On the incomputability of computable dimension

Ludwig Staiger
Martin-Luther-Universität
Halle-Wittenberg

CDMTCS-535
May 2019

Centre for Discrete Mathematics and Theoretical Computer Science
On the incomputability of computable dimension

Ludwig Staiger*
Martin-Luther-Universität Halle-Wittenberg
Institut für Informatik
von-Seckendorff-Platz 1
D–06099 Halle (Saale), Germany

Abstract
Using an iterative tree construction we show that for simple computable subsets of the Cantor space Hausdorff, constructive and computable dimensions might be incomputable.

Contents

1 Notation 2

2 Iterative Tree Construction 3
  2.1 Preliminaries .................................................. 3
  2.2 Tree construction .............................................. 4
  2.3 The infinite tree .............................................. 5
  2.4 Monotone families $\left(q_i \right)_{i \in \mathbb{N}}$ ......................... 7

3 Gales and Martingales 7

4 Incomputable dimensions 9
  4.1 Hausdorff dimension ........................................... 9
  4.2 Martingales ..................................................... 10
  4.3 Comparison of gales and martingales ....................... 11

5 Concluding remark 12

*email: staiger@informatik.uni-halle.de
Computable dimension along with constructive dimension was introduced by Lutz [Lut03] as a means for measuring the complexity of sets of infinite strings (ω-words). Since then and prior to this constructive and computable dimension were investigated in connection with Kolmogorov complexity and Hausdorff dimension. The results of [Hit05, Sta93, Sta98] show that the Hausdorff, constructive and the computable dimensions of automaton definable sets of infinite strings (regular ω-languages) is computable. In contrast to this Ko [Ko98] derived examples of computable ω-languages which have incomputable Hausdorff dimension.

In this paper we derive simple examples of computable ω-languages which have not only incomputable Hausdorff dimension but also incomputable computable dimension. To this end we use in iteration of finite trees which resembles the tree construction of Furstenberg [Fur70] (see also [MSS18]).

As a byproduct we obtain simple examples of computable ω-languages having incomputable Hausdorff dimension.

Lutz [Lut03] defines computable and constructive dimension via (super-)gales. Terwijn [Ter04, CST06] observed that this can also be done using Schnorr’s concept of martingales and (exponential) order functions [Sch71, Section 17]. For the computable ω-languages derived in this paper we can show that the latter concept is in some details more precise than Lutz’s approach.

1 Notation

In this section we introduce the notation used throughout the paper. By \( \mathbb{N} = \{0,1,2,\ldots\} \) we denote the set of natural numbers, by \( \mathbb{Q} \) the set of rational numbers, and \( \mathbb{R} \) are the real numbers.

Let \( X \) be an alphabet of cardinality \( |X| \geq 2 \). By \( X^* \) we denote the set of finite words on \( X \), including the empty word \( e \), and \( X^\omega \) is the set of infinite strings (ω-words) over \( X \). Subsets of \( X^* \) will be referred to as languages and subsets of \( X^\omega \) as ω-languages.

For \( w \in X^* \) and \( \eta \in X^* \cup X^\omega \) let \( w \cdot \eta \) be their concatenation. This concatenation product extends in an obvious way to subsets \( W \subseteq X^* \) and \( B \subseteq X^* \cup X^\omega \). We denote by \( |w| \) the length of the word \( w \in X^* \) and \( \text{pref}(B) \) is the set of all finite prefixes of strings in \( B \subseteq X^* \cup X^\omega \).

It is sometimes convenient to regard \( X^\omega \) as Cantor space, that is, as the product space of the (discrete space) \( X \). Here open sets in \( X^\omega \) are those of the form \( W \cdot X^\omega \) with \( W \subseteq X^* \). Closed are sets \( F \subseteq X^\omega \) which satisfy the condition \( F = \{ \xi : \text{pref}(\xi) \subseteq \text{pref}(F) \} \).

For a computable domain \( \mathcal{D} \), such as \( \mathbb{N}, \mathbb{Q} \) or \( X^* \), we refer to a function
On the incomputability of computable dimension

3

\( f : \mathcal{D} \to \mathbb{R} \) as left-computable (or approximable from below) provided the set \( \{ (d, q) : d \in \mathcal{D} \land q \in \mathbb{Q} \land q < f(d) \} \) is computably enumerable. Accordingly, a function \( f : \mathcal{D} \to \mathbb{R} \) is called right-computable (or approximable from above) if the set \( \{ (d, q) : d \in \mathcal{D} \land q \in \mathbb{Q} \land q > f(d) \} \) is computably enumerable, and \( f \) is computable if \( f \) is right- and left-computable.

If we refer to a function \( f : \mathcal{D} \to \mathbb{Q} \) as computable we usually mean that it maps the domain \( \mathcal{D} \) to the domain \( \mathbb{Q} \), that is, it returns the exact value \( f(d) \in \mathbb{Q} \). If \( \mathcal{D} = \mathbb{N} \) we write \( f \) as a sequence \((q_i)_{i \in \mathbb{N}}\).

A real number \( \alpha \in \mathbb{R} \) is left-computable, right computable or computable provided the constant function \( c_\alpha(t) = \alpha \) is left-computable, right-computable or computable, respectively. \( \alpha \in \mathbb{R} \) is referred to as computably approximable if \( \alpha = \lim_{i \to \infty} q_i \) for a computable sequence \((q_i)_{i \in \mathbb{N}}\) of rationals. It is well-known (see e.g. [ZW01]) that there are left-computable which are not right-computable and vice versa, and that there are computably approximable reals which are neither left-computable nor right-computable.

The following approximation property is easily verified.

**Property 1** Let \((q_i)_{i \in \mathbb{N}}\) be a computable family of rationals converging to \( \alpha \) and let \((q'_i)_{i \in \mathbb{N}}, q'_i > 0\), be a computable family of rationals converging to 0. If \( \alpha \) is not right-computable then there are infinitely many \( i \in \mathbb{N} \) such that \( \alpha - q_i > q'_i \).

For, otherwise, \( \alpha \) as the limit of \((q_i + q'_i)_{i \in \mathbb{N}}\) would be right-computable.

## 2 Iterative Tree Construction

### 2.1 Preliminaries

The aim of this section is to present how one can, given a sequence of rationals \((q_i)_{i \in \mathbb{N}}\), find sequences of natural numbers \((k_i)_{i \in \mathbb{N}}\) and \((\ell_i)_{i \in \mathbb{N}}\) with appropriate properties such that \( q_i = k_i/\ell_i \).

**Lemma 2** Let \((q_i)_{i \in \mathbb{N}}, 0 < q_i < 1, q_i \neq q_{i+1}\) be a family of positive rationals. Then there are families of natural numbers \((k_i)_{i \in \mathbb{N}}, (\ell_i)_{i \in \mathbb{N}}, (\kappa_i)_{i \in \mathbb{N}}, (p_i)_{i \in \mathbb{N}}\) and \((r_i)_{i \in \mathbb{N}}\) such that \( q_i = k_i/\ell_i, \ q_{i+1} = r_i \cdot k_i + \kappa_i \cdot \ell_i \) where \( \kappa_i = \begin{cases} 0, & \text{if } q_i > q_{i+1} \text{ and} \\ p_i, & \text{if } q_i < q_{i+1}. \end{cases} \)

Moreover, for \( 0 \leq t \leq p_i \cdot \ell_i \) we have

\[
q_i \geq \frac{r_i \cdot k_i}{r_i \cdot \ell_i + t} \geq q_{i+1}, \text{ if } q_i > q_{i+1} \text{ and } \tag{1}
\]
has the solutions

\[ q_i \leq \frac{r_i \cdot k_i + t}{r_i \cdot \ell_i + t} \leq q_{i+1}, \text{ if } q_i < q_{i+1}. \]  

(2)

Proof. Let \( q_i = k_i / \ell_i \) and \( q_{i+1} = a/b \cdot q_i = a \cdot k_i / b \cdot \ell_i \), with \( a, b \in \mathbb{N} \setminus \{0\}, a \neq b \). Since \( 1 > q_{i+1} \) we have \( b \cdot \ell_i - a \cdot k_i = a \cdot \frac{q_i}{q_{i+1}} \cdot (1 - q_{i+1}) \cdot \ell_i > 0 \).

Assume \( q_i > q_{i+1} \). Then \( b > a \) and the equation

\[ \frac{r_i \cdot k_i + \kappa_i \cdot \ell_i}{r_i \cdot \ell_i + p_i \cdot \ell_i} = \frac{a \cdot k_i}{b \cdot \ell_i} \]

(3)

has the solutions \( r_i = a \), and \( p_i = (b - a) = a \cdot \left( \frac{q_i}{q_{i+1}} - 1 \right) \) and \( \kappa_i = 0 \).

If \( q_i < q_{i+1} \) then \( a > b \) and \( r_i := b \cdot \ell_i - a \cdot k_i = a \cdot \left( \frac{q_i}{q_{i+1}} \cdot \ell_i - k_i \right) = a \cdot q_i \cdot \left( \frac{1}{q_{i+1}} - 1 \right) \cdot \ell_i \) and \( p_i = \kappa_i := (a - b) \cdot k_i = a \cdot q_i \cdot (1 - \frac{q_i}{q_{i+1}}) \cdot \ell_i \) are solutions of Eq. (3).

In view of \( \kappa_i = 0 \) Eq. (1) is obvious. Eq. (2) follows inductively from \( \frac{k+1}{\ell+1} \geq \frac{k}{\ell} \) whenever \( 0 \leq k < \ell \).

If the family \( (q_i)_{i \in \mathbb{N}} \) is a computable one then the families in Lemma 2 can be chosen to be computable. In addition, the values \( \ell_i \) and \( \ell_{i+1}/\ell_i \) can be made arbitrarily large.

### 2.2 Tree construction

We define \( F \) via the following auxiliary languages \( T_i \subseteq X^{\ell_i} \) and \( U_i \subseteq X^{p_i \cdot \ell_i} \).

Let \( T_0 := X^{k_0 \cdot 0^60 - k_0} \) or \( T_0 := 0^{60 - k_0} \cdot X^{k_0} \) and set

\[ T_{i+1} := T_i^{r_i} \cdot U_i \quad \text{with} \quad U_i := \begin{cases} X^{p_i \cdot \ell_i}, & \text{if } q_{i+1} \geq q_i \text{ and} \\ \{ u_i \}, & \text{otherwise} \end{cases} \]

(4)

where \( u_i \in X^{p_i} \) is a fixed word. Then \( \ell_{i+1} = (r_i + p_i) \cdot \ell_i \). The values \( r_i \) and \( p_i \) are referred to as repetition or prolongation factors, respectively.

By induction one proves

\[ |T_i| = |X|^{q_i \cdot \ell_i}. \]

(5)

**Property 3** The trees \( T_i \) have the following properties. Let \( \ell \leq \ell_i \).

1. Prefix property: \( \text{pref}(T_{i+1}) = \bigcup_{j=0}^{q_i - 1} T_i^j \cdot \text{pref}(T_i) \cup T_i^{r_i} \cdot \text{pref}(U_i) \),

2. Extension property: \( \text{pref}(T_i) \cap X^\ell = \text{pref}(T_{i+1}) \cap X^\ell \), and

3. Spherical symmetry: \( \text{pref}(T_i) \cap X^{\ell - 1} = \text{pref}(T_i) \cap X^{\ell - 1} \cdot X \) or \( |\text{pref}(T_i) \cap X^{\ell - 1}| = |\text{pref}(T_i) \cap X^\ell| \).
2.3 The infinite tree

Define $F := \bigcap_{i \in \mathbb{N}} T_i \cdot X^\omega$ where $(T_i)_{i \in \mathbb{N}}$ satisfies Eq. (4).

Before we proceed to further properties of the family $(T_i)_{i \in \mathbb{N}}$ and the $\omega$-language $F$ we mention the following general property.

**Lemma 4** Let $T_i \subseteq X^\omega$, $T_{i+1} \subseteq T_i \cdot X^\omega$, $T_i \subseteq \text{pref}(T_{i+1})$ and $F := \bigcap_{i \in \mathbb{N}} T_i \cdot X^\omega$. Then $\text{pref}(F) = \bigcup_{i \in \mathbb{N}} \text{pref}(T_i)$.

If, moreover, all $T_i$ are finite then $F := \{ \xi : \xi \in X^\omega \land \text{pref}(\xi) \subseteq \bigcup_{i \in \mathbb{N}} \text{pref}(T_i) \}$.

**Proof.** In view of $T_{i+1} \subseteq T_i \cdot X^\omega$ we have $T_{i+1} \cdot X^\omega \subseteq T_i \cdot X^\omega$ and also $|w| \geq i$ for $w \in T_i$.

If $w \in \text{pref}(F)$ then $w \in \text{pref}(\xi)$ where $\xi \in F \subseteq T_i \cdot X^\omega$ for $i > |w|$. Consequently, $w \in \text{pref}(T_i)$.

Using the condition $T_i \subseteq \text{pref}(T_{i+1})$, by induction we obtain that for every $w \in \text{pref}(T_i)$ there is an infinite chain $(w_j)_{j \geq 1}$ such that $w_j \in T_j$ and $w \subseteq w_i \sqcup w_{i+1} \sqcup \ldots$. Thus there is a $\xi \in F$ with $w \sqcup \xi$.

If the languages $T_i$ are finite $F = \bigcap_{i \in \mathbb{N}} T_i \cdot X^\omega$ is closed in the product topology of the space $X^\omega$ which implies $F := \{ \xi : \xi \in X^\omega \land \text{pref}(\xi) \subseteq \text{pref}(F) \}$.

Our lemma shows that $F := \{ \xi : \xi \in X^\omega \land \text{pref}(\xi) \subseteq \bigcup_{i \in \mathbb{N}} \text{pref}(T_i) \}$.

From the spherical symmetry of $T_i$ (see Property 3) the $\omega$-language $F = \bigcap_{i \in \mathbb{N}} T_i \cdot X^\omega$ inherits the following balance property.

**Lemma 5** Let $F = \bigcap_{i \in \mathbb{N}} T_i \cdot X^\omega$ where the $T_i$ are defined by Eq. (4). Then for all $w, v \in \text{pref}(F)$ with $|w| = |v|$ we have

$$|w \cdot X^k \cap \text{pref}(F)| = |v \cdot X^k \cap \text{pref}(F)|.$$

**Proof.** We proceed by induction on $k$. Let $k = 1$. Then for all $w, v \in \text{pref}(F)$ with $|w| = |v|$ either $\text{pref}(F) \cap X^{[u]+1} = (\text{pref}(F) \cap X^{[u]}) \cdot X$ or $|\text{pref}(F) \cap X^{[u]+1}| = |\text{pref}(F) \cap X^{[u]}|$ ($u \in \{w, v\}$).

In the first case we have $|w \cdot X \cap \text{pref}(F)| = |X| = |v \cdot X \cap \text{pref}(F)|$ and in the second $|w \cdot X \cap \text{pref}(F)| = 1 = |v \cdot X \cap \text{pref}(F)|$.

Let the assertion be proved for $k$ and all pairs $u, u' \in \text{pref}(F)$ of the same length. Let $w, v \in \text{pref}(F)$ with $|w| = |v|$ and consider words $w', v' \in X^k$ such that $w \cdot w', v \cdot v' \in \text{pref}(F)$. Then from the spherical symmetry we obtain either $\text{pref}(F) \cap X^{[u]+1} = (\text{pref}(F) \cap X^{[u]}) \cdot X$ or $|\text{pref}(F) \cap X^{[u]+1}| = |\text{pref}(F) \cap X^{[u]}|$ for $u \in \{w \cdot w', v \cdot v'\}$ and we proceed as above.

Since, by our assumption $|w' : |w'| = k \land w \cdot w' \in \text{pref}(F)| = |v' : |v'| = k \land v \cdot v' \in \text{pref}(F)|$, the assertion follows.

Next we investigate in more detail the structure function $s_F : \mathbb{N} \to \mathbb{N}$ where $s_F(\ell) := |\text{pref}(F) \cap X^\ell|$. 

\[ \]
First, Lemma 4 implies
\[ \text{pref}(F) \cap X^\ell = \text{pref}(T_i) \cap X^\ell \] whenever \( \ell \leq \ell_i \).
(6)

From Eqs. (4) and (5) and the properties of the tree family \((T_i)_{i \in \mathbb{N}}\) we obtain for the intervals \(\ell_i \leq \ell \leq \ell_{i+1}\):

**Lemma 6**  
1. In the interval \([j \cdot \ell_i, (j+1) \cdot \ell_i]\) where \(j \leq r_i\) we have:
\[ s_F(j \cdot \ell_i + t) = s_F(\ell_i)^j \cdot s_F(t) \text{ for } 0 \leq t \leq \ell_i, \]
and in a more detailed form in the subinterval
\([j \cdot \ell_i + j' \cdot \ell_{i-1}, j \cdot \ell_i + (j' + 1) \cdot \ell_{i-1}]\) where \(j' < r_{i-1}\)
\[
s_F(j \cdot \ell_i + j' \cdot \ell_{i-1} + t) = s_F(\ell_i)^j \cdot s_F(\ell_{i-1})^{j'} \cdot s_F(t) \text{ for } 0 \leq t \leq \ell_{i-1}. \]

2. In the interval \([r_i \cdot \ell_i, \ell_{i+1}]\) for \(0 \leq t \leq p_i \cdot \ell_i\):
\[
s_F(r_i \cdot \ell_i + t) = \begin{cases} 
  s_F(\ell_i)^{r_i}, & \text{if } |U_i| = 1 \text{ and } \\
  s_F(\ell_i)^{r_i} \cdot |X|^t, & \text{if } U_i = X^{p_i \cdot \ell_i}. 
\end{cases} \]

This yields the following connection to the values \(q_i\).

From Eqs. (6) and (5) we have
\[
\frac{\log_{|X|} s_F(j \cdot \ell)}{j \cdot \ell} = q_i. 
\]
(7)

Using the identities of Lemma 6 and Eqs. (1) and (2) we obtain the following estimates for \(\frac{\log_{|X|} s_F(\ell)}{\ell}\) in the range \(\ell_i \leq \ell \leq \ell_{i+1} = r_i \cdot \ell_i + n_i \cdot \ell_i\).

For \(\ell_i \leq \ell < r_i \cdot \ell_i\) we have \(\ell = j \cdot \ell_i + j' \cdot \ell_{i-1} + t\) where \(0 \leq t < \ell_{i-1}\) and Lemma 6.1 yields
\[
\frac{\log_{|X|} s_F(\ell)}{\ell} \geq \frac{j \cdot \ell_i \cdot q_i + j' \cdot \ell_{i-1} \cdot q_{i-1}}{\ell} \geq \frac{j \cdot \ell_i + j' \cdot \ell_{i-1}}{\ell} \cdot \min\{q_{i-1}, q_i\} \geq (1 - \frac{\ell_{i-1}}{\ell_i}) \cdot \min\{q_{i-1}, q_i\} \geq j \cdot \ell_i \cdot q_i + j' \cdot \ell_{i-1} \cdot q_{i-1} \]
(8)

If \(r_i \cdot \ell_i \leq \ell \leq \ell_{i+1}\), that is, for \(\ell = r_i \cdot \ell_i + t\) where \(t \leq \ell_{i+1} - r_i \cdot \ell_i\), following Eqs. (1) and (2), respectively, we have according to Lemma 6.2
\[
q_i \geq \frac{\log_{|X|} s_F(\ell)}{\ell} = \frac{\log_{|X|} s_F(r_i \cdot \ell_i)}{r_i \cdot \ell_i + t} \geq q_{i+1} \text{ if } q_i > q_{i+1} \]
(9)
\[
q_i \leq \frac{\log_{|X|} s_F(\ell)}{\ell} = \frac{\log_{|X|} s_F(r_i \cdot \ell_i) + t}{r_i \cdot \ell_i + t} \leq q_{i+1} \text{ if } q_i < q_{i+1} \]
(10)

The considerations in Eqs. (7), (8), (9) and (10) show the following.
Lemma 7 If the sequence \((\ell_i)_{i \in \mathbb{N}}\) is chosen in such a way that \(\liminf_{i \to \infty} \frac{\ell_{i-1}}{\ell_i} = 0\) then
\[
\liminf_{\ell \to \infty} \frac{\log|X| s_F(\ell)}{\ell} = \liminf_{i \to \infty} q_i.
\]

Proof. In view of Eq. (7) the limit cannot exceed \(\liminf_{i \to \infty} q_i\).

On the other hand, by Eqs. (8), (9) and (10), for \(\ell_i \leq \ell \leq \ell_{i+1}\), the intermediate values satisfy
\[
\frac{\log|X| s_F(\ell)}{\ell} \geq (1 - \frac{\ell_{i-1}}{\ell_i}) \cdot \min(q_{i-1}, q_i, q_{i+1}). \quad \square
\]

2.4 Monotone families \((q_i)_{i \in \mathbb{N}}\)

If the sequence \((q_i)_{i \in \mathbb{N}}\) is monotone we can simplify the above considerations of Eq. (8).

Theorem 8 Let the sequence \((q_i)_{i \in \mathbb{N}}\) be monotone and \(\lim_{i \to \infty} q_i = \alpha\).

1. If \((q_i)_{i \in \mathbb{N}}\) is decreasing and \(T_0 = X^{k_0} \cdot 0^{l_0-k_0}\) then \(s_F(\ell) \geq |X|^{q_0-\ell}\) for all \(\ell \in \mathbb{N}\).

2. If \((q_i)_{i \in \mathbb{N}}\) is increasing and \(T_0 = 0^{l_0-k_0} \cdot X^{k_0}\) then \(s_F(\ell) \leq |X|^{q_0-\ell}\) for all \(\ell \in \mathbb{N}\).

Proof. If \((q_i)_{i \in \mathbb{N}}\) is decreasing we start with \(T_0 = X^{k_0} \cdot 0^{l_0-k_0}\) and have \(s_F(\ell) \geq |X|^{q_0-\ell} \geq |X|^{q_0-\ell}\) for \(\ell \leq \ell_0\). Then we use Eqs. (6) and (4) and proceed by induction.

If \((q_i)_{i \in \mathbb{N}}\) is increasing we start with \(T_0 = 0^{l_0-k_0} \cdot X^{k_0}\) and have \(s_F(\ell) \leq |X|^{q_0-\ell} \leq |X|^{q_0-\ell}\) for \(\ell \leq \ell_0\). Again we use Eqs. (6) and (4) and proceed by induction.

3 Gales and Martingales

Hausdorff [Hau18] introduced a notion of dimension of a subset \(Y\) of a metric space which is now known as its Hausdorff dimension, \(\dim Y\); Falconer [Fal03] provides an overview and introduction to this subject. In the case of the Cantor space \(X^\omega\), Lutz [Lut03] (see also [DH10, Section 13.2]) has found an equivalent definition of Hausdorff dimension via generalisations of martingales.
Following Lutz a mapping \( d : X^* \to [0, \infty) \) will be called an \( \sigma\)-supergale provided
\[
\forall w (w \in X^* \to |X|^\sigma \cdot d(w) \geq \sum_{x \in X} d(wx)). \tag{11}
\]
A \( \sigma\)-supergale \( d \) is called an \( \sigma \)-gale if, for all \( w \in X^* \), Eq. (11) is satisfied with equality. (Super-)Martingales are 1-(super-)gales.

Observe that, for \( \sigma' \geq \sigma \) any \( \sigma \)-supergale \( d \) is also a \( \sigma' \)-supergale. We define the cut point \( \chi_d \) of a supergale \( d \) as the smallest value \( \sigma \) for which \( d \) can be an \( \sigma \)-supergale.
\[
\chi_d := \inf \{ \sigma : \forall w (|X|^\sigma \cdot d(w) \geq \sum_{x \in X} d(wx)) \} \tag{12}
\]
If \( d \) is a computable mapping then \( \chi_d \) as sup\{\( q \in \mathbb{Q} \land \exists w (|X|^q \cdot d(w) < \sum_{x \in X} d(wx)) \)\} is a left-computable real number.

Following Lutz [Lut03] we define as follows.

**Definition 1** Let \( F \subseteq X^\omega \). Then \( \alpha \) is the Hausdorff dimension \( \dim F \) of \( F \) provided
1. for all \( \sigma > \alpha \) there is a \( \sigma \)-supergale \( d \) such that \( \forall \xi (\xi \in F \to \limsup_{w \to \xi} d(w) = \infty) \), and
2. for all \( \sigma < \alpha \) and all \( \sigma \)-supergales \( d \) it holds \( \exists \xi (\xi \in F \land \limsup_{w \to \xi} d(w) < \infty) \).

For \( \omega \)-languages having a simple structure like the one in the tree construction above we can simplify the calculation of the Hausdorff dimension (see [Sta89, Theorem 4]).

**Lemma 9** Let \( F \subseteq X^\omega \) satisfy the conditions \( F = \{ \xi : \text{pref}(\xi) \subseteq \text{pref}(F) \} \) and \( s_F \cap \omega \times X^\omega = s_F \cap w \times X^\omega \) for all \( w, v \in \text{pref}(F) \) with \( |w| = |v| \). Then
\[
\dim F = \liminf_{n \to \infty} \log \frac{|X|}{\text{max} \{1, s_F(n)\}}.
\]
As a consequence we obtain the following.

**Corollary 10** Let \( F \subseteq X^\omega \) be constructed according to the tree construction of Section 2.2. Then \( \dim F = \liminf_{n \to \infty} \frac{\log |X|}{\text{max} \{1, s_F(n)\}}. \)

If we require the supergales in Definition 1 to be computable mappings we obtain the definition of computable dimension \( \dim_{\text{comp}} F \) of [Hit05, Lut03].

---

\(^1\)Here \( \limsup_{w \to \xi} d(w) \) is an abbreviation for \( \lim_{n \to \infty} \sup_{w \in \text{pref}(\xi) \land |w| \geq n} d(w) \).
Definition 2 Let $F \subseteq X^\omega$. Then $\alpha$ is the \textit{computable dimension} of $F$ provided

1. for all $\sigma > \alpha$ there is a computable $\sigma$-supergale $d$ such that $\forall \xi (\xi \in F \rightarrow \limsup_{w \to \xi} d(w) = \infty)$, and

2. for all $\sigma < \alpha$ and all computable $\sigma$-supergales $d$ it holds $\exists \xi (\xi \in F \land \limsup_{w \to \xi} d(w) < \infty)$.

Then the inequality $\dim F \leq \dim_{\text{comp}} F$ is immediate.

4 Incomputable dimensions

4.1 Hausdorff dimension

In this section we provide the announced examples. First we have the following.

Lemma 11 If the sequence $(q_i)_{i \in \mathbb{N}}$ of rationals $0 < q_i < 1, q_i \neq q_{i+1}$, is computable then one can construct an $\omega$-language $F \subseteq X^\omega$ according to the tree construction such that $\text{pref}(F)$ is a computable language.

Proof. Construct from $(q_i)_{i \in \mathbb{N}}$ the numerator and denominator sequences $(k_i)_{i \in \mathbb{N}}$ and $(\ell_i)_{i \in \mathbb{N}}$. Then in view of the results of Sections 2.2 and 2.3 the assertion is obvious.

Our lemma shows that the $\omega$-language $F \subseteq X^\omega$ has a very simple computable structure (compare with [Sta07, Section 4]).

Next we show that the Hausdorff dimension of a computable $\omega$-language $F \subseteq X^\omega$ as in Lemma 11 may be incomputable.

Theorem 12 If the sequence $(q_i)_{i \in \mathbb{N}}$ of rationals $0 < q_i < 1, q_i \neq q_{i+1}$, is computable and $\alpha = \liminf_{i \to \infty} q_i$ then there is an $\omega$-language $F \subseteq X^\omega$ such that $\text{pref}(F)$ is a computable language and $\dim F = \alpha$.

Proof. Construct from $(q_i)_{i \in \mathbb{N}}$ the numerator and denominator sequences $(k_i)_{i \in \mathbb{N}}$ and $(\ell_i)_{i \in \mathbb{N}}$ such that $\liminf_{i \to \infty} \frac{\ell_i}{k_i} = 0$. Then the assertion follows from Lemmata 7, 11 and Corollary 10.

Theorem 3.4 of [Ko98] proves a similar result where the achieved Hausdorff dimension $\alpha$ is a computably approximable number. Our result extends this range to a class of numbers beyond the computably approximable ones [ASWZ00, ZW01].
4.2 Martingales

We associate with every non-empty \( \omega \)-language \( E \subseteq X^\omega \) a martingale \( \mathcal{V}_E \) in the following way.

**Definition 3**

\[
\mathcal{V}_E(e) := 1 \\
\mathcal{V}_E(wx) := \begin{cases} 
\frac{|X|}{\text{pref}(E) \cap wx \cdot X} \cdot \mathcal{V}_E(w), & \text{if } wx \in \text{pref}(E), \text{ and} \\
0, & \text{otherwise.}
\end{cases}
\]

In view of the spherical symmetry, for \( F \) defined as in Section 2.3 we obtain

\[
\mathcal{V}_F(w) = |X|^{\text{dim}_F} s_F(|w|), \text{ if } w \in \text{pref}(F). \tag{13}
\]

Moreover, if \( \text{pref}(F) \) is computable then \( s_F \) and \( \mathcal{V}_F \) are computable mappings.

**Theorem 13** If the sequence \( (q_i)_{i \in \mathbb{N}} \) of rationals \( 0 < q_i < 1, q_i \neq q_{i+1} \), is computable and \( \alpha = \liminf_{i \to \infty} q_i \) then there is an \( \omega \)-language \( F \subseteq X^\omega \) such that \( \text{dim}_F = \alpha \).

**Proof.** We use the \( \omega \)-language \( F \) defined in the proof of Theorem 12. If \( \sigma \in (0,1) \) is a computable number then \( \mathcal{V}_F(w) \cdot |X|^{-|w| \cdot (1-\sigma)} \) is a computable \( \sigma \)-gale (see [DH10, Section 13.2]). If \( \sigma > \alpha \) then \( \limsup_{w \to \xi} \mathcal{V}_F(w) \cdot |X|^{-(1-\sigma) \cdot |w|} = \infty \) for all \( \xi \in F \). Thus \( \text{dim}_{\text{comp}} F \leq \alpha \). The other inequality follows from \( \text{dim}_F \leq \text{dim}_{\text{comp}} F \) and Theorem 12.

In certain cases we can achieve even the borderline value

\[
\frac{\mathcal{V}_F(w)}{|X|^{(1-\text{dim}_F) \cdot |w|}} = \limsup_{n \to \infty} \frac{|X|^\text{dim}_F \cdot n}{s_F(n)} = \infty \text{ for all } \xi \in F.
\]

**Theorem 14** Let \( (q_i)_{i \in \mathbb{N}}, 0 < q_i < 1, q_i \neq q_{i+1} \), be a computable sequence of rationals converging to \( \alpha \). If \( \alpha \) is not right-computable then there is an \( \omega \)-language \( F \subseteq X^\omega \) such that \( \alpha = \text{dim}_F, \text{pref}(F) \) is a computable language and

\[
\limsup_{n \to \infty} \frac{|X|^\text{dim}_F \cdot n}{s_F(n)} = \infty.
\]

**Proof.** We construct \( F \) as in the proof of Theorem 12 requiring additionally that \( \ell_i \geq i^2 \). Then \( \text{pref}(F) \) is computable and \( \text{dim}_F = \alpha \). In view of Property 1 there are infinitely many \( i \in \mathbb{N} \) with \( \alpha - \frac{1}{i^2} > q_i \) and, consequently, \( s_F(\ell_i) = |X|^{\alpha \cdot \ell_i} \leq |X|^{|\alpha| \cdot \ell_i - ^2 |i \cdot |i} \). This shows \( \limsup_{n \to \infty} \frac{|X|^\alpha \cdot n}{s_F(n)} \geq \limsup_{i \to \infty} |X|^{\ell_i} = \infty. \]
4.3 Comparison of gales and martingales

In this final part we compare the precision with which gales and martingales achieve the value of computable dimension of a subset $E \subseteq X^\omega$. To this end we define the following notion which reflects in some sense the accuracy with which a supergale or a martingale defines the computable dimension of a subset $E \subseteq X^\omega$.

**Definition 4**

1. A computable supergale $d : X^* \rightarrow [0, \infty)$ matches $E \subseteq X^\omega$ provided $d$ is a $\dim_{\text{comp}}E$-supergale and $\forall \xi (\xi \in E \rightarrow \limsup_{w \rightarrow \xi} d(w) = \infty)$.

2. A computable martingale $V : X^* \rightarrow [0, \infty)$ matches $E \subseteq X^\omega$ provided

$$\limsup_{w \rightarrow \xi} \frac{V_E(w)}{|X|^{(1-\dim_{\text{comp}}E)\cdot |w|}} = \infty$$

for all $\xi \in E$.

Since from Definition 2 it follows that a for $\sigma < \dim_{\text{comp}}E$ no computable $\sigma$-supergale satisfies $\forall \xi (\xi \in E \rightarrow \limsup_{w \rightarrow \xi} d(w) = \infty)$, the matching condition characterises “best” computable supergales for an $\omega$-language $E$. Similarly, Definition 4.2 characterises “best” computable martingales. It should be mentioned that matching supergales or martingales do not always exist.

**Lemma 15** If a computable supergale $d : X^* \rightarrow [0, \infty)$ matches $E \subseteq X^\omega$ then $\dim_{\text{comp}}E = \chi_d$.

**Proof.** By definition of $\chi_d$ we have $\chi_d \leq \dim_{\text{comp}}E$. Assume $\chi_d < \dim_{\text{comp}}E$. Then there is a rational number $q, \chi_d < q < \dim_{\text{comp}}E$, and $d$ is a computable $q$-supergale which satisfies $\forall \xi (\xi \in E \rightarrow \limsup_{w \rightarrow \xi} d(w) = \infty)$. This contradicts the definition of $\dim_{\text{comp}}E$. 

Above we mentioned that the cut point $\chi_d$ of a computable supergale $d$ is always left-computable. Therefore, if some supergale $d$ matches $E \subseteq X^\omega$ the value $\dim_{\text{comp}}E$ has to be left-left-computable.

In Theorem 12 we proved that for every computably approximable $\sigma$ there are simple computable $\omega$-languages $F \subseteq X^\omega$ with $\dim_{\text{comp}}F = \sigma$. Moreover, Theorem 14 shows that, if additionally $\dim_{\text{comp}}F = \sigma$ is not right-computable the computable martingale $V_F$ matches $F$. Since there are computably approximable reals which are neither right- nor left-computable this shows that in some cases Schnorr’s combination of martingales with (exponential) order functions (see [Sch71]) can be more precise than Lutz’s approach via supergales.
5 Concluding remark

As the constructive dimension of subsets of $X^\omega$ is sandwiched between the computable and the Hausdorff dimension ([Lut03, Hit05]) the result of Theorem 13 holds likewise for constructive dimension, too.

References


On the incomputability of computable dimension


