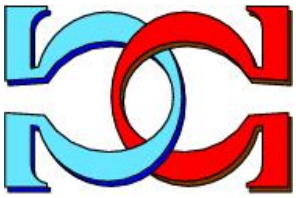

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Analytic Continuation and Randomness of a Generalized Halting Probability

Michael A. Stay
University of Auckland

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Michael A. Stay
mike@cs.auckland.ac.nz

Abstract

Tadaki's generalization of Chaitin's halting probability associates with a self-delimiting Turing machine C the partition function

$$Z_C(s) = \sum_{p \in \mathrm{dom}(C)} 2^{-s|p|}.$$

For a universal machine, $Z_C(1)$ is Chaitin's Omega number and, for every computable real $s > 1$, $Z_C(s)$ is partially random with compression rate $1/s$. For real $0 < s < 1$ the defining series diverges. We show that this divergence need not be a natural boundary phenomenon. Starting effectively with an arbitrary universal self-delimiting Turing machine V, we construct a universal self-delimiting machine U for which Z_U can be analytically continued around $s = 1$ to the real interval $0 < s < 1$. The construction has two parts. A positive bilateral Fourier kernel produces a universal machine whose normalized length-generating function continues across an open arc of its circle of convergence. A second, prefix-disjoint branch introduces a square-root singularity carrying a scaled Omega number. On either continuation to a real point $0 < s < 1$, all rounding terms remain real while the square-root term becomes imaginary. Consequently, for every computable real $s \in (0, 1)$, the absolute value of the imaginary part of the continued $Z_U(s)$ is a computably enumerable random real. In particular, below the critical value $s = 1$ the continued function contains a component of maximal algorithmic information density.

1 Introduction

Algorithmic thermodynamics regards the halting programs of a universal prefix-free Turing machine as microstates and computable quantities on programs as observables. Baez and Stay [1] developed this point of view systematically by considering Gibbs ensembles on the halting set. If program length is the only observable, the corresponding partition function has the form

$$Z = \sum_{p \in \mathrm{dom}(C)} e^{-\gamma|p|}.$$

Putting $\gamma = s \log 2$ gives precisely the one-parameter function

$$Z_C(s) = \sum_{p \in \mathrm{dom}(C)} 2^{-s|p|} \tag{1}$$

introduced by Tadaki [6]. Baez and Stay investigated the domain of convergence of such partition functions and emphasized that, on parts of its boundary, the partition

function itself can carry nontrivial algorithmic information [1]. This raises a natural complex-analytic question: what remains of that information if the partition function is analytically continued beyond its original domain of convergence?

At the critical value $s = 1$ we recover Chaitin's halting probability

$$\Omega_C = \sum_{p \in \mathrm{dom}(C)} 2^{-|p|}.$$

When C is universal, Ω_C is computably enumerable (c.e.) and random; conversely, every c.e. random real is the halting probability of a universal self-delimiting machine [5, 3, 2]. Tadaki proved that if C is universal and $s > 1$ is computable, then $Z_C(s)$ is Chaitin $1/s$-random. Calude and Stay [4] observed that this family has the formal structure of a statistical-mechanical partition function, with the randomness parameter playing the role of temperature. Baez and Stay [1] subsequently embedded this observation in a broader thermodynamic formalism involving several program observables.

For a universal machine, the real series defining $Z_C(s)$ converges for $s \geq 1$ and diverges for $0 < s < 1$. Thus $s = 1$ is simultaneously the point at which the partition function becomes Chaitin's Ω and the boundary of the real convergence region. It is natural to ask whether this boundary is forced by universality in the stronger complex-analytic sense: must the line $\Re(s) = 1$ be a natural boundary? The answer is negative. In fact, constant-overhead universality is compatible with analytic continuation around $s = 1$. More surprisingly, the continuation can be made to retain maximal algorithmic information on the other side of the critical point. In the language of algorithmic thermodynamics, the divergent real "low-s" regime can still be reached by analytic continuation, and a full random component survives there.

The main result of this paper is the following.

Theorem 1. *Let V be a universal self-delimiting Turing machine. We can effectively construct a universal self-delimiting Turing machine U with the following properties:*

1. *U contains a fixed-prefix simulation of V; in the construction below,*

$$U(000p) = V(p), \qquad p \in \mathrm{dom}(V).$$

2. *The function Z_U, initially defined by (1) in $\Re(s) > 1$, admits analytic continuation along paths around $s = 1$ to every real point $s \in (0, 1)$.*

3. *There are two conjugate continuations to each real $s \in (0, 1)$. For every computable real $s \in (0, 1)$,*

$$\left|\Im(Z_U^{\mathrm{cont}}(s))\right| = c_s \Omega_{U_0},$$

   where U_0 is a universal machine effectively constructed from V and $c_s > 0$ is a computable real.

4. *Consequently, for every computable real $s \in (0, 1)$, $|\Im(Z_U^{\mathrm{cont}}(s))|$ is c.e. and random.*

The proof is constructive. First we use a positive two-sided convolution to manufacture a spectral gap on the unit circle without destroying a fixed-prefix copy of the original universal machine. The positive-index part is realized by a self-delimiting machine; the negative-index part supplies the exterior analytic germ. A small dyadic rounding error is analytic in a larger disk. Secondly, we add a separate prefix branch based on $1 - \sqrt{1 - w}$.

The coefficient of this branch is a scaled Omega number. On continuing around the branch point, this term acquires an imaginary part, while all rounding corrections remain real. Hence cancellation of the random component is impossible.

The rest of the paper gives the details. Some final comments and open questions conclude the paper.

2 Notation and preliminaries

Let $\Sigma = \{0, 1\}$ and let Σ^* be the set of finite binary strings. A self-delimiting Turing machine is a partial computable function $C : \Sigma^* \to \Sigma^*$ whose domain is prefix-free. Its program-size complexity is

$$H_C(x) = \min\{\, |p| \mid C(p) = x \,\}.$$

For such a machine put

$$d_C(n) = \#\{p \in \mathrm{dom}(C) \mid |p| = n\}.$$

The Kraft inequality gives

$$\sum_{n \geq 0} d_C(n) 2^{-n} = \Omega_C \leq 1.$$

A self-delimiting machine U is universal if for every self-delimiting machine C there is a constant $c \geq 0$ such that for every x in the range of C,

$$H_U(x) \leq H_C(x) + c.$$

Equivalently for our purposes, a universal machine can simulate an arbitrary fixed universal machine with an additive constant in program length. The constructions below preserve universality in the strongest possible way: they contain a literal fixed-prefix copy of the source machine.

For a self-delimiting machine C define the normalized length-generating function

$$A_C(w) = \sum_{n=0}^{\infty} d_C(n) 2^{-n} w^n. \tag{2}$$

Then

$$Z_C(s) = A_C(2^{1-s}). \tag{3}$$

Indeed,

$$A_C(2^{1-s}) = \sum_{n \geq 0} d_C(n) 2^{-n} 2^{(1-s)n} = \sum_{n \geq 0} d_C(n) 2^{-sn}.$$

Thus the line $\Re(s) = 1$ corresponds to the unit circle $|w| = 1$. The point

$$s = 1 + \frac{\pi i}{\log 2}$$

corresponds to $w = -1$.

We use the standard definition of a c.e. real and of a random real. If $\alpha = 0.x_1 x_2 \cdots$ is in $(0, 1)$, we write $\alpha[n] = x_1 \cdots x_n$. A c.e. real α is random if, for one (equivalently, every) universal self-delimiting machine C, there is a constant c such that

$$H_C(\alpha[n]) \geq n - c$$

for every $n \geq 1$. This is equivalent to Martin-Löf randomness for c.e. reals [3, 2].

For a universal machine the unit circle is the actual circle of convergence of (2). Indeed, the density theorem for universal self-delimiting machines proved by Calude and Stay [4] implies

$$\liminf_{n\to\infty} \frac{1}{n} \log_2 \#\{p \in \mathrm{dom}(C) \mid |p| \leq n\} = 1.$$

Consequently, for every real $0 < s < 1$, $Z_C(s)$ diverges, while Kraft's inequality gives convergence at $s = 1$. Hence the radius of convergence of A_C is exactly one.

We recall Tadaki's result in the notation used here.

Theorem 2 (Tadaki [6])**.** *Let C be a universal self-delimiting Turing machine and let $s > 1$ be computable. Then $Z_C(s)$ is Chaitin $1/s$-random. In particular, for some constant c and every $n \geq 1$,*

$$H_C(Z_C(s)[n]) \geq \frac{n}{s} - c.$$

At $s = 1$, $Z_C(1) = \Omega_C$ and the randomness rate is one. The construction below shows that after analytic continuation around the critical point, the information rate can remain saturated at one.

3 A positive bilateral kernel

We begin with a simple Fourier-analytic observation. Define a 2π-periodic function by

$$\Theta(t) = \begin{cases} 4\left(1 - \dfrac{2|t|}{\pi}\right), & |t| \leq \dfrac{\pi}{2}, \\ 0, & \dfrac{\pi}{2} < |t| \leq \pi. \end{cases} \tag{4}$$

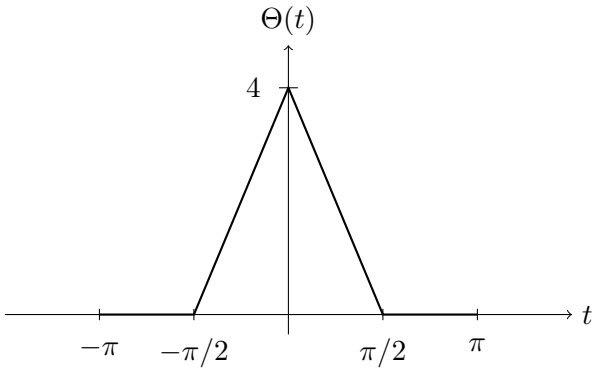

Figure 1: The function Θ on $[-\pi, \pi]$; it is extended periodically with period 2π.

Figure 1 shows one period of Θ.

Let

$$\kappa_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Theta(t) e^{-imt}\, dt, \qquad m \in \mathbb{Z}.$$

The first several coefficients are shown in Figure 2.

Lemma 3. *The sequence $(\kappa_m)_{m\in\mathbb{Z}}$ has the following properties:*

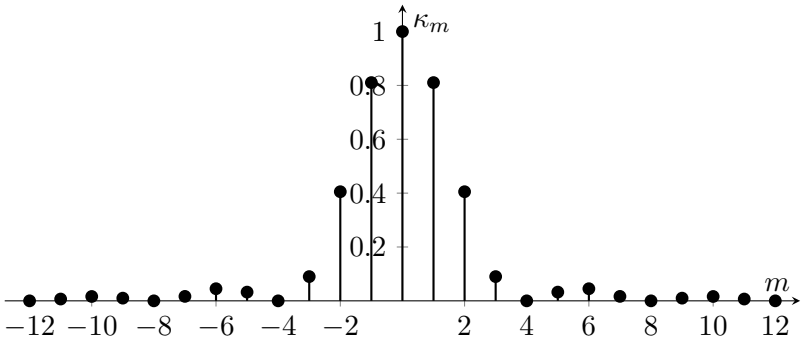

Figure 2: The Fourier coefficients κ_m for $-12 \le m \le 12$.

1. $\kappa_m \ge 0$ *for every* $m \in \mathbb{Z}$*;*
2. $\kappa_0 = 1$*;*
3. *for* $m \neq 0$*,*
$$\kappa_m = \frac{8\,(1-\cos(\pi m/2))}{\pi^2 m^2};$$
4. $\sum_{m\in\mathbb{Z}} \kappa_m = 4$*;*
5. *the absolutely convergent Fourier series*
$$\sum_{m\in\mathbb{Z}} \kappa_m e^{imt} = \Theta(t)$$
vanishes on the open left semicircle, i.e. when $\pi/2 < |t| < \pi$*.*

Proof. The formula follows by an elementary integration. The numerator $1-\cos(\pi m/2)$ is nonnegative, so all Fourier coefficients are nonnegative. The equality $\kappa_0 = 1$ follows from the area of the triangle in (4). Since $\kappa_m = O(m^{-2})$, the Fourier series converges absolutely and hence to Θ. Finally, its value at $t = 0$ gives $\sum_m \kappa_m = \Theta(0) = 4$. □

The importance of this kernel is that positivity is compatible with an exact gap in the Fourier transform. Positivity will allow the coefficients to be realized by halting programs; the gap will supply analytic continuation.

4 A universal machine with a continuation arc

Fix a universal self-delimiting machine V. Put
$$d_j = d_V(j), \qquad \nu_j = d_j 2^{-j},$$
and set $\nu_j = 0$ for $j < 0$. Hence
$$\sum_{j\ge 0} \nu_j = \Omega_V \le 1.$$
For every $n \in \mathbb{Z}$, define
$$a_n = 2^{-2} \sum_{j\ge 0} \nu_j \kappa_{n-2-j}. \tag{5}$$

The constant 2 in the exponent and the subscript is the length of a prefix we will be adding to the programs of V. All terms are nonnegative. Moreover,

$$\sum_{n\in\mathbb{Z}} a_n = 2^{-2}\Omega_V \sum_{m\in\mathbb{Z}} \kappa_m = \Omega_V. \tag{6}$$

Thus $(a_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$.

Its bilateral Fourier series factors exactly:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} a_n e^{int} &= 2^{-2}e^{i2t}\left(\sum_{j\geq 0}\nu_j e^{ijt}\right)\left(\sum_{m\in\mathbb{Z}}\kappa_m e^{imt}\right)\\ &= 2^{-2}e^{i2t}\left(\sum_{j\geq 0}\nu_j e^{ijt}\right)\Theta(t).\end{aligned} \tag{7}$$

Consequently, the bilateral Fourier series vanishes on the open left semicircle.

Define

$$A_+(w) = \sum_{n\geq 0} a_n w^n, \qquad |w| < 1,$$

and

$$A_-(w) = -\sum_{n<0} a_n w^n, \qquad |w| > 1.$$

Both functions have continuous boundary values on $|w| = 1$ because $(a_n) \in \ell^1(\mathbb{Z})$. By (7), on the open left semicircle their boundary values agree:

$$A_+(e^{it}) = A_-(e^{it}).$$

Lemma 4. *The function A_+ admits a holomorphic continuation through every compact subarc of the open left semicircle. The continued function agrees with A_- on the exterior side.*

Proof. Glue A_+ and A_- along such a compact subarc. The resulting function is continuous across the arc and holomorphic on each side. Morera's theorem shows that it is holomorphic across the arc. □

We next realize the positive-index coefficients by a self-delimiting machine without losing the continuation. The central coefficient $\kappa_0 = 1$ is the key. At $n = j + 2$, the jth source term in (5) is

$$2^{-2}\nu_j\kappa_0 = d_j 2^{-(j+2)}.$$

This is exactly the normalized weight of d_j programs obtained by prefixing the V-programs with 00.

Define

$$c_n = \begin{cases} d_{n-2}2^{-n}, & n \geq 2,\\ 0, & n < 2,\end{cases}$$

and put

$$r_n = a_n - c_n, \qquad n \geq 0.$$

The numbers r_n are nonnegative and uniformly lower semicomputable. Also, using (6),

$$\sum_{n\geq 0} r_n \leq \Omega_V - \frac{\Omega_V}{4} = \frac{3\Omega_V}{4} < \frac{3}{4}. \tag{8}$$

For every $n \geq 0$ define the integer

$$E_n = \#\{h \in \mathbf{N} \setminus \{0\} \mid h < 2^n r_n\}. \tag{9}$$

The strict inequality in (9) is deliberate: since r_n is only lower semicomputable, it ensures that E_n is uniformly lower semicomputable even when $2^n r_n$ is an integer. With a non-strict inequality, equality with an integer could not in general be certified from an increasing approximation to r_n. Put

$$\varepsilon_n = r_n - E_n 2^{-n}.$$

Then

$$0 \leq \varepsilon_n \leq 2^{-n}. \tag{10}$$

By (8),

$$\sum_{n \geq 0} E_n 2^{-n} < \frac{3}{4}.$$

In particular $E_0 = 0$. We also have $E_1 = 0$: since $c_1 = 0$,

$$r_1 = a_1 = \frac{1}{4} \sum_{j \geq 0} \nu_j \kappa_{j+1} < \frac{1}{4},$$

because $\kappa_m < 1$ for every $m \neq 0$. Hence $2r_1 < 1$.

The complement of the cylinder $00\Sigma^*$ is the computable prefix-free forest with the three level-two roots 01, 10, and 11, of total Kraft measure 3/4. Apply the usual Kraft–Chaitin construction with these three strings as the initial free roots. Since all requests have length at least two and their total Kraft weight is less than 3/4, the construction gives a c.e. prefix-free set $X \subseteq \Sigma^* \setminus 00\Sigma^*$ containing exactly E_n strings of length n for every $n \geq 2$.

Define U_0 by

$$U_0(00p) = V(p), \qquad p \in \mathrm{dom}(V), \tag{11}$$

and

$$U_0(x) = \lambda, \qquad x \in X,$$

where λ is the empty string. The two branches are prefix-disjoint, hence U_0 is self-delimiting. Equation (11) shows that U_0 is universal.

Let $D_n = d_{U_0}(n)$. By construction,

$$D_n 2^{-n} = a_n - \varepsilon_n.$$

Therefore

$$A_{U_0}(w) = A_+(w) - E(w), \qquad E(w) = \sum_{n \geq 0} \varepsilon_n w^n. \tag{12}$$

By (10), E is holomorphic in $|w| < 2$.

Theorem 5. *The machine U_0 is universal and A_{U_0} admits analytic continuation through the open left semicircle. After crossing this arc, the continuation is given in $1 < |w| < 2$ by*

$$A^{\mathrm{ext}}_{U_0}(w) = A_-(w) - E(w). \tag{13}$$

For real $1 < w < 2$, $A^{\mathrm{ext}}_{U_0}(w)$ is real.

Proof. Universality was proved above. Analytic continuation follows from Lemma 4 and the fact that E is holomorphic in $|w| < 2$. Formula (13) follows by uniqueness of analytic continuation. Finally, both $A_-(w)$ and $E(w)$ are real for real w. □

In the s-plane, the continuation can cross the line $\Re(s) = 1$ at the point corresponding to $w = -1$, namely $s = 1 + \pi i/\log 2$, and then move in the exterior $|w| > 1$ to a real point $1 < w < 2$. This reaches the interval $0 < s < 1$ without passing through the singular point $s = 1$.

5 A random square-root branch

The preceding construction gives analytic continuation, but the dyadic rounding error need not be algorithmically simple. We now add a separate branch in which the random information appears in an imaginary component and therefore cannot be cancelled by real rounding terms.

Consider

$$F(w) = 1 - \sqrt{1 - w}, \tag{14}$$

where the germ at $w = 0$ is chosen so that $\sqrt{1} = 1$. We have

$$F(w) = \sum_{n \geq 1} \rho_n w^n, \qquad \rho_n = \frac{1}{2n-1} \binom{2n}{n} 4^{-n}. \tag{15}$$

The coefficients are positive and computable, and

$$\sum_{n \geq 1} \rho_n = F(1) = 1. \tag{16}$$

Fix an integer $b \geq 2$ and put

$$\alpha = 2^{-b} \Omega_{U_0}. \tag{17}$$

Since U_0 is universal, Ω_{U_0} is c.e. and random. Hence α is a positive c.e. random real.

For $n \geq 1$ define

$$M_n = \#\{k \in \mathbf{N} \setminus \{0\} \mid k < \alpha 2^n \rho_n\}. \tag{18}$$

The sequence (M_n) is uniformly lower semicomputable and

$$M_n 2^{-n} \leq \alpha \rho_n.$$

Consequently,

$$\sum_{n \geq 1} M_n 2^{-n} \leq \alpha < 1. \tag{19}$$

By the Kraft–Chaitin theorem there is a c.e. prefix-free set Y containing exactly M_n strings of length n.

Put

$$\eta_n = \alpha \rho_n - M_n 2^{-n}.$$

Then

$$0 \leq \eta_n \leq 2^{-n}. \tag{20}$$

If

$$B(w) = \sum_{n \geq 1} M_n 2^{-n} w^n, \qquad H(w) = \sum_{n \geq 1} \eta_n w^n,$$

then, initially for $|w| < 1$,

$$B(w) = \alpha F(w) - H(w). \tag{21}$$

By (20), H is holomorphic in $|w| < 2$ and is real-valued on the real interval $(-2, 2)$.

We can now define the final machine U. Let

$$U(0q) = U_0(q), \qquad q \in \mathrm{dom}(U_0), \tag{22}$$

and

$$U(1y) = \lambda, \qquad y \in Y.$$

Again the branches are prefix-disjoint. Since $U(0q) = U_0(q)$, U is universal; combining this with (11) gives

$$U(000p) = V(p), \qquad p \in \mathrm{dom}(V).$$

The normalized generating function of U is

$$A_U(w) = \frac{w}{2}\left(A_{U_0}(w) + B(w)\right) = \frac{w}{2}\left(A_{U_0}(w) + \alpha F(w) - H(w)\right). \tag{23}$$

6 Analytic continuation around the critical point

Formula (23) separates the three analytic ingredients. The function A_{U_0} crosses the unit circle through the left semicircle by Theorem 5; H is holomorphic throughout $|w| < 2$; and F is an algebraic function with its only finite branch point at $w = 1$.

Start with a real point $0 < w < 1$. Move inside the disk to the left semicircle, cross the unit circle there using the continuation of A_{U_0}, and then move through $1 < |w| < 2$ to a real point $1 < w < 2$, avoiding $w = 1$. The germ F can be continued along the same path. There are two choices, according as the path goes around $w = 1$ on one side or the other. At a real point $w > 1$ the two resulting germs satisfy

$$\sqrt{1 - w} = \pm i\sqrt{w - 1}, \tag{24}$$

and consequently

$$F^{\mathrm{cont}}(w) = 1 \mp i\sqrt{w - 1}. \tag{25}$$

By contrast, Theorem 5 gives

$$A_{U_0}^{\mathrm{cont}}(w) \in \mathbf{R}$$

for real $1 < w < 2$, and $H(w) \in \mathbf{R}$ on the same interval. Therefore the entire imaginary part of the continued A_U comes from the square-root branch.

Theorem 6. *For real $1 < w < 2$, either continuation described above satisfies*

$$\left|\Im(A_U^{\mathrm{cont}}(w))\right| = \frac{w}{2}\alpha\sqrt{w - 1}. \tag{26}$$

Proof. Substitute (25) in (23). The other terms are real, so they do not contribute to the imaginary part. □

Return now to the variable s by putting

$$w = 2^{1-s}.$$

For a real $s \in (0,1)$ we have $1 < w < 2$. Equation (26) becomes

$$\left|\Im(Z_U^{\mathrm{cont}}(s))\right| = 2^{-s}\alpha\sqrt{2^{1-s}-1} = c_s\Omega_{U_0}, \tag{27}$$

where

$$c_s = 2^{-b-s}\sqrt{2^{1-s}-1}. \tag{28}$$

For computable $s \in (0,1)$, c_s is positive and computable.

This proves the analytic-continuation part of Theorem 1. Notice that the continuation is genuinely different from the divergent defining series in $0 < s < 1$. It is obtained by passing around the obstruction at $s = 1$. The two choices of path give conjugate values.

7 Randomness of the imaginary part

We finish the proof of Theorem 1 with the algorithmic information content of (27).

Lemma 7. *Let $\beta \in (0,1)$ be a c.e. random real and let $c > 0$ be computable, with $c\beta \in (0,1)$. Then $c\beta$ is c.e. and random.*

Proof. Since c is a positive computable real, it has an increasing computable sequence of positive rational approximations. Multiplying these by an increasing computable rational approximation to β shows that $c\beta$ is c.e.

For randomness, multiplication by a fixed positive computable constant, as well as multiplication by its reciprocal, changes prefix complexity by at most $O(1)$. Thus from the first $n+O(1)$ bits of $c\beta$ one can effectively recover the first n bits of β, and conversely. If $c\beta$ failed the Chaitin complexity criterion for randomness, then so would β. Hence $c\beta$ is random. □

Theorem 8. *For every computable real $s \in (0,1)$,*

$$\left|\Im(Z_U^{\mathrm{cont}}(s))\right|$$

is a c.e. random real. In particular, there is a constant c (depending on s and U) such that for every $n \geq 1$,

$$H\left(\left|\Im(Z_U^{\mathrm{cont}}(s))\right|[n]\right) \geq n - c. \tag{29}$$

Proof. By (27),

$$\left|\Im(Z_U^{\mathrm{cont}}(s))\right| = c_s\Omega_{U_0}.$$

The machine U_0 is universal, so Ω_{U_0} is c.e. and random. For computable $s \in (0,1)$, the number c_s in (28) is positive and computable. The result follows from Lemma 7. □

Theorem 8 gives a simple information-theoretic interpretation of the continuation. Tadaki's theorem gives compression rate $1/s$ for computable $s > 1$. At $s = 1$ the rate reaches one because $Z_U(1) = \Omega_U$. The formal continuation of the expression $1/s$ to $s < 1$ would exceed one, which is impossible for binary prefix complexity. In our construction the continued function instead contains a coordinate of full randomness: the information density saturates at the maximal value one.

There is a minor but useful distinction. On the original half-plane, for real $s > 1$, $Z_U(s)$ is real and positive because every summand in (1) is positive. The imaginary part in Theorem 8 is created by monodromy: after continuation around the branch point at $s = 1$, the real square-root term becomes imaginary. The two choices of continuation produce conjugate values.

8 Conclusion and open questions

We have shown that universality does not force the line $\Re(s) = 1$ to be a natural boundary for Tadaki's generalized halting probability. The machine U contains a fixed-prefix copy of the arbitrary universal machine V, so no weakening of universality or increase in asymptotic simulation overhead is used. Nevertheless, its partition function can be continued around $s = 1$ to the divergent side $0 < s < 1$.

The proof uses two independent ideas. The first is Fourier-analytic. A nonnegative bilateral sequence can have a Fourier transform which vanishes on an open arc. Its positive part can therefore be the coefficient sequence of a generating function which extends across that arc, while the negative part supplies the exterior analytic germ. Positivity makes the construction compatible with c.e. halting-program counts. The central coefficient of the kernel is exactly one, so a universal machine survives as a fixed-prefix submachine.

The second idea separates analytic error from algorithmic information. Direct dyadic rounding of the Fourier construction produces real errors which are analytically harmless but need not be algorithmically simple. Rather than trying to prove that these errors cannot cancel an Omega-like term, we place a scaled Omega number in a separate algebraic branch. After continuation around $w = 1$, this branch becomes imaginary on the real exterior interval, while all rounding errors remain real. The random information is therefore protected from cancellation.

Several questions remain open. First, for which universal machines is $\Re(s) = 1$ actually a natural boundary? The construction shows that universality alone cannot decide the question. Secondly, can one characterize the possible analytic continuation domains of Z_U as U ranges over universal machines? Thirdly, it would be interesting to obtain analogous results for the exact-index zeta function and for the prime product machines introduced by Calude and Stay [4]. In those cases the precise locations of programs, rather than only their lengths, enter the Dirichlet series and the convolution argument used here is no longer exact. Finally, zeros and singularities of such continued machine zeta functions may provide a useful way of measuring oscillations in program counts, in analogy with the role played by singularities and zeros of classical zeta functions.

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Competing interests

The author declares none.

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