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Liouville Numbers, Borel Normality and Martin-Löf Randomness

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Liouville, Computable, Borel Normal and Martin-Löf Random Numbers

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Abstract

We survey the current known relations between four classes of real numbers: Liouville numbers, computable reals, Borel absolutely-normal numbers and Martin-Löf random reals. The expansions of the reals will play an important role. The paper refers to the original material and does not repeat the proofs. A characterisation of Liouville numbers in terms of their expansions will be proved (Theorem 2.2) and a connection between the asymptotic complexity of the expansion of a real and its irrationality exponent will be used to show that Martin-Löf random reals have irrationality exponent 2 (Corollary 3.4). Finally we discuss an *open problem*: are there computable, Borel absolutely-normal, non-Liouville numbers?

1 Introduction

The origin of this paper was a question posed by J. Borwein [5] regarding the relations between Liouville numbers and "random" reals. Here we consider the following two mathematical definitions of "random" reals: Borel absolutely-normal numbers – the number-theoretic random ones – and Martin-Löf random reals, arguably the most important class of algorithmic random numbers. To get a complete answer we survey the current known relations between the above classes of random reals, Liouville numbers and computable reals. Many results presented here are known. However, they are scattered through the literature of various areas: we will not present proofs for them. A characterisation of Liouville numbers in terms of their expansions will be proved.

1.1 Notation

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. Its elements will be usually denoted by letters $i, \ldots, n$. The set $A_b = \{0, 1, \ldots, b-1\}$, where $b \geq 2$ is a positive integer, is called the *b–base*; the elements of A_b are called b–digits. By A_b^* we denote the set of all finite strings (words) with ε denoting the empty string; A_b^ω is the set of all (infinite) sequences over A_b. Sequences (infinite strings) are usually denoted by $\mathbf{x}, \mathbf{y}$; the prefix of length n of the sequence $\mathbf{x}$ is denoted by $\mathbf{x} \upharpoonright n$. The length of a finite or infinite string η over A_b is denoted by $|\eta|$; the nth element of η is denoted by $\eta(n)$.

For $w \in A_b^*$ and $\beta \in A_b^* \cup A_b^\omega$ let $w \cdot \beta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $L \subseteq A_b^*$ and $B \subseteq A_b^* \cup A_b^\omega$. If $w \in A_b^*$ and $i \geq 0$ is an integer, then w^i is the concatenation $ww \cdots w$ (i times) and w^ω is the infinite concatenation $ww \cdots w \cdots$. The $\cdot$ operator can be omitted where the meaning is clear, as in $w\beta$.

By $w \sqsubseteq u$ and $w \sqsubset \mathbf{y}$ we denote that w is a prefix of u and $\mathbf{y}$, respectively, and a prefix-free set $L \subset A_b^*$ is a set with the property that for all strings $p, q \in A_b^*$, if $p, pq \in L$ then $p = pq$.

1.2 Preliminary definitions

In this section we define the four classes of real numbers to be studied in the paper.

A real number α is called a *Liouville number* [26] if it is irrational and for every positive integer k, there exist integers p_k and q_k with $q_k > 1$ such that

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^k}.$$

The *irrationality exponent* of a real number α is a measure of how "closely" α can be approximated by rationals [6]:

$$\inf \left\{ \mu \geq 0 : \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^\mu} \text{ has finitely many solutions for all } p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Thus, Liouville numbers are reals having infinite irrationality exponents.

After having defined the class of Liouville numbers $\mathscr{L}$, we now introduce the following three classes of numbers: the Borel absolutely-normal numbers $\mathscr{N}$, the Martin-Löf random numbers $\mathscr{M}$ and the computable numbers $\mathscr{C}$.

A *b-ary expansion* ($b \geq 2$) of the real $\alpha \in [0,1]$ is an infinite sequence $\mathbf{x} = x_1x_2\cdots$ with $x_i \in A_b$, such that $\alpha = \sum_{i\geq 1} x_i \cdot b^{-i}$. Here we will use also the notation $\alpha = 0.x_1x_2\cdots$. It is well-known that the b-ary expansion of $\alpha \in [0,1]$ is unique unless α is a rational of the form $\alpha = p/b^i, i > 0, 0 < p < b^i$, in which case $\alpha = \frac{p-1}{b^i} + \sum_{k>i} \frac{b-1}{b^k}$.

A real number $\alpha \in [0,1]$ is referred to as *(Borel) normal number in base b* if it has a b-ary expansion $\mathbf{x} \in A_b^\omega$ which is uniformly distributed, i.e. each b–digit has the same natural density $1/b$, every string of two b–digits has the same natural density b^{-2}, and, in general, every string of k digits has the same natural density b^{-k}. More precisely, for all $w \in A_b^*$ we have:

$$\lim_{n\to\infty} \frac{|\{i : 1 \leq i \leq n \wedge \mathbf{x} \upharpoonright i \in A_b^* \cdot w\}|}{n} = b^{-|w|}.$$

If the base is clear we will simply say that the number is normal. Normality was introduced by Borel [4] as a model of randomness, sometimes referred to "number-theoretical randomness". Obviously, numbers normal in some base b are irrational.

Champernowne's number $0.0123456789101112\ldots$ is normal in base 10 and computable [14, 2]. A *(Borel) -normal* number is a real which is normal in every base.

For the definition of Martin-Löf random numbers we adopt a characterisation of randomness based on description complexity (see [9, 17]).

Recall that the plain (Kolmogorov) complexity of a string $x \in A_b^*$ w.r.t. a partially computable function $\varphi : A_b^* \to A_b^*$ is $K_\varphi(x) = \inf\{|p| : \varphi(p) = x\}$. It is well-known that there is a universal partially computable function $U_b : A_b^* \to A_b^*$ such that $K_{U_b}(x) \leq K_\varphi(x) + c_\varphi$ holds for all strings $x \in A_b^*$. Here the constant c_φ depends only on U_b and φ but not on the particular string $x \in A_b^*$. We will denote the complexity K_{U_b} simply by K_b, or K if the alphabet is clear from the context. Furthermore, in the case that one considers only computable partial functions with prefix-free domain, there are also universal ones among them and the corresponding complexity, called *prefix complexity*, is denoted by H; like K, the prefix-free complexity H depends only up to a constant on the given choice of the underlying universal machine.

Martin-Löf [29] introduced the notion of the random sequences in terms of tests. Here we adopt the following complexity-theoretic characterisation. An infinite sequence $\mathbf{x} \in A_b^\omega$ is *Martin-Löf random* if there is a constant c such that $H(\mathbf{x} \upharpoonright n) \geq n - c$, for all $n \geq 1$. A real $\alpha \in [0,1]$ is *Martin-Löf random* if there is a base b such that its b-ary expansion is Martin-Löf random.

Unlike the case of Borel normality this property is base-independent, that is, if $\mathbf{x} \in A_b^\omega$ is Martin-Löf random and $0.\mathbf{x} = 0.\mathbf{y}$ for $\mathbf{y} \in A_{b'}^\omega$, then $\mathbf{y}$ is also Martin-Löf random (cf. [11, 21, 36]).

A real $\alpha \in [0,1]$ is called *computable* if it has a b-ary expansion which is computable, that is, there is a computable function f_α such that $f_\alpha(n) = x_n$, for all $n \geq 1$. This condition is equivalent to the requirement that there is a computable sequence of rationals $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ such that $|\alpha - \frac{p_n}{q_n}| \leq \frac{1}{2^n}$, for all $n \in \mathbb{N}$. This shows that the b-ary expansions of α are computable, for all bases b.

2 Expansions of Liouville numbers

2.1 Expansion characterisation of Liouville numbers

In the previous section we defined three of the four classes of numbers under consideration by properties of their expansions. Here we show that also Liouville numbers have similar characterisations.

We start with a technical result.

Lemma 2.1 *Let $\alpha \in [0,1]$ and assume that its b-ary expansion starts with $v \cdot w^\ell$ where $v, w \in A_b^*, |w| > 0$, and $l \in \mathbb{N}$. Then there exist two non-negative integers p, q such that*

$$\left|\alpha - \frac{p}{q}\right| < q^{-\frac{|v|+\ell\cdot|w|}{|v|+|w|}}. \tag{1}$$

Proof. Consider the rational number $\frac{p}{q}$ whose b-ary expansion is $v \cdot w^\omega$, that is

$$\frac{p}{q} = \frac{p_v(b^{|w|}-1) + p_w}{b^{|v|}(b^{|w|}-1)}, \tag{2}$$

where p_v and p_w are the natural numbers whose b-ary expansions are v and w, respectively. Then the denominator q satisfies $q \leq b^{|v|}(b^{|w|}-1)$.

Since the b-ary expansion of α starts with $v \cdot w^\ell$ we have

$$\left|\alpha - \frac{p}{q}\right| \leq b^{-(|v|+\ell\cdot|w|)}. \tag{3}$$

Using the inequality

$$b^{|v|+\ell\cdot|w|} = \left(b^{|v|}\cdot b^{|w|}\right)^{\frac{|v|+\ell\cdot|w|}{|v|+|w|}} > \left(b^{|v|}\cdot(b^{|w|}-1)\right)^{\frac{|v|+\ell\cdot|w|}{|v|+|w|}} \geq q^{\frac{|v|+\ell\cdot|w|}{|v|+|w|}}. \qquad (4)$$

we obtain Eq. (1) from (3). □

Theorem 2.2 *Let $\alpha \in [0,1]$ be an irrational. Then, α is a Liouville number if and only if for every integer $k > 1$ there exists a base $b = b_{\alpha,k} \geq 2$ and words $v, w \in A_b^*$, $|v| \leq |w|$, $|w| > 0$, such that the b-ary expansion $\mathbf{x} = x_1 x_2 \cdots$ of α satisfies $\mathbf{x} = v \cdot w^k \cdot \mathbf{x}'$ for some $\mathbf{x}' \in A_b^\omega$.*

Proof. Let k be a positive integer and choose b and $v, w \in A_b^*$ to satisfy the assumptions of the theorem. Then, in view of $|v| \leq |w|$, Lemma 2.1 with $\ell = k$ implies the inequality

$$\left|\alpha - \frac{p}{q}\right| \leq q^{-k/2},$$

for suitable positive integers p, q (see (2)). Since k is arbitrary and α is irrational, then α is a Liouville number.

Now let α be a Liouville number. Since α is irrational, there exists an integer $k_0 > 1$ such that $2^{-k_0} < \alpha < 1 - 2^{-k_0}$. Let $k \geq k_0$ such that $|\alpha - \frac{p}{q}| < q^{-k}$ and set $b = q$. Then $1 \leq p < q$ and $(p-1), p \in A_q$. If $\alpha > \frac{p}{q}$ then the q-ary expansion $\mathbf{x}$ of α starts with $p0^k \in A_q^*$, and if $\alpha < \frac{p}{q}$ then $\mathbf{x}$ starts with $(p-1)(q-1)^k \in A_q^*$. □

Examining the last part of the previous proof we obtain the following property of expansions of Liouville numbers.

Corollary 2.3 *If $\alpha \in [0,1]$ is a Liouville number, then for every integer $k > 1$ there exist a base $b = b_{\alpha,k} \geq 2$ such that the b-ary expansion $\mathbf{x} = x_1 x_2 \cdots$ of α starts with $x_1 0^k$ or $x_1 (b-1)^k$.*

2.2 Maillet's construction of Liouville numbers

Based on patterns of their expansions we describe a simple 'construction' of Liouville numbers which will be used in the sequel. This construction was sketched, without proof by Maillet in [27] (see also [34, Kapitel 1]).

Using finitely or infinitely many strings $w_i \in A_b^*$ we can construct b-ary expansions of real numbers. Let $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$; by $\Lambda_{j=0}^{\infty} w_j^{f(j)}$ we denote the concatenation of w_0 ($f(0)$ times), w_1 ($f(1)$ times),w_2 ($f(2)$ times)...

Lemma 2.4 *Let $(w_i)_{i\in\mathbb{N}}$ be a family of non-empty strings $w_i \in A_b^*$, $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, and $n_i = \sum_{j=0}^{i} f(j) \cdot |w_j|$. If $\liminf_{i\to\infty} \frac{n_{i-1}+|w_i|}{n_i} = 0$, then $\mathbf{x} = \Lambda_{j=0}^{\infty} w_j^{f(j)}$ is the b-ary expansion of a rational or a Liouville number.*

Proof. Let $v_i = \Lambda_{j=0}^{i-1} w_j^{f(j)}$ and observe that $n_{i-1} = \left|\Lambda_{j=0}^{i-1} w_j^{f(j)}\right| = |v_i|$. Next, choose two positive integers i,k such that $(n_{i-1}+|w_i|)\cdot k < n_i$ and consider the b-ary expansion $\mathbf{y}_i = \Lambda_{j=0}^{i-1} w_j^{f(j)} \cdot w_i^{\omega} = v_i \cdot w_i^{\omega}$. Then $\mathbf{x}$ and $\mathbf{y}_i$ both start with $\Lambda_{j=0}^{i} w_j^{f(j)} = v_i \cdot w_i^{f(i)}$.

Using Lemma 2.1 we get positive integers p_i, q_i such that

$$\left|0.\mathbf{x} - \frac{p_i}{q_i}\right| \le q_i^{-\frac{n_{i-1}+f(i)\cdot|w_i|}{n_{i-1}+|w_i|}} = q_i^{-\frac{n_i}{n_{i-1}+|w_i|}} \le q_i^{-k}. \qquad \square$$

Remark. Maillet [27] requested that the repetition factors $f(i)$ (K_i in the terminology of [34, p. 411] and k_m in [27]) increase rapidly (‘*Si k_m croît assez vite avec m*’ [27]). In Lemma 2.4 the function f does not have to be increasing. For example, the above construction with $w_{2i} = 0$, $w_{2i+1} = 1$, $f(2i) = i!$ and $f(2i+1) = 1$ yields the Liouville number with the expansion $\Lambda_{j=0}^{\infty} 0^{j!}1$.

2.3 A construction of Liouville numbers normal in a given base

In [7] Bugeaud proved that there exist Borel absolutely-normal Liouville numbers. His existence proof uses complicated measure-theoretical considerations. The paper [31] uses de Bruijn sequences to give a simple construction of Liouville numbers normal in *a given base b* via their b-ary expansions; a similar construction was given in [37] for the construction of numbers normal in a given base.

A *de Bruijn word* of *order r* is a b-ary string w of length $b^r + r - 1$ over the alphabet A_b such that any string of length r occurs as a substring of w (exactly once). It is well-known that de Bruijn words of any order r and every alphabet size b exist, and have an explicit construction [16, 38] or [20, Ch. 9]. For example, 00110 and 0001011100 are binary de Bruijn words of orders 2 and 3, respectively, and 0010221120 and 0011021220 are ternary de Bruijn words of order 2.

Note that de Bruijn words are derived in a circular way, hence their prefix of length $r-1$ coincides with the suffix of length $r-1$. Denote by $B(b,r)$ the prefix of length b^r of a de Bruijn word of order r over A_b. To be precise we assume that $B(b,r)$ is the lexicographically first b^r-length prefix among all b-ary de Bruijn words of order r. Thus $B(b,r)$ starts with 0^r1 and ends on a symbol different from 0, and, consequently, $B(b,r)$ is not a prefix of $B(b,r+1)$.

From the examples of binary de Bruijn words of orders 2 and 3 previously presented the strings $B(2,2) = 0011$ and $B(2,3) = 00010111$ are derived. Thus the string $B(b,r) \cdot B'(b,r)$, where $B'(b,r)$ is the length $r-1$ prefix of $B(b,r)$, contains every b-ary string of length r exactly once as a substring.

Theorem 2.5 ([31]) *Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $f(i) \geq i^i$, for all $i \geq 1$. Let $B(b,r)$ the prefix of length b^r of a de Bruijn word of order r. Then every sequence of the form*

$$\mathbf{x}_f = \Lambda_{i=1}^{\infty} B(b,i)^{f(i)} = B(b,1)^{f(1)} B(b,2)^{f(2)} \cdots B(b,i)^{f(i)} \cdots \qquad (5)$$

is normal in base b.

Moreover, one obtains that certain of the numbers of the form of Eq. (5) are Liouville numbers.

Theorem 2.6 ([31]) *If the family $\big(B(b,i)\big)_{i \in \mathbb{N}}$ and f satisfy the hypothesis of Lemma 2.4, then the real $\alpha_f = 0.\mathbf{x}_f$ is a Liouville number.*

Remark. Since the set of strings $\{B(b,i) : i \in \mathbb{N}, i \geq 1\}$ is, by construction, prefix-free, it is easy to see that the number $\alpha_f = 0.\mathbf{x}_f$ is computable if and only if the function f is computable.

3 Relations between $\mathscr{L}$, $\mathscr{C}$, $\mathscr{N}$ and $\mathscr{M}$

In this section we explore the relations between the classes $\mathscr{L}$, $\mathscr{C}$, $\mathscr{N}$ and $\mathscr{M}$.

First, how large are the classes $\mathscr{L}, \mathscr{C}, \mathscr{N}, \mathscr{M}$ from the points of view of measure and category (in Baire sense, cf. [32])? While $\mathscr{C}$ is countable, all the other classes have the cardinality of the continuum. The class $\mathscr{L}$ is a dense G_δ-set (hence co-meagre), measure zero set [32, 6]; it has Hausdorff dimension zero [22]. The classes $\mathscr{N}$ and $\mathscr{M}$ are constructive measure one [29], but constructively meagre in the Cantor space [32, 10] (a constructive meagre set is a meagre set covered by a computably enumerable union of computably enumerable nowhere dense subsets; a constructive meagre set is "smaller" than a meagre set). The latter property depends on the topology chosen: In [12] it is shown that $\mathscr{M}$ is co-meagre for a suitably chosen metric topology refining the topology of the Cantor space.

3.1 Complexity-theoretic properties

In this section we present results from [36] which provide some sufficient criteria for numbers being Borel absolutely-normal or non-Liouville using tools of Algorithmic Information Theory. To this end we use the asymptotic complexities $\liminf_{n\to\infty} K_b(\mathbf{x}\restriction n)/n$ and $\limsup_{n\to\infty} K_b(\mathbf{x}\restriction n)/n$ of their b-ary expansions. Our criteria, however, imply that the numbers fulfilling them are highly incomputable.

These asymptotic complexities are base-invariant (cf. [36]), that is,

$$\liminf_{n\to\infty} K_b(\mathbf{x}\restriction n)/n = \liminf_{n\to\infty} K_{b'}(\mathbf{y}\restriction n)/n,$$

(and likewise for $\limsup$) whenever $0.\mathbf{x} = 0.\mathbf{y}$ for $\mathbf{x}\in A_b$ and $\mathbf{y}\in A_{b'}$.

In view of the inequalities $K(w)\le H(w)+c\le K(w)+2\cdot\log|w|+c'$ for suitable constants c,c' (cf. [9, 17]) in what follows we could replace the plain complexity K by the prefix-free complexity H.

From the definitions of Martin-Löf random and computable reals we obtain immediately the following well-known facts (see [9, 17]).

Fact 3.1 (a) *Let* $\mathbf{x}$ *be the b-ary expansion of a Martin-Löf random number. Then* $\liminf_{n\to\infty} K(\mathbf{x}\restriction n)/n = 1$. (b) *If* $\mathbf{x}$ *is the b-ary expansion of a computable number then* $\limsup_{n\to\infty} K(\mathbf{x}\restriction n)/n = 0$.

Using a result of Kolmogorov [25] and the base-invariance of the asymptotic complexity, Fact 3.1.(a) can, to a certain extent, be reversed yielding a sufficient condition for absolute normality.

Lemma 3.2 ([36, Corollary 9]) *Let* $\mathbf{x}$ *be the b-ary expansion of a real* $\alpha\in[0,1]$. *If* $\liminf_{n\to\infty} K(\mathbf{x}\restriction n)/n = 1$, *then* α *is Borel absolutely-normal.*

Generalising the proof of Lemma 10 of [36] we can establish a close connection between the irrationality exponent of a real $\alpha\in[0,1]$ and the asymptotic complexity of its b-ary expansions.

Lemma 3.3 *Let* $\alpha\in[0,1]$ *be an irrational number with irrationality exponent* $\mu\ge 2$ *and let* $\mathbf{x}$ *be its b-ary expansion. Then* $\liminf_{n\to\infty} K(\mathbf{x}\restriction n)/n \le 2/\mu$.

Proof. We fix a rational number $m<\mu$. Then there are infinitely many positive integers p,q such that $|\alpha-\frac{p}{q}|<q^{-m}$, so α is an interior point of the interval $(\frac{p}{q}-\frac{1}{q^m},\frac{p}{q}+\frac{1}{q^m})$. Proceeding as in the proof of Lemma 10 in [36], we note that

this interval can be covered by at most two intervals of the form $[\frac{a}{b^k}, \frac{a+1}{b^k})$, where $k = \lfloor \log_b(q^m/2) \rfloor$ (see [36, Fact 4]). The b-ary expansions of the reals in these intervals start with words $w_0(p,q) \in A_b^k$ and $w_1(p,q) \in A_b^k$, respectively, and these words can be effectively computed from p and q and the fixed rational number m.

Next we encode the numbers p and q via a prefix-free encoding $\mathsf{code} : \mathbb{N} \to A_b^*$ satisfying $|\mathsf{code}(\ell)| \leq \log_b \ell + 2\log_b \log_b \ell + c$ for a suitable constant $c \in \mathbb{N}$ (see e.g. [9, Example 2.5]), and let $\pi_{p,q}(i) = i \cdot \mathsf{code}(p)\mathsf{code}(q)$, $i \in A_b$. Since $p \leq q$ we have $|\pi_{p,q}(i)| \leq 2 \cdot (log_b q + 2\log_b \log_b q + c)$

Further define for the fixed rational m the mapping $\psi_m : A_b^* \to A_b^*$ by

$$\psi_m(\pi_{p,q}(i)) = \begin{cases} w_i(p,q), & \text{for } i = 0,1, \\ \infty, & \text{otherwise.} \end{cases}$$

Then $|\psi_m(\pi_{p,q}(i))| = \lfloor \log_b(q^m/2) \rfloor \geq m \cdot \log_b q - 3$. Hence, there exist infinitely many prefixes $w_i(p,q)$ of the b-ary expansion $\mathbf{x}$ of α such that

$$\frac{K_\psi(w_i(p,q))}{|w_i(p,q)|} \leq \frac{2 \cdot (\log_b q + 2\log_b \log_b q + c)}{m \cdot \log_b q - 3} \to 2/m.$$

As m can be chosen arbitrarily close to μ the statement of the lemma follows. $\square$

From the Fact 3.1.(a) we deduce:

Corollary 3.4 *Every Martin-Löf random real has irrationality exponent* 2.

Lemma 10 and Corollary 11 in [36] are now corollaries.

Corollary 3.5 ([36, Lemma 10]) *Every b-ary expansion* $\mathbf{x}$ *of a Liouville number has* $\liminf_{n\to\infty} K(\mathbf{x} \restriction n)/n = 0$.

Corollary 3.6 ([36, Corollary 11]) *The sets of Liouville numbers and Martin-Löf random numbers are disjoint.*

Is the bound $2/\mu$ in Lemma 3.3 the best upper bound for reals having irrationality exponent μ? The following considerations based on results by Jarník [22] and Ryabko [33] show that the bound $2/\mu$ cannot be in general improved. To this aim we use the Hausdorff dimension for subsets of $[0,1]$ and A_b^ω. It is well-known that $\dim M = \dim\{\mathbf{x} : \mathbf{x}$ is a b-ary expansion of $\alpha \in M\}$ and dim is monotone with respect to set inclusion, see [18, 19]).

Theorem 3.7 ([22]) *Let* $\mu \geq 2$. *Then*

$$\dim\{\alpha : \alpha \in [0,1] \text{has irrationality exponent } \mu\} = 2/\mu\,.$$

Theorem 3.8 ([33, 35]) *Let $\gamma > 1$. Then*

$$\dim\{\mathbf{x} : \liminf_{n\to\infty} K(\mathbf{x}\upharpoonright n)/n \leq \gamma\} = \gamma.$$

According to Ryabko's Theorem 3.8, for every $\varepsilon > 0$ the set $\{\mathbf{x} : \liminf_{n\to\infty} K(\mathbf{x}\upharpoonright n)/n \leq 2/\mu - \varepsilon\}$ has Hausdorff dimension $2/\mu - \varepsilon < 2/\mu$. Thus, this set cannot contain the set $F_\mu = \{\mathbf{x} : 0.\mathbf{x}$ has irrationality exponent equal to $\mu\}$ which has $\dim F_\mu = 2/\mu$ by Jarník's Theorem 3.7. Consequently, there is a sequence $\mathbf{y} \in F_\mu$ such that $\liminf_{n\to\infty} K(\mathbf{y}\upharpoonright n)/n > 2/\mu - \varepsilon$.

3.2 Empty intersections

Here we summarise known relations between our four classes of real numbers. Except for the one in Eq. (8) which is Corollary 3.6 these relations are folklore, however, one can deduce them also from the results in the previous section.

Fact 3.9

$$\mathscr{M} \subseteq \mathscr{N} \tag{6}$$
$$\mathscr{M} \cap \mathscr{C} = \emptyset \tag{7}$$
$$\mathscr{M} \cap \mathscr{L} = \emptyset \tag{8}$$

If we consider all possible Boolean combinations between the four classes of numbers $\mathscr{L}$, $\mathscr{N}$, $\mathscr{M}$ and $\mathscr{C}$ we obtain that out of 16 possible combinations the following seven sets are empty:[1]

$\bar{\mathscr{L}} \cap \bar{\mathscr{C}} \cap \bar{\mathscr{N}} \cap \mathscr{M}$, $\bar{\mathscr{L}} \cap \mathscr{C} \cap \bar{\mathscr{N}} \cap \mathscr{M}$, $\mathscr{L} \cap \bar{\mathscr{C}} \cap \bar{\mathscr{N}} \cap \mathscr{M}$, $\mathscr{L} \cap \mathscr{C} \cap \bar{\mathscr{N}} \cap \mathscr{M}$ (all because $\mathscr{M} \subseteq \mathscr{N}$), $\bar{\mathscr{L}} \cap \mathscr{C} \cap \mathscr{N} \cap \mathscr{M}$, $\mathscr{L} \cap \mathscr{C} \cap \mathscr{N} \cap \mathscr{M}$ (both because $\mathscr{C} \cap \mathscr{M} = \emptyset$), and $\mathscr{L} \cap \bar{\mathscr{C}} \cap \mathscr{N} \cap \mathscr{M}$ (because $\mathscr{M} \cap \mathscr{L} = \emptyset$).

3.3 Non-empty intersections

Next we show that eight of the nine remaining intersections are non-empty, but they are all "small" in measure or/and category. For the remaining intersection $\bar{\mathscr{L}} \cap \mathscr{C} \cap \mathscr{N} \cap \bar{\mathscr{M}}$ we give an indication that it might be non-empty: a result in [15] shows that there are computable non-Liouville numbers which have a binary expansion which is normal.

[1]We denote by $\bar{S}$ the complement of the set S.

The first three non-emptiness results use complicated constructions of Liouville numbers [7]. Liouville numbers normal to a fixed base $b, b \geq 2$, can be constructed in a simpler way (see Section 2.3 above).

3.3.1 $\mathscr{L} \cap \bar{\mathscr{C}} \cap \mathscr{N} \cap \bar{\mathscr{M}}$

In [7, Theorem 2] it is proved that the set $\mathscr{L} \cap \mathscr{N}$ is uncountable, thus it contains incomputable reals.

3.3.2 $\mathscr{L} \cap \mathscr{C} \cap \mathscr{N} \cap \bar{\mathscr{M}}$

Based on the proof of Theorem 2 of [7] a construction of a computable Borel absolutely-normal number was given in [3].

3.3.3 $\mathscr{L} \cap \bar{\mathscr{C}} \cap \bar{\mathscr{N}} \cap \bar{\mathscr{M}}$

Incomputable Liouville numbers not normal to a certain base have been constructed in [23, Proposition 7]. The required example can be obtained from Lemma 2.4 with $w_{2i} = 0$, $w_{2i+1} = a_i \in A_b$, $f(2i+1) = 1$ and $f(2i) = i! - (i-1)! - 1$, where the sequence $(a_i)_{i \in \mathbb{N}}$ is incomputable. This yields incomputable Liouville numbers $\sum_{i=1}^{\infty} a_i \cdot b^{-i!}$ with the expansion $\Lambda_{j=0}^{\infty} 0^{f(2j)} a_j$ which are not normal in base b.

A stronger existence result is Theorem 1 of [7] which shows that there are uncountably many Liouville numbers not normal to any base.

3.3.4 $\mathscr{L} \cap \mathscr{C} \cap \bar{\mathscr{N}} \cap \bar{\mathscr{M}}$

Liouville's 'classical' number $\sum_{i=1}^{\infty} b^{-i!}$ is computable, and not normal in base b.

A more interesting example was given by Martin [28] who constructed a Liouville number not normal in any base.

3.3.5 $\bar{\mathscr{L}} \cap \bar{\mathscr{C}} \cap \bar{\mathscr{N}} \cap \bar{\mathscr{M}}$

Let $\alpha = 0.x_1 x_2 \ldots x_n \ldots$, $x_i \in A_b$ be Martin-Löf random (given by a b-ary expansion) and let $\beta = 0.\mathbf{y}$, where $\mathbf{y} = x_1 00 x_2 00 \ldots x_n 00 \ldots$. Then β is not normal in base b because it contains at least $2/3$ more zeroes than other letters. It is not computable, for otherwise α would be computable. Finally, since $\liminf_{n\to\infty} K(\mathbf{y} \upharpoonright n)/n = 1/3$ (actually β is 1/3–Martin-Löf random in the sense of [13]), Lemma 3.5 shows that β is not a Liouville number.

3.3.6 $\bar{\mathscr{L}} \cap \bar{\mathscr{C}} \cap \mathscr{N} \cap \mathscr{M}$

Here $\bar{\mathscr{L}} \cap \bar{\mathscr{C}} \cap \mathscr{N} \cap \mathscr{M} = \mathscr{M} \neq \emptyset$ follows from Fact 3.9.

An interesting example of real in this class is Chaitin Omega number $\Omega = \sum_{n \geq 1} 2^{-H(\mathrm{bin}(n))}$ (see [9, 17]), where $\mathrm{bin}(n)$ is the nth binary string in quasi-lexicographical order.

3.3.7 $\bar{\mathscr{L}} \cap \mathscr{C} \cap \bar{\mathscr{N}} \cap \bar{\mathscr{M}}$

Every rational number is computable but neither Liouville nor normal.

The computable irrational number $\alpha_b = \sum_{i \in \mathbb{N}} b^{-2^i}$ is non-Liouville (see [24]). It is readily seen that α_b is not normal in base b.

3.3.8 $\bar{\mathscr{L}} \cap \bar{\mathscr{C}} \cap \mathscr{N} \cap \bar{\mathscr{M}}$

Let $\alpha = 0.x_1x_2 \ldots x_n \ldots$ be a Martin-Löf random real (given by a b-ary expansion) and let $\mathbf{y}(2^i) = 0$ and $\mathbf{y}(j) = x_j$, otherwise. Then $\liminf_{n\to\infty} K(\mathbf{y} \upharpoonright n)/n = 1$ (see [30, Example 4.1]) and, in view of Lemmas 3.2 and 3.5, the real $\beta = 0.\mathbf{y}$ is an Borel absolutely-normal non-Liouville number.

To show that $\beta = 0.\mathbf{y}$ is not Martin-Löf random we use the following property (cf. [9, Corollary 6.4.2] or [17, Theorem 7.2.23]):

Fact 3.10 *For the b-ary expansion $\alpha = 0.x_1x_2 \ldots x_n \ldots$ of a Martin-Löf random real the set $\{j : x_j = 0\}$ cannot contain an infinite computable subset $M \subseteq \mathbb{N}$, in particular not the set $\{2^i : i \in \mathbb{N}\}$.*

3.3.9 $\bar{\mathscr{L}} \cap \mathscr{C} \cap \mathscr{N} \cap \bar{\mathscr{M}}$

It is an *open problem* whether there exist computable, Borel absolutely-normal, non-Liouville numbers. There exist computable, non-Liouville numbers, normal to base 2, but not Borel absolutely-normal. Any Stoneham number $F(1/2) = \sum_{i=1}^{\infty} 2^{-k^i} \cdot k^{-i}$ (where $k \in \mathbb{N}$ is odd ,$k \geq 3$) is computable, normal in base 2 (but not in base 6, see [15]), and, by [15, Theorem 1], has irrationality exponent $\mu(F(1/2)) = k$, thus, it is not Liouville.

The set $\mathscr{L} \cap \bar{\mathscr{C}} \cap \bar{\mathscr{N}} \cap \bar{\mathscr{M}}$ is co-meagre and has measure zero and the set $\bar{\mathscr{L}} \cap \bar{\mathscr{C}} \cap \mathscr{N} \cap \mathscr{M} = \mathscr{M}$ has constructive measure one and is meagre. The remaining non-empty sets are meagre and have constructive measure zero.

4 Concluding remark

We surveyed relations between four classes of real numbers: Liouville numbers, computable reals, Borel absolutely-normal numbers and Martin-Löf random reals. The results showed that several properties of real numbers are incompatible. For the remaining possibilities – up to one case – we found real numbers possessing these properties. We note that the existence results come from quite different realms of mathematics and theoretical computer science.

The following *open* question was discussed: the existence of computable, Borel absolutely-normal, non-Liouville numbers (Section 3.3.9).

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