CDMTCS
Research
Report
Series

On the incomputability of computable dimension

Ludwig Staiger
Martin-Luther-Universität
Halle-Wittenberg

CDMTCS-535
May 2019

Centre for Discrete Mathematics and
Theoretical Computer Science

On the incomputability of computable dimension

Ludwig Staiger*

Martin-Luther-Universität Halle-Wittenberg
Institut für Informatik
von-Seckendorff-Platz 1
D–06099 Halle (Saale), Germany

Abstract

Using an iterative tree construction we show that for simple computable subsets of the Cantor space Hausdorff, constructive and computable dimensions might be incomputable.

Contents

*email: `staiger@informatik.uni-halle.de`

Computable dimension along with constructive dimension was introduced by Lutz [Lut03] as a means for measuring the complexity of sets of infinite strings (ω-words). Since then and prior to this constructive and computable dimension were investigated in connection with Kolmogorov complexity and Hausdorff dimension. The results of [Hit05, Sta93, Sta98] show that the Hausdorff, constructive and the computable dimensions of automaton definable sets of infinite strings (regular ω-languages) is computable. In contrast to this Ko [Ko98] derived examples of computable ω-languages which have incomputable Hausdorff dimension.

In this paper we derive simple examples of computable ω-languages which have not only incomputable Hausdorff dimension but also incomputable computable dimension. To this end we use in iteration of finite trees which resembles the tree construction of Furstenberg [Fur70] (see also [MSS18])

As a byproduct we obtain simple examples of computable ω-languages having incomputable Hausdorff dimension.

Lutz [Lut03] defines computable and constructive dimension via (super-) gales. Terwijn [Ter04, CST06] observed that this can also be done using Schnorr's concept of martingales and (exponential) order functions [Sch71, Section 17]. For the computable ω-languages derived in this paper we can show that the latter concept is in some details more precise than Lutz's approach.

1 Notation

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers, by $\mathbb{Q}$ the set of rational numbers, and $\mathbb{R}$ are the real numbers.

Let X be an alphabet of cardinality $|X| \geq 2$. By X^* we denote the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X. Subsets of X^* will be referred to as *languages* and subsets of X^ω as *ω-languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. We denote by $|w|$ the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$.

It is sometimes convenient to regard X^ω as Cantor space, that is, as the product space of the (discrete space) X. Here *open* sets in X^ω are those of the form $W \cdot X^\omega$ with $W \subseteq X^*$. *Closed* are sets $F \subseteq X^\omega$ which satisfy the condition $F = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$.

For a computable domain $\mathscr{D}$, such as $\mathbb{N}$, $\mathbb{Q}$ or X^*, we refer to a function

$f:\mathscr{D}\to\mathbb{R}$ as *left-computable* (or *approximable from below*) provided the set $\{(d,q): d\in\mathscr{D}\wedge q\in\mathbb{Q}\wedge q<f(d)\}$ is computably enumerable. Accordingly, a function $f:\mathscr{D}\to\mathbb{R}$ is called *right-computable* (or *approximable from above*) if the set $\{(d,q): d\in\mathscr{D}\wedge q\in\mathbb{Q}\wedge q>f(d)\}$ is computably enumerable, and f is *computable* if f is right- and left-computable. If we refer to a function $f:\mathscr{D}\to\mathbb{Q}$ as computable we usually mean that it maps the domain $\mathscr{D}$ to the domain $\mathbb{Q}$, that is, it returns the exact value $f(d)\in\mathbb{Q}$. If $\mathscr{D}=\mathbb{N}$ we write f as a sequence $(q_i)_{i\in\mathbb{N}}$.

A real number $\alpha\in\mathbb{R}$ is left-computable, right computable or computable provided the constant function $c_\alpha(t)=\alpha$ is left-computable, right-computable or computable, respectively. $\alpha\in\mathbb{R}$ is referred to as *computably approximable* if $\alpha=\lim_{i\to\infty}q_i$ for a computable sequence $(q_i)_{i\in\mathbb{N}}$ of rationals. It is well-known (see e.g. [ZW01]) that there are left-computable which are not right-computable and vice versa, and that there are computably approximable reals which are neither left-computable nor right-computable.

The following approximation property is easily verified.

Property 1 *Let $(q_i)_{i\in\mathbb{N}}$ be a computable family of rationals converging to α and let $(q'_i)_{i\in\mathbb{N}}, q'_i>0$, be a computable family of rationals converging to 0. If α is not right-computable then there are infinitely many $i\in\mathbb{N}$ such that $\alpha-q_i>q'_i$.*

For, otherwise, α as the limit of $(q_i+q'_i)_{i\in\mathbb{N}}$ would be right-computable.

2 Iterative Tree Construction

2.1 Preliminaries

The aim of this section is to present how one can, given a sequence of rationals $(q_i)_{i\in\mathbb{N}}$, find sequences of natural numbers $(k_i)_{i\in\mathbb{N}}$ and $(\ell_i)_{i\in\mathbb{N}}$ with appropriate properties such that $q_i=k_i/\ell_i$.

Lemma 2 *Let $(q_i)_{i\in\mathbb{N}}$, $0<q_i<1$, $q_i\neq q_{i+1}$, be a family of positive rationals. Then there are families of natural numbers $(k_i)_{i\in\mathbb{N}}$, $(\ell_i)_{i\in\mathbb{N}}$, $(\kappa_i)_{i\in\mathbb{N}}$, $(p_i)_{i\in\mathbb{N}}$ and $(r_i)_{i\in\mathbb{N}}$, such that $q_i=k_i/\ell_i$, $q_{i+1}=\dfrac{r_i\cdot k_i+\kappa_i\cdot\ell_i}{r_i\cdot\ell_i+p_i\cdot\ell_i}$ where $\kappa_i=\begin{cases}0, & \text{if } q_i>q_{i+1} \text{ and}\\ p_i, & \text{if } q_i<q_{i+1}.\end{cases}$*

Moreover, for $0\le t\le p_i\cdot\ell_i$ we have

$$q_i\ge\frac{r_i\cdot k_i}{r_i\cdot\ell_i+t}\quad\ge\quad q_{i+1},\ \text{if } q_i>q_{i+1} \text{ and} \tag{1}$$

$$q_i \le \frac{r_i \cdot k_i + t}{r_i \cdot \ell_i + t} \quad \le \quad q_{i+1},\ \textit{if}\ q_i < q_{i+1}. \tag{2}$$

Proof. Let $q_i = k_i/\ell_i$ and $q_{i+1} = a/b \cdot q_i = \dfrac{a \cdot k_i}{b \cdot \ell_i}$, with $a, b \in \mathbb{N} \setminus \{0\}, a \ne b$. Since $1 > q_{i+1}$ we have $b \cdot \ell_i - a \cdot k_i = a \cdot \frac{q_i}{q_{i+1}} \cdot (1 - q_{i+1}) \cdot \ell_i > 0$.

Assume $q_i > q_{i+1}$. Then $b > a$ and the equation

$$\frac{r_i \cdot k_i + \kappa_i \cdot \ell_i}{r_i \cdot \ell_i + p_i \cdot \ell_i} = \frac{a \cdot k_i}{b \cdot \ell_i} \tag{3}$$

has the solutions $r_i = a$, and $p_i = (b-a) = a \cdot (\frac{q_i}{q_{i+1}} - 1)$ and $\kappa_i = 0$.

If $q_i < q_{i+1}$ then $a > b$ and $r_i := b \cdot \ell_i - a \cdot k_i = a \cdot (\frac{q_i}{q_{i+1}} \cdot \ell_i - k_i) = a \cdot q_i \cdot (\frac{1}{q_{i+1}} - 1) \cdot \ell_i$ and $p_i = \kappa_i := (a-b) \cdot k_i = a \cdot q_i \cdot (1 - \frac{q_i}{q_{i+1}}) \cdot \ell_i$ are solutions of Eq. (3).

In view of $\kappa_i = 0$ Eq. (1) is obvious. Eq. (2) follows inductively from $\frac{k+1}{\ell+1} \ge \frac{k}{\ell}$ whenever $0 \le k < \ell$. ❑

If the family $(q_i)_{i\in\mathbb{N}}$ is a computable one then the families in Lemma 2 can be chosen to be computable. In addition, the values ℓ_i and ℓ_{i+1}/ℓ_i can be made arbitrarily large.

2.2 Tree construction

We define F via the following auxiliary languages $T_i \subseteq X^{\ell_i}$ and $U_i \subseteq X^{p_i \cdot \ell_i}$.

Let $T_0 := X^{k_0} \cdot 0^{\ell_0 - k_0}$ or $T_0 := 0^{\ell_0 - k_0} \cdot X^{k_0}$ and set

$$T_{i+1} := T_i^{r_i} \cdot U_i \text{ with } U_i := \begin{cases} X^{p_i \cdot \ell_i}, & \text{if } q_{i+1} \ge q_i \text{ and} \\ \{u_i\}, & \text{otherwise} \end{cases} \tag{4}$$

where $u_i \in X^{p_i}$ is a fixed word. Then $\ell_{i+1} = (r_i + p_i) \cdot \ell_i$. The values r_i and p_i are referred to as repetition or prolongation factors, respectively.

By induction one proves

$$|T_i| = |X|^{q_i \cdot \ell_i}. \tag{5}$$

Property 3 *The trees T_i have the following properties. Let $\ell \le \ell_i$.*

1. Prefix property: $\mathbf{pref}(T_{i+1}) = \bigcup_{j=0}^{r_i-1} T_i^j \cdot \mathbf{pref}(T_i) \cup T_i^{r_i} \cdot \mathbf{pref}(U_i)$,
2. Extension property: $\mathbf{pref}(T_i) \cap X^\ell = \mathbf{pref}(T_{i+1}) \cap X^\ell$, *and*
3. Spherical symmetry: $\mathbf{pref}(T_i) \cap X^{\ell-1} = (\mathbf{pref}(T_i) \cap X^\ell) \cdot X$ *or* $|\mathbf{pref}(T_i) \cap X^{\ell-1}| = |\mathbf{pref}(T_i) \cap X^\ell|$.

2.3 The infinite tree

Define $F := \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ where $(T_i)_{i\in\mathbb{N}}$ satisfies Eq. (4).

Before we proceed to further properties of the family $(T_i)_{i\in\mathbb{N}}$ and the ω-language F we mention the following general property.

Lemma 4 *Let $T_i \subseteq X^*$, $T_{i+1} \subseteq T_i \cdot X \cdot X^*$, $T_i \subseteq \mathbf{pref}(T_{i+1})$ and $F := \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$. Then $\mathbf{pref}(F) = \bigcup_{i\in\mathbb{N}} \mathbf{pref}(T_i)$.*

If, moreover, all T_i are finite then $F := \{\xi : \xi \in X^\omega \wedge \mathbf{pref}(\xi) \subseteq \bigcup_{i\in\mathbb{N}} \mathbf{pref}(T_i)\}$.

Proof. In view of $T_{i+1} \subseteq T_i \cdot X \cdot X^*$ we have $T_{i+1} \cdot X^\omega \subseteq T_i \cdot X^\omega$ and also $|w| \geq i$ for $w \in T_i$.

If $w \in \mathbf{pref}(F)$ then $w \in \mathbf{pref}(\xi)$ where $\xi \in F \subseteq T_i \cdot X^\omega$ for $i > |w|$. Consequently, $w \in \mathbf{pref}(T_i)$.

Using the condition $T_i \subseteq \mathbf{pref}(T_{i+1})$, by induction we obtain that for every $w \in \mathbf{pref}(T_i)$ there is an infinite chain $(w_j)_{j\geq i}$ such that $w_j \in T_j$ and $w \sqsubseteq w_i \sqsubset w_{i+1} \sqsubset \cdots$. Thus there is a $\xi \in F$ with $w \sqsubset \xi$.

If the languages T_i are finite $F = \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ is closed in the product topology of the space X^ω which implies $F := \{\xi : \xi \in X^\omega \wedge \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$. ❑

Our lemma shows that $F := \{\xi : \xi \in X^\omega \wedge \mathbf{pref}(\xi) \subseteq \bigcup_{i\in\mathbb{N}} \mathbf{pref}(T_i)\}$.

From the spherical symmetry of T_i (see Property 3) the ω-language $F = \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ inherits the following balance property.

Lemma 5 *Let $F = \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ where the T_i are defined by Eq. (4). Then for all $w, v \in \mathbf{pref}(F)$ with $|w| = |v|$ we have*

$$|w \cdot X^k \cap \mathbf{pref}(F)| = |v \cdot X^k \cap \mathbf{pref}(F)|.$$

Proof. We proceed by induction on k. Let $k = 1$. Then for all $w, v \in \mathbf{pref}(F)$ with $|w| = |v|$ either $\mathbf{pref}(F) \cap X^{|u|+1} = (\mathbf{pref}(F) \cap X^{|u|}) \cdot X$ or $|\mathbf{pref}(F) \cap X^{|u|+1}| = |\mathbf{pref}(F) \cap X^{|u|}|$ $(u \in \{w, v\})$.

In the first case we have $|w \cdot X \cap \mathbf{pref}(F)| = |X| = |v \cdot X \cap \mathbf{pref}(F)|$ and in the second $|w \cdot X \cap \mathbf{pref}(F)| = 1 = |v \cdot X \cap \mathbf{pref}(F)|$.

Let the assertion be proved for k and all pairs $u, u' \in \mathbf{pref}(F)$ of the same length. Let $w, v \in \mathbf{pref}(F)$ with $|w| = |v|$ and consider words $w', v' \in X^k$ such that $w \cdot w', v \cdot v' \in \mathbf{pref}(F)$. Then from the spherical symmetry we obtain either $\mathbf{pref}(F) \cap X^{|u|+1} = (\mathbf{pref}(F) \cap X^{|u|}) \cdot X$ or $|\mathbf{pref}(F) \cap X^{|u|+1}| = |\mathbf{pref}(F) \cap X^{|u|}|$ for $u \in \{w \cdot w', v \cdot v'\}$ and we proceed as above.

Since, by our assumption $|\{w' : |w'| = k \wedge w \cdot w' \in \mathbf{pref}(F)\}| = |\{v' : |v'| = k \wedge v \cdot v' \in \mathbf{pref}(F)\}|$, the assertion follows. ❑

Next we investigate in more detail the structure function $s_F : \mathbb{N} \to \mathbb{N}$ where $s_F(\ell) := |\mathbf{pref}(F) \cap X^\ell|$.

First, Lemma 4 implies

$$\mathbf{pref}(F)\cap X^{\ell}=\mathbf{pref}(T_i)\cap X^{\ell} \text{ whenever } \ell\le\ell_i\,. \tag{6}$$

From Eqs. (4) and (5) and the properties of the tree family $(T_i)_{i\in\mathbb{N}}$ we obtain for the intervals $\ell_i\le\ell\le\ell_{i+1}$:

Lemma 6 *1. In the interval* $[j\cdot\ell_i,(j+1)\cdot\ell_i]$ *where* $j<r_i$ *we have:*

$$s_F(j\cdot\ell_i+t)=s_F(\ell_i)^j\cdot s_F(t) \text{ for } 0\le t\le\ell_i\,,$$

and in a more detailed form in the subinterval $[j\cdot\ell_i+j'\cdot\ell_{i-1},j\cdot\ell_i+(j'+1)\cdot\ell_{i-1}]$ *where* $j'<r_{i-1}$

$$s_F(j\cdot\ell_i+j'\cdot\ell_{i-1}+t)=s_F(\ell_i)^j\cdot s_F(\ell_{i-1})^{j'}\cdot s_F(t) \text{ for } 0\le t<\ell_{i-1}\,.$$

2. In the interval $[r_i\cdot\ell_i,\ell_{i+1}]$ *for* $0\le t\le p_i\cdot\ell_i$*:*

$$s_F(r_i\cdot\ell_i+t)=\begin{cases} s_F(\ell_i)^{r_i}, & \text{if } |U_i|=1 \text{ and}\\ s_F(\ell_i)^{r_i}\cdot|X|^t, & \text{if } U_i=X^{p_i\cdot\ell_i}\,.\end{cases}$$

This yields the following connection to the values q_i.

From Eqs. (6) and (5) we have

$$\frac{\log_{|X|}s_F(j\cdot\ell)}{j\cdot\ell}=q_i\,. \tag{7}$$

Using the identities of Lemma 6 and Eqs. (1) and (2) we obtain the following estimates for $\frac{\log_{|X|}s_F(\ell)}{\ell}$ in the range $\ell_i\le\ell\le\ell_{i+1}=r_i\cdot\ell_i+n_i\cdot\ell_i$.

For $\ell_i\le\ell<r_i\cdot\ell_i$ we have $\ell=j\cdot\ell_i+j'\cdot\ell_{i-1}+t$ where $0\le t<\ell_{i-1}$ and Lemma 6.1 yields

$$\begin{aligned}\frac{\log_{|X|}s_F(\ell)}{\ell} &\ge \frac{j\cdot\ell_i}{\ell}\cdot q_i+\frac{j'\cdot\ell_{i-1}}{\ell}\cdot q_{i-1}\\ &\ge \frac{j\cdot\ell_i+j'\cdot\ell_{i-1}}{\ell}\cdot\min\{q_{i-1},q_i\}\\ &\ge (1-\frac{\ell_{i-1}}{\ell_i})\cdot\min\{q_{i-1},q_i\}\end{aligned} \tag{8}$$

If $r_i\cdot\ell_i\le\ell\le\ell_{i+1}$, that is, for $\ell=r_i\cdot\ell_i+t$ where $t\le\ell_{i+1}-r_i\cdot\ell_i$, following Eqs. (1) and (2), respectively, we have according to Lemma 6.2

$$q_i\ge\frac{\log_{|X|}s_F(\ell)}{\ell} = \frac{\log_{|X|}s_F(r_i\cdot\ell_i)}{r_i\cdot\ell_i+t}\ge q_{i+1} \text{ if } q_i>q_{i+1} \tag{9}$$

$$q_i\le\frac{\log_{|X|}s_F(\ell)}{\ell} = \frac{\log_{|X|}s_F(r_i\cdot\ell_i)+t}{r_i\cdot\ell_i+t}\le q_{i+1} \text{ if } q_i<q_{i+1} \tag{10}$$

The considerations in Eqs. (7), (8), (9) and (10) show the following.

Lemma 7 *If the sequence* $(\ell_i)_{i\in\mathbb{N}}$ *is chosen in such a way that* $\liminf_{i\to\infty}\frac{\ell_{i-1}}{\ell_i}=0$ *then*

$$\liminf_{\ell\to\infty}\frac{\log_{|X|}s_F(\ell)}{\ell}=\liminf_{i\to\infty}q_i\,.$$

Proof. In view of Eq. (7) the limit cannot exceed $\liminf_{i\to\infty}q_i$.

On the other hand, by Eqs. (8), (9) and (10), for $\ell_i\le\ell\le\ell_{i+1}$, the intermediate values satisfy $\frac{\log_{|X|}s_F(\ell)}{\ell}\ge(1-\frac{\ell_{i-1}}{\ell_i})\cdot\min\{q_{i-1},q_i,q_{i+1}\}$. ❑

2.4 Monotone families $(q_i)_{i\in\mathbb{N}}$

If the sequence $(q_i)_{i\in\mathbb{N}}$ is monotone we can simplify the above considerations of Eq. (8).

Theorem 8 *Let the sequence* $(q_i)_{i\in\mathbb{N}}$ *be monotone and* $\lim_{i\to\infty}q_i=\alpha$.

1. *If* $(q_i)_{i\in\mathbb{N}}$ *is decreasing and* $T_0=X^{k_0}\cdot 0^{\ell_0-k_0}$ *then* $s_F(\ell)\ge|X|^{\alpha\cdot\ell}$, *for all* $\ell\in\mathbb{N}$.

2. *If* $(q_i)_{i\in\mathbb{N}}$ *is increasing and* $T_0=0^{\ell_0-k_0}\cdot X^{k_0}$ *then* $s_F(\ell)\le|X|^{\alpha\cdot\ell}$, *for all* $\ell\in\mathbb{N}$.

Proof. If $(q_i)_{i\in\mathbb{N}}$ is decreasing we start with $T_0=X^{k_0}\cdot 0^{\ell_0-k_0}$ and have $s_F(\ell)\ge|X|^{q_0\cdot\ell}\ge|X|^{\alpha\cdot\ell}$ for $\ell\le\ell_0$. Then we use Eqs. (6) and (4) and proceed by induction.

$s_F(j\cdot\ell_i+t)=s_F(j\cdot\ell_i)\cdot s_F(t)\ge|X|^{q_i\cdot\ell_i}\cdot|X|^{\alpha\cdot t}\ge|X|^{\alpha\cdot\ell}$ for $j<r_i$. In the range $r_i\cdot\ell_i\le\ell\le\ell_{i+1}$ we have according to Eq. (9) $s_F(\ell)\ge|X|^{q_{i+1}\cdot\ell}\ge|X|^{\alpha\cdot\ell}$.

If $(q_i)_{i\in\mathbb{N}}$ is increasing we start with $T_0=0^{\ell_0-k_0}\cdot X^{k_0}$ and have $s_F(\ell)\ge|X|^{q_0\cdot\ell}\le|X|^{\alpha\cdot\ell}$ for $\ell\le\ell_0$. Again we use Eqs. (6) and (4) and proceed by induction.

$s_F(j\cdot\ell_i+t)=s_F(j\cdot\ell_i)\cdot s_F(t)\le|X|^{q_i\cdot\ell_i}\cdot|X|^{\alpha\cdot t}\le|X|^{\alpha\cdot\ell}$ for $j<r_i$. In the range $r_i\cdot\ell_i\le\ell\le\ell_{i+1}$ we have according to Eq. (10) $s_F(\ell)\le|X|^{q_{i+1}\cdot\ell}\le|X|^{\alpha\cdot\ell}$. ❑

3 Gales and Martingales

Hausdorff [Hau18] introduced a notion of dimension of a subset Y of a metric space which is now known as its *Hausdorff dimension*, $\dim Y$; Falconer [Fal03] provides an overview and introduction to this subject. In the case of the Cantor space X^ω, Lutz [Lut03] (see also [DH10, Section 13.2]) has found an equivalent definition of Hausdorff dimension via generalisations of martingales.

Following Lutz a mapping $d : X^* \to [0,\infty)$ will be called an *σ-supergale* provided

$$\forall w(w \in X^* \to |X|^\sigma \cdot d(w) \geq \sum_{x\in X} d(wx)). \tag{11}$$

A σ-supergale d is called an σ-gale if, for all $w \in X^*$, Eq. (11) is satisfied with equality. (*Super-)Martingales* are 1-(super-)gales.

Observe that, for $\sigma' \geq \sigma$ any σ-supergale d is also a σ'-supergale. We define the *cut point* χ_d of a supergale d as the smallest value σ for which d can be an σ-supergale.

$$\chi_d := \inf\left\{\sigma : \forall w \left(|X|^\sigma \cdot d(w) \geq \sum_{x\in X} d(wx)\right)\right\}. \tag{12}$$

If d is a computable mapping then χ_d as $\sup\{q : q \in \mathbb{Q} \wedge \exists w(|X|^q \cdot d(w) < \sum_{x\in X} d(wx))\}$ is a left-computable real number.

Following Lutz [Lut03] we define as follows.

Definition 1 Let $F \subseteq X^\omega$. Then α is the *Hausdorff dimension* $\dim F$ of F provided[1]

1. for all $\sigma > \alpha$ there is a σ-supergale d such that $\forall \xi(\xi \in F \to \limsup_{w\to\xi} d(w) = \infty)$, and
2. for all $\sigma < \alpha$ and all σ-supergales d it holds $\exists \xi(\xi \in F \wedge \limsup_{w\to\xi} d(w) < \infty)$.

For ω-languages having a simple structure like the one in the tree construction above we can simplify the calculation of the Hausdorff dimension (see [Sta89, Theorem 4]).

Lemma 9 *Let $F \subseteq X^\omega$ satisfy the conditions $F = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$ and $s_{F\cap w\cdot X^\omega} = s_{F\cap v\cdot X^\omega}$ for all $w, v \in \mathbf{pref}(F)$ with $|w| = |v|$. Then*

$$\dim F = \liminf_{n\to\infty} \frac{\log_{|X|} \max\{1, s_F(n)\}}{n}.$$

As a consequence we obtain the following.

Corollary 10 *Let $F \subseteq X^\omega$ be constructed according to the tree construction of Section 2.2. Then* $\dim F = \liminf_{n\to\infty} \frac{\log_{|X|} s_F(n)}{n}$.

If we require the supergales in Definition 1 to be computable mappings we obtain the definition of computable dimension $\dim_{\mathrm{comp}} F$ of [Hit05, Lut03].

[1] Here $\limsup_{w\to\xi} d(w)$ is an abbreviation for $\lim_{n\to\infty} \sup\{d(w) : w \in \mathbf{pref}(\xi) \wedge |w| \geq n\}$.

Definition 2 Let $F \subseteq X^\omega$. Then α is the *computable dimension* of F provided

1. for all $\sigma > \alpha$ there is a computable σ-supergale d such that $\forall\xi(\xi \in F \to \limsup_{w\to\xi} d(w) = \infty)$, and

2. for all $\sigma < \alpha$ and all computable σ-supergales d it holds $\exists\xi(\xi \in F \wedge \limsup_{w\to\xi} d(w) < \infty)$.

Then the inequality $\dim F \leq \dim_{\mathrm{comp}} F$ is immediate.

4 Incomputable dimensions

4.1 Hausdorff dimension

In this section we provide the announced examples. First we have the following.

Lemma 11 *If the sequence $(q_i)_{i\in\mathbb{N}}$ of rationals $0 < q_i < 1, q_i \neq q_{i+1}$, is computable then one can construct an ω-language $F \subseteq X^\omega$ according to the tree construction such that* **pref**(F) *is a computable language.*
Proof. Construct from $(q_i)_{i\in\mathbb{N}}$ the numerator and denominator sequences $(k_i)_{i\in\mathbb{N}}$ and $(\ell_i)_{i\in\mathbb{N}}$. Then in view of the results of Sections 2.2 and 2.3 the assertion is obvious. ❑

Our lemma shows that the ω-language $F \subseteq X^\omega$ has a very simple computable structure (compare with [Sta07, Section 4]).

Next we show that the Hausdorff dimension of a computable ω-language $F \subseteq X^\omega$ as in Lemma 11 may be incomputable.

Theorem 12 *If the sequence $(q_i)_{i\in\mathbb{N}}$ of rationals $0 < q_i < 1, q_i \neq q_{i+1}$, is computable and $\alpha = \liminf_{i\to\infty} q_i$ then there is an ω-language $F \subseteq X^\omega$ such that* **pref**(F) *is a computable language and* $\dim F = \alpha$.
Proof. Construct from $(q_i)_{i\in\mathbb{N}}$ the numerator and denominator sequences $(k_i)_{i\in\mathbb{N}}$ and $(\ell_i)_{i\in\mathbb{N}}$ such that $\liminf_{i\to\infty} \frac{\ell_i}{\ell_{i+1}} = 0$. Then the assertion follows from Lemmata 7, 11 and Corollary 10. ❑

Theorem 3.4 of [Ko98] proves a similar result where the achieved Hausdorff dimension α is a computably approximable number. Our result extends this range to a class of numbers beyond the computably approximable ones [ASWZ00, ZW01].

4.2 Martingales

We associate with every non-empty ω-language $E \subseteq X^\omega$ a martingale $\mathcal{V}_E$ in the following way.

Definition 3

$$\begin{array}{rcl} \mathcal{V}_E(e) & := & 1 \\ \mathcal{V}_E(wx) & := & \begin{cases} \frac{|X|}{|\mathbf{pref}(E)\cap w\cdot X|}\cdot \mathcal{V}_E(w), & \textit{if } wx \in \mathbf{pref}(E), \textit{ and} \\ 0, & \textit{otherwise.} \end{cases} \end{array}$$

In view of the spherical symmetry, for F defined as in Section 2.3 we obtain

$$\mathcal{V}_F(w) = |X|^{|w|}/s_F(|w|), \text{ if } w \in \mathbf{pref}(F). \tag{13}$$

Moreover, if $\mathbf{pref}(F)$ is computable then s_F and $\mathcal{V}_F$ are computable mappings.

Theorem 13 *If the sequence* $(q_i)_{i\in\mathbb{N}}$ *of rationals* $0 < q_i < 1, q_i \neq q_{i+1}$, *is computable and* $\alpha = \liminf_{i\to\infty} q_i$ *then there is an* ω*-language* $F \subseteq X^\omega$ *such that* $\mathbf{pref}(F)$ *is a computable language and* $\dim F = \dim_{\mathrm{comp}} F = \alpha$.
Proof. We use the ω-language F defined in the proof of Theorem 12. If $\sigma \in (0,1)$ is a computable number then $\mathcal{V}_F(w)\cdot|X|^{-|w|\cdot(1-\sigma)}$ is a computable σ-gale (see [DH10, Section 13.2]). If $\sigma > \alpha$ then $\limsup_{w\to\xi} \mathcal{V}_F(w)\cdot|X|^{-(1-\sigma)\cdot|w|} = \infty$ for all $\xi \in F$. Thus $\dim_{\mathrm{comp}} F \le \alpha$. The other inequality follows from $\dim F \le \dim_{\mathrm{comp}} F$ and Theorem 12. ❑

In certain cases we can achieve even the borderline value

$$\frac{\mathcal{V}_F(w)}{|X|^{(1-\dim F)\cdot|w|}} = \limsup_{n\to\infty} \frac{|X|^{\dim F\cdot n}}{s_F(n)} = \infty \text{ for all } \xi \in F.$$

Theorem 14 *Let* $(q_i)_{i\in\mathbb{N}}, 0 < q_i < 1, q_i \neq q_{i+1}$, *be a computable sequence of rationals converging to* α. *If* α *is not right-computable then there is an* ω*-language* $F \subseteq X^\omega$ *such that* $\alpha = \dim F$, $\mathbf{pref}(F)$ *is a computable language and* $\limsup_{n\to\infty} \frac{|X|^{\dim F\cdot n}}{s_F(n)} = \infty$.
Proof. We construct F as in the proof of Theorem 12 requiring additionally that $\ell_i \ge i^2$. Then $\mathbf{pref}(F)$ is computable and $\dim F = \alpha$. In view of Property 1 there are infinitely many $i \in \mathbb{N}$ with $\alpha - \frac{1}{i} > q_i$ and, consequently, $s_F(\ell_i) = |X|^{q_i\cdot\ell_i} \le |X|^{\alpha\cdot\ell_i - \ell_i/i}$. This shows $\limsup_{n\to\infty} \frac{|X|^{\alpha\cdot n}}{s_F(n)} \ge \limsup_{i\to\infty} |X|^{\ell_i/i} = \infty$. ❑

4.3 Comparison of gales and martingales

In this final part we compare the precision with which gales and martingales achieve the value of computable dimension of a subset $E \subseteq X^{\omega}$. To this end we define the following notion which reflects in some sense the accuracy with which a supergale or a martingale defines the computable dimension of a subset $E \subseteq X^{\omega}$.

Definition 4 1. A computable supergale $d : X^* \to [0,\infty)$ *matches* $E \subseteq X^{\omega}$ provided d is a $\dim_{\mathrm{comp}} E$-supergale and $\forall \xi(\xi \in E \to \limsup_{w \to \xi} d(w) = \infty)$.

2. A computable martingale $\mathcal{V} : X^* \to [0,\infty)$ *matches* $E \subseteq X^{\omega}$ provided $\limsup_{w \to \xi} \frac{\mathcal{V}_E(w)}{|X|^{(1-\dim_{\mathrm{comp}} E)\cdot|w|}} = \infty$ for all $\xi \in E$.

Since from Definition 2 it follows that a for $\sigma < \dim_{\mathrm{comp} E}$ no computable σ-supergale satisfies $\forall \xi(\xi \in E \to \limsup_{w \to \xi} d(w) = \infty)$, the matching condition characterises "best" computable supergales for an ω-language E. Similarly, Definition 4.2 characterises "best" computable martingales. It should be mentioned that matching supergales or martingales do not always exist.

Lemma 15 *If a computable supergale* $d : X^* \to [0,\infty)$ *matches* $E \subseteq X^{\omega}$ *then* $\dim_{\mathrm{comp}} E = \chi_d$.

Proof. By definition of χ_d we have $\chi_d \le \dim_{\mathrm{comp}} E$. Assume $\chi_d < \dim_{\mathrm{comp}} E$. Then there is a rational number $q, \chi_d < q < \dim_{\mathrm{comp}} E$, and d is a computable q-supergale which satisfies $\forall \xi(\xi \in E \to \limsup_{w \to \xi} d(w) = \infty)$. This contradicts the definition of $\dim_{\mathrm{comp}} E$. ❏

Above we mentioned that the cut point χ_d of a computable supergale d is always left-computable. Therefore, if some supergale d matches $E \subseteq X^{\omega}$ the value $\dim_{\mathrm{comp}} E$ has to be left left-computable.

In Theorem 12 we proved that for every computably approximable σ there are simple computable ω-languages $F \subseteq X^{\omega}$ with $\dim_{\mathrm{comp}} F = \sigma$. Moreover, Theorem 14 shows that, if additionally $\dim_{\mathrm{comp}} F = \sigma$ is not right-computable the computable martingale $\mathcal{V}_F$ matches F. Since there are computably approximable reals which are neither right- not left-computable this shows that in some cases Schnorr's combination of martingales with (exponential) order functions (see [Sch71]) can be more precise than Lutz's approach via supergales.

5 Concluding remark

As the constructive dimension of subsets of X^ω is sandwiched between the computable and the Hausdorff dimension ([Lut03, Hit05]) the result of Theorem 13 holds likewise for constructive dimension, too.

References

[ASWZ00] Klaus Ambos-Spies, Klaus Weihrauch, and Xizhong Zheng. Weakly computable real numbers. *J. Complexity*, 16(4):676–690, 2000.

[CST06] Cristian S. Calude, Ludwig Staiger, and Sebastiaan A. Terwijn. On partial randomness. *Ann. Pure Appl. Logic*, 138(1-3):20–30, 2006.

[DH10] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Theory and Applications of Computability. Springer-Verlag, New York, 2010.

[Fal03] Kenneth Falconer. *Fractal geometry. Mathematical foundations and applications*. John Wiley & Sons, Inc., Hoboken, N.J., second edition, 2003.

[Fur70] Harry Furstenberg. Intersections of Cantor sets and transversality of semigroups. In Robert C. Gunning, editor, *Problems in Analysis*, pages 41–59. Princeton University Press, Princeton, N.J., 1970.

[Hau18] Felix Hausdorff. Dimension und äußeres Maß. *Math. Ann.*, 79(1-2):157–179, 1918.

[Hit05] John M. Hitchcock. Correspondence principles for effective dimensions. *Theory Comput. Syst.*, 38(5):559–571, 2005.

[Ko98] Ker-I. Ko. On the computability of fractal dimensions and Hausdorff measure. *Ann. Pure Appl. Logic*, 93(1-3):195–216, 1998.

[Lut03] Jack H. Lutz. The dimensions of individual strings and sequences. *Inform. and Comput.*, 187(1):49–79, 2003.

[MSS18] Birzhan Moldagaliyev, Ludwig Staiger, and Frank Stephan. On the values for factor complexity. In Cezar Câmpeanu, editor, *Implementation and Application of Automata - 23rd International*

Conference, CIAA 2018, Charlottetown, PE, Canada, July 30 - August 2, 2018, Proceedings, volume 10977 of *Lecture Notes in Computer Science*, pages 274–285. Springer, Cham, 2018.

[Sch71] Claus-Peter Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitstheorie*. Lecture Notes in Mathematics, Vol. 218. Springer-Verlag, Berlin, 1971.

[Sta89] Ludwig Staiger. Combinatorial properties of the Hausdorff dimension. *J. Statist. Plann. Inference*, 23(1):95–100, 1989.

[Sta93] Ludwig Staiger. Kolmogorov complexity and Hausdorff dimension. *Inform. and Comput.*, 103(2):159–194, 1993.

[Sta98] Ludwig Staiger. A tight upper bound on Kolmogorov complexity and uniformly optimal prediction. *Theory Comput. Syst.*, 31(3):215–229, 1998.

[Sta07] Ludwig Staiger. The Kolmogorov complexity of infinite words. *Theoret. Comput. Sci.*, 383(2-3):187–199, 2007.

[Ter04] Sebastiaan A. Terwijn. Complexity and randomness. *Rend. Semin. Mat., Torino*, 62(1):1–37, 2004.

[ZW01] Xizhong Zheng and Klaus Weihrauch. The arithmetical hierarchy of real numbers. *MLQ Math. Log. Q.*, 47(1):51–65, 2001.