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A note on Automatic BAIRE property

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CDMTCS-584
May 2025

Centre for Discrete Mathematics and
Theoretical Computer Science

A note on Automatic Baire property

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Abstract

Automatic Baire property is a variant of the usual Baire property which is fulfilled for subsets of the Cantor space accepted by finite automata. We consider the family $\mathcal{A}$ of subsets of the Cantor space having the Automatic Baire property. In particular we show that not all finite subsets have the Automatic Baire property, and that already a slight increase of the computational power of the accepting device may lead beyond the class $\mathcal{A}$.

In [Fin20, Fin21] Finkel introduced an automata-theoretic variant of the topological Baire property for subsets of the Cantor space. He showed that this Automatic Baire property is valid for regular ω-languages, that is, for subsets of the Cantor space definable by finite automata.

In this note we investigate which ω-languages beyond regular ones have the the Automatic Baire property. We get a full characterisation of ω-languages of first Baire category as well as of finite ω-languages having

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the Automatic Baire property. In this respect, disjunctive ω-words, that is, ω-words random w.r.t. to finite automata in the measure-theoretic approach (cf. [Sta18]) play a major rôle. Here, as a tool, we use the measure-category coincidence for regular ω-languages (see [Sta76], Theorem 3 of [Sta98], [VV06], or Section 9.4 of [VV12]).

Moreover, we show that, besides definability by finite automata, other computational constraints do not imply Automatic Baire property. To this end we derive ω-languages closed or open in the topology of the Cantor space definable by simple one-counter automata not having the Automatic Baire property.

1 Preliminaries

1.1 Notation

We introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. Its elements will be usually denoted by letters $i, \ldots, n$. Let X be an alphabet of cardinality $|X| \geqslant 2$. Then X^* is the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X. Subsets of X^* will be referred to as *languages* and subsets of X^ω as *ω-languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a language W let $W^* := \bigcup_{i \in \mathbb{N}} W^i$, and $W^\omega := \{w_1 \cdots w_i \cdots : w_i \in W \setminus \{e\}\}$ be the set of infinite strings formed by concatenating non-empty words in W. Furthermore, $|w|$ is the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\{\eta\})$ ($\eta \in X^* \cup X^\omega$) by $w \sqsubseteq \eta$.

An ω-word $\xi \in X^\omega$ is *ultimately periodic* if there are words $w, v \in X^*$ such that $\xi = w \cdot v^\omega = w \cdot v \cdot v \cdots$, and an ω-word $\zeta \in X^\omega$ is *disjunctive* (or *rich* , [Sta98]) if every $w \in X^*$ is an infix of ζ, that is, $\zeta \in \bigcap_{w \in X^*} X^* \cdot w \cdot X^*$.

1.2 Regular ω-languages

As usual, a language $W \subseteq X^*$ is *regular* if it is obtained from finite languages via the operations union, concatenation and star. An ω-language $F \subseteq X^\omega$ is *regular* if it is of the form $F = \bigcup_{i=1}^{n} W_i \cdot V_i^\omega$ where $W_i, V_i \subseteq X^*$ are regular languages.

We assume the reader to be familiar with the basic facts of the theory of regular languages and finite automata. For more details on ω-languages and regular ω-languages see the books [PP04, TB73] or the survey papers [Sta97, Tho90].

The following is well-known.

Theorem 1 *The family of regular ω-languages is a Boolean algebra, and every non-empty regular ω-language contains an ultimately periodic ω-word.*

1.3 The Cantor space

We consider X^ω as a topological space (Cantor space). The *closure* (smallest closed set containing F) $\mathcal{C}(F)$ of a subset $F \subseteq X^\omega$ is described as $\mathcal{C}(F) := \{\xi : \mathbf{pref}(\{\xi\}) \subseteq \mathbf{pref}(F)\}$. The *open sets* in Cantor space are the ω-languages of the form $W \cdot X^\omega$.

Next we recall some topological notions, see [Kur66, Oxt80]. As usual, an ω-language $F \subseteq X^\omega$ is *dense in* X^ω if $\mathcal{C}(F) = X^\omega$. This is equivalent to $\mathbf{pref}(F) = X^*$. An ω-language $F \subseteq X^\omega$ is *nowhere dense in* X^ω if its closure $\mathcal{C}(F)$ does not contain a non-empty open subset. This property is equivalent to the fact that for all $v \in \mathbf{pref}(F)$ there is a $w \in X^*$ such that $v \cdot w \notin \mathbf{pref}(F)$. If a regular ω-language $F \subseteq X^\omega$ is nowhere dense then there is a word $w \in X^*$ such that $F \subseteq X^* \cdot w \cdot X^\omega$ [Sta76].

Moreover, a subset $F \subseteq X^\omega$ is *meagre* or of *first Baire category* if it is a countable union of nowhere dense sets.

Any subset of a nowhere dense set is nowhere dense, hence, every subset of a meagre set is again meagre. A finite union of nowhere dense sets is nowhere dense, and a countable union of meagre sets is meagre.

The following property is a consequence of the fact that in Cantor space no non-empty open subset is of first Baire category.

Property 2 *Let $F \subseteq X^\omega$ be of first Baire category and $E \subseteq X^\omega$ be open. If $F \,\Delta\, E$ is of first Baire category then $E = \emptyset$.*

2 Measure and Category

In this section we consider the relation between measures on Cantor space and topological density.

For every $w \in X^*$ the ball $w \cdot X^\omega = \bigcup_{x \in X} wx \cdot X^\omega$ is a disjoint union of its sub-balls. Thus $\mu(w \cdot X^\omega) = \sum_{x \in X} \mu(wx \cdot X^\omega)$ for every measure μ on X^ω. The *support* of a measure μ on X^ω, $\mathbf{supp}(\mu)$, is the smallest closed subset of X^ω such that $\mu(\mathbf{supp}(\mu)) = \mu(X^\omega)$.

As measures μ on X^ω we consider finite non-null measures ($0 < \mu(X^\omega) < \infty$) having the following property that the measure of a non-null sub-ball $wx \cdot X^\omega$ does not deviate too much from $\mu(w \cdot X^\omega)$ (cf. [Sta98, VV12]).

Definition 1 (Balance condition) A measure μ on X^ω is referred to as *balanced* (or *bounded away from zero [VV12]*) provided there is a constant $c_\mu > 0$ depending only on μ such that for all words $w \in X^*$ and every $x \in X$ we have $\mu(wx \cdot X^\omega) = 0$ or $c_\mu \cdot \mu(w \cdot X^\omega) \leqslant \mu(wx \cdot X^\omega)$.

In the book by Oxtoby [Oxt80] analogies between topological density and measure, in particular, the "duality" between measure and category are discussed. The papers [Sta76, Sta98, VV06] and [VV12] show that for regular ω-languages in Cantor space measure and category coincide.

Theorem 3 (Theorem 3 of [Sta98]) *Let $F \subseteq X^\omega$ be a regular ω-language. Then the following conditions are equivalent:*

1. *No $\zeta \in F$ is a disjunctive ω-word.*
2. *F is of first Baire category.*
3. *For all measures μ with* $\mathbf{supp}(\mu) = X^\omega$ *satisfying the balance condition it holds $\mu(F) = 0$.*
4. *There is a measure μ with* $\mathbf{supp}(\mu) = X^\omega$ *satisfying the balance condition such that $\mu(F) = 0$.*

Theorem 3.1 shows that the union of all regular ω-languages of first Baire category $\mathbf{R}_0$ can be characterised as follows (see e.g. [Sta76, Korollar 8]).

$$\mathbf{R}_0 = \bigcup\nolimits_{w \in X^*} (X^\omega \setminus X^* \cdot w \cdot X^\omega) \tag{1}$$

3 Baire property and Automatic Baire property

Automatic Baire property was introduced by Finkel [Fin20, Fin21]. Here we define this variant of the usual Baire property and derive several of its properties. First we recall the following (see e.g. [Kur66, Oxt80]).

Definition 2 A subset $F \subseteq X^\omega$ has the *Baire property* if there is an open set $E \subseteq X^\omega$ such that their symmetric difference $F \,\Delta\, E$ is of first Baire category.

Theorem 4 *Every Borel set of the Cantor space has the Baire property.*

The Automatic Baire property requires the sets E and $F \Delta E$ to be restricted in some sense to regular ω-languages.

Definition 3 (Automatic Baire property) A subset $F \subseteq X^\omega$ has the *Automatic Baire property* if

$$F \Delta E \subseteq F', \tag{2}$$

where E is a regular and open ω-language and F' a regular ω-language of first Baire category.

Then it holds the following.

Theorem 5 ([Fin20, Fin21]) *Every regular ω-language has the Automatic Baire property.*

We derive some properties of the class $\mathcal{A}$ of all ω-languages having the Automatic Baire property. It is obvious that every ω-language which has the Automatic Baire property has also the Baire property.

Lemma 6 *$\mathcal{A}$ is a Boolean algebra.*

Proof. This follows from $(F_1 \cup E_1) \Delta (F_2 \cup E_2) \subseteq (F_1 \Delta E_1) \cup (F_2 \Delta E_2)$ and $(X^\omega \setminus F) \Delta (X^\omega \setminus E) = F \Delta E$ and the fact that the union of two regular ω-languages of first Baire category is also regular and of first Baire category. ❑

We derive a necessary condition for sets to be of first Baire category.

Lemma 7 *Let $F \Delta E \subseteq F'$ where $E \subseteq X^\omega$ is open and $F' \subseteq X^\omega$ a regular ω-language of first Baire category. Then for every measure μ with support* $\mathbf{supp}(\mu) = X^\omega$ *satisfying the balance condition it holds $\mu(F) = 0$ if and only if F is of first Baire category.*

Proof. Let $F \Delta E \subseteq F'$ where E is open and F' is regular and of first Baire category. According to Theorem 3 we have $\mu(F') = 0$.

If $\mu(F) = 0$ then $\mu(E) = \mu(E) - \mu(F) \leqslant \mu(E \setminus F) \leqslant \mu(E \Delta F) \leqslant \mu(F') = 0$ implies $E = \emptyset$. Thus $F = E \Delta F$ is of first Baire category.

If F and $E \Delta F$ are of first Baire category then $E \subseteq (E \Delta F) \cup F$ is also of first Baire category. Thus $E = \emptyset$. Consequently, $\mu(F) = \mu(E \Delta F) = 0$. ❑

Remark. Observe that in Lemma 7 we did not use the fact that the open set E is regular.
The proof of Lemma 7 shows also the following.

Corollary 8 *Let $F \subseteq X^\omega$ be of first Baire category. Then $F \in \mathcal{A}$ if and only if $F \subseteq F'$ for some regular ω-language of first Baire category.*

Finite ω-languages in $\mathcal{A}$ are characterised as follows.

Corollary 9 *Let* $F \subseteq X^\omega$ *be finite. Then* $F \in \mathcal{A}$ *if and only if* F *does not contain a disjunctive* ω*-word.*

Proof. If F is finite then F is of first Baire category. Now Corollary 8 and Theorem 3 imply that F does not contain a disjunctive ω-word.

If F is finite and does not contain a disjunctive ω-word then for every $\xi \in F$ there is a w_ξ such that $\xi \notin X^* \cdot w_\xi \cdot X^\omega$. Then $F \subseteq \bigcup_{\xi \in F}(X^\omega \setminus X^* \cdot w_\xi \cdot X^\omega)$ which is a regular and nowhere dense ω-language. ❏

Besides finite ω-languages containing disjunctive ω-words, examples of sets not satisfying the Automatic Baire property are the following ones.

Lemma 10 *If* $F \subseteq X^\omega$, $\mathsf{Ult} \subseteq F \subseteq \mathbf{R}_0$, *then* F *does not have the Automatic Baire property.*

Proof. Since $\mathsf{Ult} \subseteq F \subseteq \mathbf{R}_0$, the set F is of first Baire category. Now Property 2 shows that the symmetric difference $E \,\Delta\, F$ with a non-empty open set E is not of first Baire category. Hence $E = \emptyset$ and $F \subseteq F'$ for some regular ω-language F'

Then $X^\omega \setminus F' \subseteq X^\omega \setminus \mathsf{Ult}$ does not contain any ultimately periodic ω-word. Consequently, $F' = X^\omega$ which is not of first Baire category. ❏

Corollary 11 *The family* $\mathcal{A}$ *is not closed under countable union.*

Proof. As $\mathbf{R}_0 = \bigcup_{w \in X^*}(X^\omega \setminus X^* \cdot w \cdot X^\omega)$ and every ω-language $X^\omega \setminus X^* \cdot w \cdot X^\omega$ is regular and nowhere dense in X^ω (cf. [Sta76]), the assertion follows immediately. ❏

4 Simple counter-examples

In Corollary 9 we have seen that there are even finite ω-languages having the Baire property but not the Automatic Baire property. Those finite ω-languages contain ω-words $\xi \notin \mathsf{Ult}$ and are, therefore, not context-free (e.g. [EH93, Sta97]), that is accepted by push-down automata.

In this part we show that also a slight increase of the computational power of accepting devices results in open or closed ω-languages not having the Automatic Baire property.

As measure in Cantor space we use the equidistribution. For a language $W \subseteq X^*$ we set $\sigma_X(W) := \sum_{w \in W} |X|^{-|w|}$. Then $\mu_=(W \cdot X^\omega) = \sigma_X(W)$, if $W \subseteq X^*$ prefix-free, that is, $w \sqsubseteq v$ and $w, v \in W$ imply $w = v$.

Since $\sigma_X(W)$ is rational for regular languages $W \subseteq X^*$, we have the following (see [Tak01, Theorem 4.16]).

Theorem 12 *The measure $\mu_=(F)$ of a regular ω-language is rational.*

We consider the language $V_3 \subseteq \{a,b\}^*$ defined by the equation $V_3 = a \cup b \cdot V_3$ which is known to be accepted by a deterministic one-counter automaton using empty-storage acceptance (cf. [ABB97]). Accordingly the ω-languages $V_3 \cdot \{a,b\}^\omega$, $F := \{a,b\}^\omega \setminus V_3 \cdot \{a,b\}^\omega$ and $V_3 \cdot c \cdot \{a,b,c\}^\omega$ are also accepted by deterministic one-counter automata [EH93, Sta97].

Since V_3 is prefix-free, the measure of these ω-languages can be easily computed from the value $\sigma_X(V_3)$ which in turn is the minimum positive solution $t_{|X|}$ of the equation (cf. [Sta05, Theorem 3.1])

$$t = |X|^{-1} \cdot (1 + t^3)\,. \tag{3}$$

The minimum positive solutions $t_2 = \frac{\sqrt{5}-1}{2} < 1$ and $0 < t_3 < 1$ are irrational[1].

The first example presents an open ω-language accepted by a deterministic one-counter automaton not satisfying the Automatic Baire property.

Example 1 *We consider the open ω-language $F_1 := V_3 \cdot c \cdot \{a,b,c\}^\omega \subseteq \{a,b,c\}^\omega$. Since $\mu_=(\{a,b\}^\omega) = 0$ in $\{a,b,c\}^\omega$, we obtain $\mu_=(F_1) = \mu_=(F_1 \cup \{a,b\}^\omega) = t_3/3$ which is irrational. Observe, that $F_1 \cup \{a,b\}^\omega$ is closed.*

If $E \subseteq \{a,b,c\}^\omega$ is open and regular then $\mathcal{C}(E) \setminus E$ is regular and nowhere dense, hence $\mu_=(\mathcal{C}(E) \setminus E) = 0$ by Theorem 3. Now according to Theorem 12 $\mu_=(E) = \mu_=(\mathcal{C}(E))$ is rational. Thus $\mu_=(F_1) \neq \mu_=(E)$.

If $\mu_=(F_1) > \mu_=(E) = \mu_=(\mathcal{C}(E))$ then $F_1 \setminus \mathcal{C}(E)$ is non-empty and open; if $\mu_=(E) < \mu_=(F_1) = \mu_=(F_1 \cup \{a,b\}^\omega)$ then $E \setminus (F_1 \cup \{a,b\}^\omega) \subseteq E \setminus F_1$ is non-empty and open. In both cases $F_1 \,\Delta\, E$ contains a non-empty open subset, hence F_1 cannot have the Automatic Baire property.

Next we present a closed ω-language accepted by a deterministic one-counter automaton not having the Automatic Baire property.

Example 2 (Example 3 of [Sta98]) *Define $F_2 = \{a,b\}^\omega \setminus V_3 \cdot \{a,b\}^\omega$ as a subset of the space $X^\omega = \{a,b\}^\omega$. Then F_2 is closed and has, according to the value of t_2, measure $\mu_=(F_2) = 1 - t_2 = \frac{3-\sqrt{5}}{2} > 0$. Moreover, we have $w \cdot b^{2\cdot|w|} \in V_3 \cdot \{a,b\}^* \subseteq X^* \setminus$ **pref**(F) which shows that F is nowhere dense.*

The measure $\mu_=$ trivially satisfies the balance condition. Now Lemma 7 shows that F_2 cannot have the Automatic Baire property.

[1] In case of t_3 assume $t_3 = p/q$ where $p \neq q$ are natural numbers having no common prime divisor. Then Eq. (3) yields $3 \cdot p \cdot q^2 = p^3 + q^3$ which is impossible.

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