = \emptyset$ . Consequently,  $\mu(F) = \mu(E \Delta F) = 0$ .  $\square$

*Remark.* Observe that in Lemma 7 we did not use the fact that the open set  $E$  is regular.

The proof of Lemma 7 shows also the following.

**Corollary 8** Let  $F \subseteq X^\omega$  be of first Baire category. Then  $F \in \mathcal{A}$  if and only if  $F \subseteq F'$  for some regular  $\omega$ -language of first Baire category.

Finite  $\omega$ -languages in  $\mathcal{A}$  are characterised as follows.

**Corollary 9** *Let  $F \subseteq X^\omega$  be finite. Then  $F \in \mathcal{A}$  if and only if  $F$  does not contain a disjunctive  $\omega$ -word.*

**Proof.** If  $F$  is finite then  $F$  is of first Baire category. Now Corollary 8 and Theorem 3 imply that  $F$  does not contain a disjunctive  $\omega$ -word.

If  $F$  is finite and does not contain a disjunctive  $\omega$ -word then for every  $\xi \in F$  there is a  $w_\xi$  such that  $\xi \notin X^* \cdot w_\xi \cdot X^\omega$ . Then  $F \subseteq \bigcup_{\xi \in F} (X^\omega \setminus X^* \cdot w_\xi \cdot X^\omega)$  which is a regular and nowhere dense  $\omega$ -language.  $\square$

Besides finite  $\omega$ -languages containing disjunctive  $\omega$ -words, examples of sets not satisfying the Automatic Baire property are the following ones.

**Lemma 10** *If  $F \subseteq X^\omega$ ,  $\text{Ult} \subseteq F \subseteq \mathbf{R}_0$ , then  $F$  does not have the Automatic Baire property.*

**Proof.** Since  $\text{Ult} \subseteq F \subseteq \mathbf{R}_0$ , the set  $F$  is of first Baire category. Now Property 2 shows that the symmetric difference  $E \Delta F$  with a non-empty open set  $E$  is not of first Baire category. Hence  $E = \emptyset$  and  $F \subseteq F'$  for some regular  $\omega$ -language  $F'$

Then  $X^\omega \setminus F' \subseteq X^\omega \setminus \text{Ult}$  does not contain any ultimately periodic  $\omega$ -word. Consequently,  $F' = X^\omega$  which is not of first Baire category.  $\square$

**Corollary 11** *The family  $\mathcal{A}$  is not closed under countable union.*

**Proof.** As  $\mathbf{R}_0 = \bigcup_{w \in X^*} (X^\omega \setminus X^* \cdot w \cdot X^\omega)$  and every  $\omega$ -language  $X^\omega \setminus X^* \cdot w \cdot X^\omega$  is regular and nowhere dense in  $X^\omega$  (cf. [Sta76]), the assertion follows immediately.  $\square$

## 4 Simple counter-examples

In Corollary 9 we have seen that there are even finite  $\omega$ -languages having the Baire property but not the Automatic Baire property. Those finite  $\omega$ -languages contain  $\omega$ -words  $\xi \notin \text{Ult}$  and are, therefore, not context-free (e.g. [EH93, Sta97]), that is accepted by push-down automata.

In this part we show that also a slight increase of the computational power of accepting devices results in open or closed  $\omega$ -languages not having the Automatic Baire property.

As measure in Cantor space we use the equidistribution. For a language  $W \subseteq X^*$  we set  $\sigma_X(W) := \sum_{w \in W} |X|^{-|w|}$ . Then  $\mu_=(W \cdot X^\omega) = \sigma_X(W)$ , if  $W \subseteq X^*$  prefix-free, that is,  $w \sqsubseteq v$  and  $w, v \in W$  imply  $w = v$ .

Since  $\sigma_X(W)$  is rational for regular languages  $W \subseteq X^*$ , we have the following (see [Tak01, Theorem 4.16]).

**Theorem 12** *The measure  $\mu_=(F)$  of a regular  $\omega$ -language is rational.*

We consider the language  $V_3 \subseteq \{a, b\}^*$  defined by the equation  $V_3 = a \cup b \cdot V_3$  which is known to be accepted by a deterministic one-counter automaton using empty-storage acceptance (cf. [ABB97]). Accordingly the  $\omega$ -languages  $V_3 \cdot \{a, b\}^\omega$ ,  $F := \{a, b\}^\omega \setminus V_3 \cdot \{a, b\}^\omega$  and  $V_3 \cdot c \cdot \{a, b, c\}^\omega$  are also accepted by deterministic one-counter automata [EH93, Sta97].

Since  $V_3$  is prefix-free, the measure of these  $\omega$ -languages can be easily computed from the value  $\sigma_X(V_3)$  which in turn is the minimum positive solution  $t_{|X|}$  of the equation (cf. [Sta05, Theorem 3.1])

$$t = |X|^{-1} \cdot (1 + t^3). \quad (3)$$

The minimum positive solutions  $t_2 = \frac{\sqrt{5}-1}{2} < 1$  and  $0 < t_3 < 1$  are irrational<sup>1</sup>.

The first example presents an open  $\omega$ -language accepted by a deterministic one-counter automaton not satisfying the Automatic Baire property.

**Example 1** *We consider the open  $\omega$ -language  $F_1 := V_3 \cdot c \cdot \{a, b, c\}^\omega \subseteq \{a, b, c\}^\omega$ . Since  $\mu_=(\{a, b\}^\omega) = 0$  in  $\{a, b, c\}^\omega$ , we obtain  $\mu_=(F_1) = \mu_=(F_1 \cup \{a, b\}^\omega) = t_3/3$  which is irrational. Observe, that  $F_1 \cup \{a, b\}^\omega$  is closed.*

*If  $E \subseteq \{a, b, c\}^\omega$  is open and regular then  $\mathcal{C}(E) \setminus E$  is regular and nowhere dense, hence  $\mu_=(\mathcal{C}(E) \setminus E) = 0$  by Theorem 3. Now according to Theorem 12  $\mu_=(E) = \mu_=(\mathcal{C}(E))$  is rational. Thus  $\mu_=(F_1) \neq \mu_=(E)$ .*

*If  $\mu_=(F_1) > \mu_=(E) = \mu_=(\mathcal{C}(E))$  then  $F_1 \setminus \mathcal{C}(E)$  is non-empty and open; if  $\mu_=(E) < \mu_=(F_1) = \mu_=(F_1 \cup \{a, b\}^\omega)$  then  $E \setminus (F_1 \cup \{a, b\}^\omega) \subseteq E \setminus F_1$  is non-empty and open. In both cases  $F_1 \Delta E$  contains a non-empty open subset, hence  $F_1$  cannot have the Automatic Baire property.*

Next we present a closed  $\omega$ -language accepted by a deterministic one-counter automaton not having the Automatic Baire property.

**Example 2 (Example 3 of [Sta98])** *Define  $F_2 = \{a, b\}^\omega \setminus V_3 \cdot \{a, b\}^\omega$  as a subset of the space  $X^\omega = \{a, b\}^\omega$ . Then  $F_2$  is closed and has, according to the value of  $t_2$ , measure  $\mu_=(F_2) = 1 - t_2 = \frac{3-\sqrt{5}}{2} > 0$ . Moreover, we have  $w \cdot b^{2 \cdot |w|} \in V_3 \cdot \{a, b\}^* \subseteq X^* \setminus \text{pref}(F)$  which shows that  $F$  is nowhere dense.*

*The measure  $\mu_=(F_2)$  trivially satisfies the balance condition. Now Lemma 7 shows that  $F_2$  cannot have the Automatic Baire property.*

<sup>1</sup>In case of  $t_3$  assume  $t_3 = p/q$  where  $p \neq q$  are natural numbers having no common prime divisor. Then Eq. (3) yields  $3 \cdot p \cdot q^2 = p^3 + q^3$  which is impossible.



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