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Maximal Towers and Ultrafilter Bases in Computability Theory

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MAXIMAL TOWERS AND ULTRAFILTER BASES IN COMPUTABILITY THEORY

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Abstract. The tower number $\mathfrak{t}$ and the ultrafilter number $\mathfrak{u}$ are cardinal characteristics from set theory based on combinatorial properties of classes of subsets of ω and the almost inclusion relation $\subseteq^*$ between such subsets. We consider analogs of these cardinal characteristics in computability theory.

We say that a sequence $\langle G_n \rangle_{n\in\omega}$ of computable sets is a *tower* if $G_0 = \omega$, $G_{n+1} \subseteq^* G_n$, and $G_n \smallsetminus G_{n+1}$ is infinite for each n. A tower is *maximal* if there is no infinite computable set contained in all G_n. A tower $\langle G_n \rangle_{n\in\omega}$ is an *ultrafilter base* if for each computable R, there is n such that $G_n \subseteq^* R$ or $G_n \subseteq^* \overline{R}$; this property implies maximality of the tower. A sequence $\langle G_n \rangle_{n\in\omega}$ of sets can be encoded as the "columns" of a set $G \subseteq \omega$. Our analogs of $\mathfrak{t}$ and $\mathfrak{u}$ are the mass problems of sets encoding maximal towers, and of sets encoding towers that are ultrafilter bases, respectively. The relative position of a cardinal characteristic broadly corresponds to the relative computational complexity of the mass problem. We mainly use Medvedev reducibility to formalize relative computational complexity, and thus to compare such mass problems to known ones.

We show that the mass problem corresponding to ultrafilter bases is equivalent to the mass problem of computing a function that dominates all computable functions, and hence, by Martin's characterization, it captures highness. On the other hand, the mass problem for maximal towers is below the mass problem of computing a non-low set. We also show that no 1-generic Δ^0_2-oracle computes a maximal tower. In fact, no *index predictable oracle* does so. Here, index predictability of an oracle captures the ability to guess in a limit-wise way a characteristic index for a computable set given by a Turing reduction to the oracle.

We finally consider the mass problems of maximal almost disjoint, and of maximal independent families. We show that they are Medvedev equivalent to maximal towers, and to ultrafilter bases, respectively.

Contents

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1. Introduction

Cardinal characteristics measure how far the set-theoretic universe deviates from satisfying the continuum hypothesis. They are natural cardinals greater than $\aleph_0$ and at most $2^{\aleph_0}$. For instance, the bounding number $\mathfrak{b}$ is the least size of a collection of functions $f: \omega \to \omega$ such that no single function dominates the entire collection. Here, for functions $f, g: \omega \to \omega$, we say that g *dominates* f if $g(n) \geq f(n)$ for sufficiently large n. An important program in set theory is to prove inequalities between characteristics in ZFC, and to separate them in suitable forcing extensions.

Analogs of cardinal characteristics in computability theory were first studied by Rupprecht [14, 15] and further investigated by Brendle, Brooke-Taylor, Ng, and Nies [2]. The recent article by Greenberg, Kuyper, and Turetsky [7] provides a systematic approach to the two connected settings of set theory and computability. The relevant characteristics are given by relations, such as the domination relation $\leq^*$ between functions, and ordered by reducibilities that measure relative computability. For each relation R between two types of objects, one introduces the bounding number $\mathfrak{b}(R)$ and the dominating number $\mathfrak{d}(R)$, and their analogs in computability, which are highness and being of hyperimmune degree .

In this paper, we consider cardinal characteristics not given by relations. They are defined in the setting of subsets of ω up to almost inclusion $\subseteq^*$. We initiate the study of analogs of such cardinal characteristics in computability. We study analogs of the ultrafilter, tower, and independence number discussed shortly. A general reference in set theory is the survey paper by Blass [1]. The recent brief survey by Soukup [18] contains a diagram displaying the ZFC inequalities between the most important characteristics in this setting, along with $\mathfrak{b}(\leq^*)$ and $\mathfrak{d}(\leq^*)$.

The *ultrafilter number* $\mathfrak{u}$ is the least size of a subset of $[\omega]^\omega$ with upward closure a nonprincipal ultrafilter on ω. We note that one cannot in general require here that the subset is linearly ordered by $\subseteq^*$. Recall that an ultrafilter F on ω is a P-point if for each partition $\langle C_n \rangle$ of ω such that $C_n \notin F$ for each n, there is $A \in F$ such that $C_n \cap A$ is finite for each n. An ultrafilter with a linear base is a P-point. Shelah has shown that the non-existence of P-points is consistent with ZFC. So a version of $\mathfrak{u}$ relying on linear bases would be undefined.

The *tower number* $\mathfrak{t}$ is the minimum size of a subset of $[\omega]^\omega$ that is linearly ordered by $\subseteq^*$ and cannot be extended by adding a new element below all given elements. To define the *pseudointersection number* $\mathfrak{p}$, the requirement in the definition of towers that the sets in the class be linearly ordered under $\subseteq^*$ is weakened to requiring that every finite subset of the class has an infinite intersection). So trivially $\mathfrak{p} \leq \mathfrak{t}$. ZFC proves that $\mathfrak{p} = \mathfrak{t}$ as shown by Malliaris and Shelah [12] (also see [18]). It is not hard to see that ZFC proves $\mathfrak{t} \leq \mathfrak{u}$. It is consistent that $\mathfrak{t} < \mathfrak{u}$ (see [1] for both statements).

A class $\mathcal{C}$ of subsets of ω is *independent* if any intersection of finitely many sets in $\mathcal{C}$ or their complements is infinite. The *independence number* $\mathfrak{i}$ is the least cardinal of a maximal independent family (m.i.f.). There has been much work recently on $\mathfrak{i}$ in set theory, in particular, the descriptive complexity of m.i.f.'s, such as in Brendle, Fischer and Khomskii [3].

1.1. Comparing the complexity of the analogs in computability.

The main setting for our analogy is given by the Boolean algebra of computable sets modulo finite differences. We consider maximal towers, the closely related maximal almost disjoint sets, and thereafter ultrafilter bases and maximal independent sets. As already demonstrated in the above-mentioned papers [14, 15, 2, 7], the relative position of cardinal characteristics corresponds to the relative computational complexity of the associated class of objects.

Note that the usual formal definitions of computation relative to an oracle directly only apply to functions $f\colon \omega \to \omega$, and to subsets of ω (simply called sets from now on, and identified with their characteristic functions). The complexity of other objects is studied indirectly, via names that are functions on ω that give discrete representations of the object in question. A particular choice of names has to be made. For instance, real numbers can be named by rapidly converging Cauchy sequences of rational numbers.

The witnesses for cardinal characteristics are always uncountable. In contrast, in our setting, the analogous objects are countable. They will be considered as sequences of sets rather than unordered collections. A single set X can be used as a name for a sequence of sets: Let $X^{[n]}$ denote the "column" $\{u\colon \langle u, n\rangle \in X\}$.[1] To every set X, we can associate a sequence $\langle X_n\rangle_{n\in\omega}$ in a canonical way by setting $X_n = X^{[n]}$. (When introducing terminology, we will sometimes ignore the difference between $\langle X_n\rangle_{n\in\omega}$ and X.)

With this naming system, one can now use sequences as oracles in computations. We view the combinatorial classes of sequences as mass problems. To measure their relative complexity, we compare them via Medvedev reducibility $\leq_s$: Let $\mathcal{C}$ and $\mathcal{D}$ be sets of functions on ω, also known as *mass problems*. We say that $\mathcal{C}$ is Medvedev reducible to $\mathcal{D}$ and write $\mathcal{C} \leq_s \mathcal{D}$ if there is a Turing functional Θ such that $\Theta^g \in \mathcal{C}$ for each $g \in \mathcal{D}$. One also says that a function $g \in \mathcal{D}$ uniformly computes a function in $\mathcal{C}$. We will also refer to the weaker Muchnik reducibility: $\mathcal{C} \leq_w \mathcal{D}$ if each function $g \in \mathcal{D}$ computes a function in $\mathcal{C}$.

With subsequent research in mind, we will set up our framework to apply to general countable Boolean algebras rather than merely the Boolean algebra of the computable sets. Throughout, we fix a countable Boolean algebra $\mathbb{B}$ of subsets of ω closed under finite differences. Our basic objects will be sequences of sets in $\mathbb{B}$. We will obtain meaningful results already when we fix a countable Turing ideal $\mathcal{I}$ and let $\mathbb{B}$ be the sets with degree in $\mathcal{I}$. We mainly study the case that $\mathbb{B}$ consists of the computable sets, but will also consider the case of K-trivial sets, and even the primitive recursive sets at the end of the paper.

1.2. The mass problem $\mathcal{T}$ of maximal towers.

Definition 1.1. We say that a sequence $\langle G_n\rangle_{n\in\omega}$ of sets in $\mathbb{B}$ is a $\mathbb{B}$-*tower* if $G_0 = \omega$, $G_{n+1} \subseteq^* G_n$, and $G_n - G_{n+1}$ is infinite for each n. If $\mathbb{B}$ consists of the computable sets, we use the term *tower of computable sets*.

[1] For definiteness, we employ the usual computable Cantor pairing function $\langle x, n\rangle$. Note that $\langle x, n\rangle \geq x, n$. This property is useful in simplifying notation in some of the constructions below.

Definition 1.2. We say that a function p is *associated* with a tower G if p is increasing and $p(n) \in \bigcap_{i \le n} G_i$ for each n.

The following fact is elementary.

Fact 1.3. *A tower G uniformly computes a function p associated with it.*

Proof. Let Φ be the Turing functional such that $\Phi^G(0) = \min(G_0)$, and $\Phi^G(n+1)$ is the least number in $\bigcap_{i \le n+1} G_i$ greater than $\Phi^G(n)$. This Φ establishes the required Medvedev reduction. $\square$

Definition 1.4. Given a countable Boolean algebra of sets $\mathbb{B}$, the mass problem $\mathcal{T}_\mathbb{B}$ is the class of sets G such that $\langle G_n \rangle_{n \in \omega}$ is a $\mathbb{B}$-tower that is *maximal*, i.e., such that for each infinite set $R \in \mathbb{B}$, there is n such that $R - G_n$ is infinite.

Clearly, this implies that no associated function is computable. In particular, a maximal tower is never computable. (Note that towers by definition start with $G_0 = \omega$, and so our notion of maximality only requires that the tower cannot be extended from below, in keeping with our set-theoretic analogy.)

1.3. **The mass problem $\mathcal{U}_\mathbb{B}$ of ultrafilter bases.** We define the mass problem $\mathcal{U}_\mathbb{B}$ corresponding to the ultrafilter number. Since all filters of our Boolean algebras are countable, any base will compute a linearly ordered base by taking finite intersection. So we can restrict ourselves to linearly ordered bases. In the set theory setting this is not possible as discussed in the introduction.

Definition 1.5. Given a countable Boolean algebra of sets $\mathbb{B}$, let $\mathcal{U}_\mathbb{B}$ be the class of sets F such that F is a $\mathbb{B}$-tower as in Definition 1.1 and for each set $R \in \mathbb{B}$, there is n such that $F_n \subseteq^* \overline{R}$ or $F_n \subseteq^* R$. We will call a set F in $\mathcal{U}_\mathbb{B}$ a $\mathbb{B}$-*ultrafilter base*.

Each ultrafilter base is a maximal tower. In the cardinal setting, one has $\mathfrak{t} \le \mathfrak{u}$. Correspondingly, since $\mathcal{U}_\mathbb{B} \subseteq \mathcal{T}_\mathbb{B}$, we trivially have $\mathcal{T}_\mathbb{B} \le_s \mathcal{U}_\mathbb{B}$ via the identity reduction. The following indicates that for many natural Boolean algebras, the former notion is much stronger.

Proposition 1.6. *Suppose that the degrees of sets in $\mathbb{B}$ form a Turing ideal $\mathcal{K}$. Then for each $\mathbb{B}$-ultrafilter base F and associated function p in the sense of Definition 1.2, the function p is not dominated by a function with Turing degree in $\mathcal{K}$.*

Proof. Assume that there is a function $f \ge p$ in $\mathcal{K}$. The conditions $n_0 = 0$ and $n_{k+1} = f(n_k) + 1$ define a sequence that is computable from some oracle in $\mathcal{K}$, and for every k we have that $[n_k, n_{k+1})$ contains an element of $\bigcap_{i \le k} F_i$. So the set

$$E = \bigcup_i \, [n_{2i}, n_{2i+1})$$

is in $\mathcal{K}$. Clearly $F_n \not\subseteq^* E$ and $F_n \not\subseteq^* \overline{E}$ for each n. So F is not a $\mathbb{B}$-ultrafilter base. $\square$

Again in the cardinal setting, $\mathfrak{t} < \mathfrak{u}$ is consistent with ZFC. We will prove that $\mathcal{U}_\mathbb{B} \not\le_w \mathcal{T}_\mathbb{B}$ in the case that $\mathbb{B}$ consists of the computable sets: We will show in Theorem 3.1 below that each non-low set computes a set in $\mathcal{T}_\mathbb{B}$. Since non-low oracles can be computably dominated, there must be a member of $\mathcal{T}_\mathbb{B}$ that does not compute any member of $\mathcal{U}_\mathbb{B}$.

2. Basics of the mass problems $\mathcal{T}_{\mathbb{B}}$

2.1. **The equivalent mass problems $\mathcal{T}_{\mathbb{B}}$ and $\mathcal{A}_{\mathbb{B}}$.** Recall that in set theory, the almost disjointness number, denoted $\mathfrak{a}$, is the least possible size of a maximal almost disjoint (MAD) family of subsets of ω. In our analogous setting, we call a sequence $\langle F_n \rangle_{n\in\omega}$ of sets in $\mathbb{B}$ *almost disjoint* (AD) if each F_n is infinite and $F_n \cap F_k$ is finite for distinct n and k.

Definition 2.1. In the context of a Boolean algebra of sets $\mathbb{B}$, the mass problem $\mathcal{A}_{\mathbb{B}}$ is the class of sets F such that $\langle F_n \rangle_{n\in\omega}$ is a *maximal almost disjoint* (MAD) family in the computable sets. Namely, the sequence is AD, and for each infinite set $R \in \mathbb{B}$, there is n such that $R \cap F_n$ is infinite.

Fact 2.2. $\mathcal{A}_{\mathbb{B}} \leq_s \mathcal{T}_{\mathbb{B}} \leq_s \mathcal{A}_{\mathbb{B}}$.

Proof. We suppress the subscript $\mathbb{B}$. To check that $\mathcal{A} \leq_s \mathcal{T}$, given a set G, let $\mathrm{Diff}(G)$ be the set D such that $D_n = G_n - G_{n+1}$. Clearly, the operator Diff can be seen as a Turing functional. If G is a maximal tower then $D = \mathrm{Diff}(G)$ is MAD. For, if R is infinite computable, then $R - G_n$ is infinite for some n, and hence $R \cap D_i$ is infinite for some $i < n$.

For $\mathcal{T} \leq_s \mathcal{A}$, given a set F, let $G = \mathrm{Cp}(F)$ be the set such that

$$x \in G_n \leftrightarrow \forall i < n\, [x \notin F_n].$$

Again, Cp is a Turing functional. If F is AD, then G is a tower, and if F is MAD, then G is a maximal tower. □

Recall that a maximal tower is not computable. Hence no MAD set is computable. (This corresponds to the cardinal characteristics being uncountable.)

2.2. **Descriptive complexity and index complexity for maximal towers.** For the rest of this section, as well as the subsequent two sections, we will mainly be interested in the case that $\mathbb{B}$ is the Boolean algebra of all computable sets. We will omit the parameter $\mathbb{B}$ when we name the mass problems. In the final section, we will consider other Boolean algebras.

Besides looking at the relative complexity of mass problems such as $\mathcal{T}$ and $\mathcal{U}$, one can also look at the individual complexity of their members (as sets encoding sequences). Recall that a characteristic index for a set M is a number e such that $\chi_M = \varphi_e$. The following two questions arise:

(1) How low in the arithmetical hierarchy can the set be located?
(2) How hard is it to find characteristic indices for the sequence members?

(1) Arithmetical complexity.

Fact 2.3. *No maximal tower G is c.e. No MAD set is co-c.e.*

Proof. For the first statement, note that otherwise there is a computable function p associated with G. The range of p would extend the tower G.

For the second statement, note that the reduction Cp, introduced above to show that $\mathcal{T} \leq_s \mathcal{A}$, turns a co-c.e. set F into a c.e. set G. □

We will return to Question (1) in Section 5, where we show that c.e. MAD sets exist in every nonzero c.e. degree, and that an ultrafilter base can be co-c.e.

(2) Complexity of finding characteristic indices for the sequence members.
In several constructions of towers $\langle G_n \rangle_{n\in\omega}$ below, such as in Corollary 5.4 and

Theorem 5.5, the oracle $\emptyset''$ is able to compute, given n, a characteristic index for G_n. The oracle $\emptyset'$ does not suffice by the following

Proposition 2.4. *Suppose that G is a maximal tower. The oracle $\emptyset'$ is not able to compute, from input n, a characteristic index for G_n.*

Proof. Assume the contrary. Then there is a computable function f such that $\varphi_{\lim_s f(n,s)}$ is the characteristic function of G_n. Let $\widehat{G}$ be defined as follows. Given n and x, compute the least $s > x$ such that $\varphi_{f(n,s),s}(x) \downarrow$. If the output is not 0, put x into $\widehat{G}_n$. Clearly $\widehat{G}$ is computable and $G_n =^* \widehat{G}_n$ for each n. So $\widehat{G}$ is a maximal tower, contrary to Fact 2.3. □

3. Complexity of $\mathcal{T}$ and of $\mathcal{U}$

In this section, we compare our two principal mass problems, maximal towers and ultrafilter bases, to well-known benchmark mass problems: non-lowness, and highness. We also define index predictability. No index predictable oracle computes a maximal tower. We show that each 1-generic Δ^0_2-set is index predictable.

As we said above, we restrict ourselves to the case that $\mathbb{B}$ is the Boolean algebra of computable sets.

3.1. **Maximal towers, non-lowness, and index predictability.** We now show that each non-low oracle computes a set in $\mathcal{T}$. The result is uniform in the sense of mass problems. Let NonLow denote the class of oracles X such that $X' \not\leq_T \emptyset'$.

Theorem 3.1. $\mathcal{T} \leq_s \mathrm{NonLow}$.

Proof. In the following, x, y, and z denote binary strings; we identify such a string x with the number whose binary expansion is $1x$. For example, the string 000 is identified with 8, the number with binary representation 1000. Define a Turing functional Θ for the Medvedev reduction as follows: Set $\Theta^Z = G$, where for each n,

$$G_n = \{x\colon\ n \leq s := |x| \wedge Z'_s \restriction n = x \restriction n\}.$$

Here Z' denotes the jump of Z, which is computably enumerated relative to Z in some standard way. Note that, for each n, for sufficiently large s, the string $Z'_s \restriction n$ settles. So it is clear that for each n, we have $G_{n+1} \subseteq^* G_n$ and $G_n - G_{n+1}$ is infinite. Also G_n is computable.

Suppose now that R is an infinite set such that $R \subseteq^* G_n$ for each n. Then for each k,

$$Z'(k) = \lim_{x \in G_k, |x|>k} x(k) = \lim_{x \in R, |x|>k} x(k),$$

and hence $Z' \leq_T R'$. So if $Z \in \mathrm{NonLow}$ then R cannot be computable, and hence $\Theta^Z \in \mathcal{T}$. □

Remark 3.2. The proof above yields a more general result. Suppose $\mathcal{K}$ is a countable Turing ideal and $\mathbb{B}$ is the Boolean algebra of sets with degree in $\mathcal{K}$. Then $\mathcal{T}_{\mathbb{B}} \leq_s \mathrm{NonLow}_{\mathcal{K}}$, where $\mathrm{NonLow}_{\mathcal{K}} := \{Z \colon \forall R \in \mathcal{K}\, [Z' \not\leq_T R']\}$.

We next introduce a property of oracles we call index predictability, which implies that the oracle does not compute a maximal tower. As usual, let $\langle \Phi_e \rangle_{e \in \omega}$ be an effective list of the Turing functionals with one input, and write φ_e for $\Phi^{\emptyset}_e$. Note that for a Δ^0_2-oracle L, $\emptyset''$ can compute from e a characteristic index for Φ^L_e in case that the function Φ^L_e is computable. To be index predictable means that $\emptyset'$ suffices.

Definition 3.3. We call an oracle L *index predictable* if $\emptyset'$ can compute from e an index for Φ_e^L whenever Φ_e^L is a computable function. In other words, there is a functional Γ such that

$$\Phi_e^L \text{ is computable} \Rightarrow \Phi_e^L = \varphi_{\Gamma(\emptyset';e)}.$$

No assumption is made on the convergence of $\Gamma(\emptyset'; e)$ in case Φ_e^L is not a computable function.

Clearly, being index predictable is closed downward under $\leq_T$. A total function is computable iff its graph is computable, in a uniform way. So for index predictability of L, it suffices that there is a Turing functional Γ such that $\Gamma(\emptyset'; e)$ provides an index for Φ_e^L in case it is a computable $\{0,1\}$-valued function.

Every index predictable oracle D is low. To see this, for $i \in \omega$, let $B_i = \{t\colon J_t^D(i)\downarrow\}$. If $i \in D'$ then B_i is cofinite, otherwise $B_i = \emptyset$. There is a computable function g such that $\Phi_{g(i)}^D$ is the characteristic function of B_i. To show that $D' \leq_T \emptyset'$, on input i, let $\emptyset'$ compute a characteristic index $r(i)$ for B_i. Now use $\emptyset'$ again to determine $\lim_k \varphi_{r(i)}(k)$, which equals $D'(i)$.

By Proposition 2.4, an index predictable oracle D does not compute a maximal tower. The following provides examples of such oracles.

Theorem 3.4. *If L is Δ_2^0 and 1-generic, then L is index predictable.*

Proof. Suppose $F = \Phi_e^L$ and F is a computable set. Let S_e be the c.e. set of strings σ above which there is a Φ_e-splitting in the sense that

$$S_e = \{\sigma\colon \exists p(\exists \tau_1 \succ \sigma)(\exists \tau_2 \succ \sigma)\,[\Phi_e^{\tau_1}(p) \neq \Phi_e^{\tau_2}(p)]\}.$$

Suppose that S_e is dense along L. Then we claim that the set

$$C_e = \{\tau\colon \exists p\,[\Phi_e^{\tau}(p) \neq F(p)]\}$$

is also dense along L, i.e., for every k, there is some $\tau \succeq L \upharpoonright k$ such that $\tau \in C_e$. Indeed, let $\sigma \succeq L \upharpoonright k$ be a member of S_e and let p, τ_1 and τ_2 witness this. Let τ_i for $i = 1$ or 2 be such that $\Phi_e^{\tau_i}(p) \neq F(p)$. Then $\tau_i \succeq L \upharpoonright k$ is in C_e. The set C_e is c.e. and hence L meets C_e, contradicting our assumption that $F = \Phi_e^L$.

It follows that S_e is not dense along L. In other words, there is some least k_e such that there is no splitting of Φ_e above $L \upharpoonright k_e$. On input e, the oracle $\emptyset'$ can compute k_e and $L \upharpoonright k_e$. This allows $\emptyset'$ to find an index for F, given by the following procedure: To compute $F(p)$, find the least $\tau \succeq L \upharpoonright k_e$ such that $\Phi_e^\tau(p)\downarrow$ (in $|\tau|$ many steps). Such a τ exists because $\Phi_e^L(p)\downarrow$. By our choice of k_e, it follows that $\Phi_e^\tau(p) = \Phi_e^L(p) = F(m)$. $\square$

We summarize the known implications:

1-generic Δ_2^0-$\Rightarrow$ index predictable $\Rightarrow$ computes no maximal tower $\Rightarrow$ low.

The last arrow doesn't reverse by Theorem 5.2 below; the others might. In particular, we ask whether the converse of the implication "index predictable $\Rightarrow$ computes no maximal tower" holds. This would also strengthen Theorem 3.1. Note that the following potential weakening of index predictability of L still implies that the oracle computes no maximal tower: For each $S \leq_T L$ such that each S_n is computable, there is binary computable f such that $S_n = \varphi_{\lim_s f(n,s)}$ for each n.

Aside. We pause briefly to mention a potential connection of our topic to computational learning theory. One says that a class S of computable functions is EX-learnable if there is a total Turing machine M such that $\lim_s M(f \upharpoonright s)$ exists for each $f \in S$ and is an index for f. For an oracle A, one says that S is EX[A]-learnable if there is an oracle machine M that is total for each oracle and such that $\lim_s M^A(f \upharpoonright s)$ exists for each $f \in S$ and is an index for f. One calls an oracle A *EX-trivial* if EX $=$ EX[A]. Slaman and Solovay [16] showed that A is EX-trivial iff A is 1-generic and Δ^0_2. This used an earlier result of Haught that the Turing degrees of the 1-generic Δ^0_2-sets are closed downward.

3.2. **Ultrafilter bases and highness.** Let $\mathrm{Tot} = \{e \colon \varphi_e \text{ is total}\}$. Let DomFcn denote the mass problem of functions h that dominate every computable function and also satisfy $h(s) \geq s$ for all s. Note that a set F is high iff $\mathrm{Tot} \leq_T F'$. To represent highness by a mass problem in the Medvedev degrees, one can equivalently choose the set of functions dominating each computable function, or the set of approximations to Tot, i.e., the $\{0,1\}$-valued binary functions f such that $\lim_s f(e,s) = \mathrm{Tot}(e)$. This follows from the next fact; we omit the standard proof.

Fact 3.5. DomFcn *is Medvedev equivalent to the mass problem of approximations to* $\mathrm{Tot} = \{e \colon \varphi_e \text{ is total}\}$.

We show that exactly the high oracles compute ultrafilter bases, and that the reductions are uniform. By the Fact 3.5, it suffices to show that $\mathcal{U} \equiv_s$ DomFcn. We will obtain the two Medvedev reductions through separate theorems, with proofs that are unrelated.

Theorem 3.6. *Every ultrafilter base uniformly computes a dominating function. In other words,* $\mathcal{U} \geq_s$ DomFcn.

Our proof is directly inspired by a proof of Jockusch [8, Theorem 1, (iv) $\Longrightarrow$ (i)], who showed that any family of sets containing exactly the computable sets must have high degree.

Lemma 3.7. *There is a uniformly computable sequence $P_0, P_1, \ldots$ of nonempty Π^0_1-classes such that for every e,*

- *if φ_e is total, then P_e contains a single element, and*
- *if φ_e is not total, then P_e contains only bi-immune elements.*

Proof. Note that each Martin-Löf (or even Kurtz) random set is bi-immune: For an infinite computable set R, the class of sets containing R is a Π^0_1 null class and hence determines a Kurtz test. A similar fact holds for the class of sets disjoint from R.

For each s, let n_s be the largest number such that φ_e converges on $[0, n_s)$. We build the Π^0_1-class P_e in stages, where $P_{e,s}$ is the nonempty clopen set we have before stage s of the construction. Let $P_{e,0} = 2^\omega$.

Stage 0. Start constructing P_e as a nonempty Π^0_1-class containing only Martin-Löf random elements.

Stage s. If $n_s = n_{s-1}$, continue the construction that is currently underway, which will produce a nonempty Π^0_1-class of random elements.

On the other hand, if $n_s > n_{s-1}$, fix a string σ such that $[\sigma] \subseteq P_{e,s}$ and $|\sigma| > s$. Let $P_{e,s+1} = [\sigma]$. End the construction that we have been following and start a new construction for P_e, starting at stage $s+1$, as a nonempty Π^0_1-subclass of $[\sigma]$ containing only Martin-Löf random elements.

It is clear that if φ_e is total, then P_e will be a singleton. Otherwise, there will be a final construction of a nonempty Π^0_1-class of randoms which will run without further interruption. $\square$

Of course, when P_e is a singleton, its lone element must be computable.

Proof of Theorem 3.6. Let F be an ultrafilter base. Then, for any set C, let $P_C = \{X \in 2^\omega : C \subseteq X\}$. Note that if C is computable (or even merely c.e.), then P_C is a Π^0_1-class. Let $Q_e = \{X : \overline{X} \in P_e\}$ be the Π^0_1-class of complements of elements of P_e. Now we have that

$$\begin{aligned} \varphi_e \text{ is total} &\iff (\exists i)(\exists n)\ [F_i - [0,n] \text{ is a subset of some} \\ &\qquad\qquad X \in P_e \text{ or its complement}] \\ &\iff (\exists i)(\exists n)\ [P_e \cap P_{F_i - [0,n]} \neq \emptyset \text{ or } Q_e \cap P_{F_i-[0,n]} \neq \emptyset] \end{aligned}$$

Nonemptiness of a Π^0_1-class is a Π^0_1-property, hence $\mathrm{Tot} = \{e : \varphi_e \text{ is total}\}$ is $\Sigma^0_2[F]$. Note that the Σ^0_2-index does not depend on F. Since Tot is also Π^0_2, it is $\Delta^0_2[F]$ via a fixed pair of indices, and hence Turing below F' via a fixed reduction. One direction of the usual proof of the (relativized) Limit Lemma now shows that we can uniformly compute an approximation to Tot from F. Hence, from F we can uniformly compute a dominating function by Fact 3.5. □

Theorem 3.8. *Every dominating function uniformly computes an ultrafilter base. In other words, $\mathcal{U} \leq_s \mathrm{DomFcn}$.*

Proof. Let $\langle \psi_e \rangle_{e\in\omega}$ be an effective listing of the $\{0,1\}$-valued partial computable functions defined on an initial segment of ω. Let $V_{e,k} = \{x : \psi_e(x) = k\}$ so that $\langle (V_{e,0}, V_{e,1}) \rangle$ is an effective listing that contains all pairs of computable sets and their complements.

Let $T = \{0,1,2\}^{<\infty}$. Uniformly in $\alpha \in T$, we will define a set S_α. We first explain the basic idea and the modify it to make it work. The basic idea is that $S_{\alpha\frown k} = S_\alpha \cap V_{e.k}$ for $k = 0,1$, that is, we split S_α according to the listing above. We then consider the leftmost path h so that $S_{g\upharpoonright e}$ is infinite for each e. A dominating function h can eventually discover each initial segment of this path and uses this to compute a set F such that $F_e =^* S_{g\upharpoonright e}$ for each e.

The problem is that both $S_\alpha \cap V_{e,0}$ and $S_\alpha \cap V_{e,1}$ could be finite (because e is not a proper index of a computable set). In this case we still need to make sure that $F_n \setminus F_{e+1}$ is infinite. So the rightmost option at level n is a set $S_{\alpha\frown 2} = \widetilde{S}_\alpha$ which simply removes every other element from S_α. The sets $S_{\alpha\frown k}$ for $k \leq 1$ will be subsets of $\widetilde{S}_\alpha$.

We now provide the details. The set S_α is enumerated in increasing fashion and possibly finite. So each S_α is computable, but not uniformly in α. All the sets and functions defined below can be interpreted at stages.

Let $S_{\emptyset,s} = [0,s)$. If we have defined (at stage s) the set $S_\alpha = \{r_0 < \ldots < r_k\}$, let $\widetilde{S}_\alpha$ contain the numbers of the form r_{2i}. Let $S_{\alpha\frown 2} = \widetilde{S}_\alpha$ (this redundancy is convenient to conform with notation in the proof of the related Theorem 5.5 below). Let $S_{\alpha\frown k} = \widetilde{S}_\alpha \cap V_{e,k}$ for $k = 0,1$, $e = |\alpha|$. We define a uniform list of Turing functionals Γ_e so that the sequence $\left\langle \Gamma^h_e(t) \right\rangle_{t\in\omega}$ is nondecreasing for each e and each oracle function h such that $h(s) \geq s$ for each s. We will let $F_e = \{\Gamma^h_e(t) : t \in \omega\}$.

Definition of Γ_e. Given an oracle function h, we will write a_s for $\Gamma^h_e(s)$. Let $a_0 = 0$. Suppose $s > 0$ and a_{s-1} has been defined.

Check if there is $\alpha \in T$ of length e such that $|S_{\alpha,h(s)}| \geq s$. If there is no such α, let $a_s = a_{s-1}$. Otherwise, let α be leftmost such. If $\max S_{\alpha,h(s)} > a_{s-1}$, let $a_s = \max S_{\alpha,h(s)}$. Otherwise, again let $a_s = a_{s-1}$.

Note that the sequence $\{a_s\}_{s<\omega}$ is unbounded because for the rightmost string $\alpha \in T$ of length e, the set $S_{\alpha,t}$ consists of the numbers in $[0,t)$ divisible by 2^e. We may combine the functionals Γ_e to obtain a functional Ψ such that $(\Psi^h)_e = F_e$ for each h with $h(s) \geq s$ for each s.

Claim 3.9. *If $h \in \mathrm{DomFcn}$ then $F = \Psi^h \in \mathcal{U}$.*

To verify this, let $g \in 2^\omega$ denote the leftmost path in $\{0,1,2\}^\omega$ such that the set $S_{g\restriction e}$ is infinite for every e. Note that g is an infinite path, because for every α, if the set S_α is infinite then so is $S_{\alpha\frown 2}$.

Fix e and let $\alpha = g \restriction e$. Let $p(s)$ be the least stage t such that $S_{\alpha,t}$ has at least s elements. Since h dominates the computable function p, we will eventually always pick α in the definition of $a_s = \Gamma_e^h(s)$. Hence $F_e =^* S_\alpha$, and so, in particular, F_e is computable.

Clearly, if S_α is infinite then $S_\alpha \supset_\infty S_\beta$ for $\alpha \prec \beta$. So $F_{e+1} \subset_\infty F_e$.

Let R be a computable set. Pick e such that $R = V_{e,0}$ and $\overline{R} = V_{e,1}$. If $g(e) = 0$, then $S_{g\restriction e+1} \subseteq^* V_{e,0}$ and hence $F_{e+1} \subseteq^* R$. Otherwise, $S_{g\restriction e+1} \subseteq^* V_{e,1}$ and hence $F_{e+1} \subseteq^* \overline{R}$. □

4. Maximally independent families in computability

In this short section, we determine the complexity of the computability-theoretic analog of the independence number $\mathfrak{i}$ for the Boolean algebra of computable sets. It turns out that *maximally independent families* behave within the computable sets similarly to ultrafilter bases.

Given a sequence $\langle F_n \rangle_{n\in\omega}$, for each binary string σ we write

$$F_\sigma = \bigcap_{\sigma(i)=1} F_i \cap \bigcap_{\sigma(i)=0} \overline{F}_i. \tag{1}$$

We call (a set F encoding) such a sequence *independent* if each set F_σ is infinite.

Definition 4.1. Given a Boolean algebra of sets $\mathbb{B}$, the mass problem $\mathcal{I}_\mathbb{B}$ is the class of sets F such that $\langle F_n \rangle_{n\in\omega}$ is a family that is *maximally independent*, namely, it is independent, and for each set $R \in \mathbb{B}$, there is σ such that $F_\sigma \subseteq^* R$ or $F_\sigma \subseteq^* \overline{R}$. We abbreviate maximally independent family by m.i.f.

In the following, we let $\mathbb{B}$ be the Boolean algebra of computable sets, and we drop the parameter $\mathbb{B}$ as usual. An easy modification of the proof of Theorem 3.6 yields the following

Theorem 4.2. *Every m.i.f. F uniformly computes a dominating function. In other words, $\mathcal{I} \geq_s \mathrm{DomFcn}$.*

Proof. Define the Π^0_1-classes P_e as in Lemma 3.7. As before let $Q_e = \{X : \overline{X} \in P_e\}$ be the Π^0_1-class of complements of elements of P_e. Recall that for any set C, we let $P_C = \{X \in 2^\omega : C \subseteq X\}$. Now we have that

$$\begin{aligned}\varphi_e \text{ is total } &\iff (\exists\sigma)(\exists n)\,[F_\sigma - [0,n] \text{ is a subset of some} \\ &\qquad\qquad X \in P_e \text{ or its complement}] \\ &\iff (\exists\sigma)(\exists n)\,[P_e \cap P_{F_\sigma - [0,n]} \neq \emptyset \text{ or } Q_e \cap P_{F_\sigma-[0,n]} \neq \emptyset]\end{aligned}$$

As before, this shows that from F one can uniformly compute a dominating function. □

Theorem 4.3. *Every dominating function h uniformly computes a maximally independent family. In other words, $\mathcal{I} \leq_s$ DomFcn.*

In fact, we will prove that a dominating function h uniformly computes a set F such that the equivalence classes of the sets F_e freely generate the Boolean algebra of computable sets modulo finite sets.

Proof. As in the proof of Theorem 3.8, let $\langle \psi_e \rangle_{e\in\omega}$ be an effective listing of the $\{0,1\}$-valued partial computable functions defined on an initial segment of ω, and let $V_{e,k} = \{x \colon \psi_e(x) = k\}$ for $k = 0, 1$.

In Phase e of the construction, we will define a computable set F_e such that $F_e = \Theta_e^h$ for a Turing functional Θ_e determined uniformly in e. Suppose we have defined Θ_i for $i < e$, and thereby the sets F_σ defined in (1), where σ is a string of length e.

The idea for building F_e is to try to follow $V_{e,0}$ while maintaining independence from the previous sets. We apply this strategy separately on each F_σ. We define inductively a sufficiently fast growing increasing sequence $\langle r_n^e \rangle_{n\in\omega}$ computable from h and carry out the attempts on intervals $[r_n^e, r_{n+1}^e)$. If $V_{e,0}$ appears to split F_σ on the current interval, then we follow it; otherwise, we merely make sure that F_e remains independent from F_σ on the interval by putting one number in and leaving another one out. To decide which case holds, we consult the dominating function.

We now provide the details for Phase e. As e is fixed, we drop the superscripts e. By σ we will always denote a string of length e. Let $r_0 = 0$. If r_n has been defined, let $r_{n+1} > r_n$ be the least number r such that for each σ

(a) $|[r_n, r) \cap F_\sigma| \geq 2$, and
(b) if there are $u, w \in \mathrm{dom}(\psi_{e,h(r_n)}) \cap F_\sigma$ with $r_n \leq u < w$ such that $\psi_e(u) = 1$ and $\psi_e(w) = 0$, then $r > w$ for the least such w.

We define $F_e(x) = \Theta_e^h(x)$ for $x \in [r_n, r_{n+1})$ as follows. For each σ,

- if condition (b) holds and ψ_e is defined on $[r_n, r_{n+1})$, then let $F_e(x) = \psi_e(x)$;
- otherwise, if $x = \min([r_n, r_{n+1}) \cap F_\sigma)$, let $F_e(x) = 1$, else let $F_e(x) = 0$.

Verification. By induction on e, one verifies that F_σ is infinite for each σ with $|\sigma| = e$, and that the sequence $\langle r_n^e \rangle_{n\in\omega}$ defined in Phase e of the construction is infinite. Thus Θ_e^h is total for each function h. So $F \leq_T h$ where $F_e = \Theta_e^h$, and F is an independent family.

Claim 4.4. *Each set F_e is computable.*

We verify this by induction on e. Suppose it holds for each $i < e$. So F_σ is computable for $|\sigma| = e$.

First assume that ψ_e is partial. Then for sufficiently large n, condition (b) does not apply, and so the sequence $\langle r_n \rangle_{n\in\omega}$ and hence F_e are computable.

Now assume that ψ_e is total. Let

$$D_e = \{\sigma \colon |\sigma| = e \wedge |F_\sigma \cap V_{e,0}| = |F_\sigma \cap V_{e,1}| = \infty\}.$$

Define a function p by letting $p(m)$ be the least stage s such that for each $\sigma \notin D$, condition (a) holds with $r_n = m, r = s$, and for each $\sigma \in D_e$, there are $u, w \in \mathrm{dom}(\psi_{e,s})$ such that $m \leq u < w$ as in condition (b). (Let $p(m) = 0$ if m is not of the form r_n.) Since F_σ is computable for each σ of length e, the function p is computable. Since h dominates p, for sufficiently large n, we will define r_{n+1} by checking ψ_e at a stage $h(r_n) \geq p(r_n)$; since we chose the witnesses minimal, r_{n+1} is determined by stage $p(r_n)$. So we might as well check ψ_e at that stage and don't need h. Hence the sequence $\langle r_n \rangle_{n \in \omega}$ and therefore F_e are computable.

Claim 4.5. *Suppose that ψ_e is total. Then for each string $\tau = \sigma\widehat{\ }a$ of length $e+1$, $F_\tau \subseteq^* V_{e,0}$ or $F_\tau \cap V_{e,0} =^* \emptyset$.*

If $\sigma \notin D_e$, then this is immediate since $F_\sigma \subseteq^* V_{e,i}$ for some i. Otherwise, Phase e of the construction ensures that $F_{\sigma\widehat{\ }0} =^* F_\sigma \cap V_{e,0}$.

By the last claim, the $=^*$-equivalence classes of the F_e freely generate the Boolean algebra of the computable sets modulo finite sets. In particular, F is a maximally independent family. □

We don't know at present whether there is a "natural" Medvedev equivalence between the two mass problems $\mathcal{U}$ and $\mathcal{I}$. This would require direct conceptual proofs avoiding the detour via the mass problem of dominating functions.

5. The co-computably enumerable case

Recall from Fact 2.3 that no tower, and in particular no ultrafilter base, can be computably enumerable. In contrast, in this section we will see that even ultrafilter bases can have computably enumerable complement.

Recall that a coinfinite c.e. set A is called *simple* if it meets every infinite c.e. (or, equivalently, computable) set; A is called *r-maximal* if $\overline{A} \subseteq^* \overline{R}$ or $\overline{A} \subseteq^* R$ for each computable set R. Each r-maximal set is simple. For more background, see, e.g., Soare [17].

We note that to some extent, the co-c.e. maximal towers behave like the complements of simple sets, and the co-c.e. ultrafilter bases behave like the complements of r-maximal sets. Firstly, from a co-c.e. tower one can canonically obtain a set of the required kind, namely, an infinite co-c.e. (but usually noncomputable) set extending the tower, as we will show shortly in Proposition 5.1. Secondly, our constructions of co-c.e. towers with the respective properties resemble constructions of such sets. However, our constructions add "more detail", namely, structure inside the simple set.

Proposition 5.1. *(i) From a co-c.e. tower G, one can uniformly compute an infinite co-c.e. set B such that $B \subseteq^* G_n$ for each n. Moreover, the index for B is also obtained uniformly.*
(ii) If G is a maximal tower, then B is immune.
(iii) If G is an ultrafilter base, then B is r-cohesive, and hence $\overline{B}$ is r-maximal.

Proof. (i) We may assume that $G_n \supseteq G_{n+1}$ for each n. We have $G_n = \bigcap_s G_{n,s}$ where the $G_{n,s}$ are uniformly cofinite sets given by strong indices, descending in s, and $G_{n,s} \supseteq G_{n+1,s}$ for all n and s. Let $\gamma_{n,s}$ be the n-th element of $G_{n,s}$. Clearly, the double sequence $\langle \gamma_{n,s} \rangle$ is computable, monotonic in s, and strictly monotonic in n. Let $\gamma_n = \lim_s \gamma_{n,s}$. The set $B = \{\gamma_n \colon n \in \omega\}$ is as required.

(ii) and (iii) follow from the definitions. E.g., for (iii), clearly $B \subseteq^* R$ or $B \subseteq^* \overline{R}$ for each computable set R. □

5.1. **Computably enumerable MAD sets, and co-c.e. towers.** We will show that there is a co-c.e. maximal tower G Turing reducible to any given noncomputable c.e. set A. Given that it is more standard to build c.e. rather than co-c.e.

sets, it will be convenient to first build a MAD set $F \le_T A$. We employ a priority construction with requirements that act only finitely often.

Theorem 5.2. *For each noncomputable c.e. set A, there is a MAD c.e. set $F \le_T A$.*

Proof. As mentioned above, the construction is akin to Post's construction of a simple set. In particular, it is compatible with permitting.

Let $\langle M_e \rangle_{e\in\omega}$ be a uniformly c.e. sequence of sets such that $M_{2e} = W_e$ and $M_{2e+1} = \omega$ for each e. We will build an auxiliary c.e. set $H \le_T A$ and let the c.e. set $F \le_T A$ be defined by $F^{[e]} = H^{[2e]} \cup H^{[2e+1]}$. The role of the M_{2e+1} is to make the sets $H^{[2e+1]}$, and hence the $F^{[e]}$, infinite. The construction also ensures that H, and hence F, is AD, and that $\bigcup_n H^{[n]}$ is coinfinite.

As usual, we will write H_e for $H^{[e]}$. We provide a stage-by-stage construction to meet the requirements

$$P_n\colon M_e - \bigcup_{i<n} H_i \text{ infinite } \Rightarrow |H_e \cap M_e| \ge k, \text{ where } n = \langle e, k\rangle.$$

At stage s, we say that P_n is *satisfied* if $|H_{e,s} \cap M_{e,s}| \ge k$.

Construction.

Stage $s > 0$. For each $n < s$ such that P_n is not satisfied where $n = \langle e, k\rangle$, if there is $x \in M_{e,s} - \bigcup_{i<n} H_{i,s}$ such that

$$x > \max(H_{e,s-1}),\ x \ge 2n \text{ and } A_s \upharpoonright x \ne A_{s-1} \upharpoonright x,$$

then put $\langle x, e\rangle$ into H (i.e., put x into H_e).

Verification. Each H_e is enumerated in increasing fashion and hence computable.

Each P_n is active at most once. This ensures that $\bigcup_e H_e$ is coinfinite: for each N, if $x < 2N$ enters this union, then this is due to the action of a requirement P_n with $n \le N$, so there are at most N many such x.

To see that a requirement P_n for $n = \langle e, k\rangle$ is met, suppose that its hypothesis holds. Then there are potentially infinitely many candidates x that can go into H_e. Since A is noncomputable, one of them will be permitted.

Now, by the choice of M_{2e+1}, each H_{2e+1}, and hence each F_e, is infinite. By construction, for $e < m$, we have $|H_e \cap H_m| \le m$. So the family described by H, and therefore also the one described by F, is almost disjoint.

To show that F is MAD, it suffices to verify that if M_e is infinite then $M_e \cap F_p$ is infinite for some p. If all the $P_{e,k}$ are satisfied during the construction, we let $p = e$. Otherwise, we let k be least such that P_n is never satisfied where $n = \langle e, k\rangle$. Then its hypothesis fails, so $M_e \subseteq^* \bigcup_{i<n} H_i$. □

Since an index predictable set computes no MAD set, we obtain the following

Corollary 5.3. *No noncomputable, c.e. set L is index predictable.*

Downey and Nies have given a direct proof of this fact, see [6].

Corollary 5.4. *For each noncomputable c.e. set A, there is a co-c.e. set $G \le_T A$ such that $G \in \mathcal{T}$, i.e., $\langle G_n\rangle_{n\in\omega}$ is a maximal tower.*

Proof. Let F be the MAD set obtained above. Recall the Turing reduction Cp showing $\mathcal{T} \le_s \mathcal{A}$ in Fact 2.2. The set $G = \mathrm{Cp}(F)$, given by

$$x \in G_n \leftrightarrow \forall i < n\, [x \notin F_n]$$

is as required. □

5.2. **Co-c.e. ultrafilter bases.** To build a co-c.e. ultrafilter base F we can't simply adapt the Friedberg construction of a maximal set. If we did so, for each c.e. set W, we would have $F_n \subseteq^* \overline{W}$ or $F_n \subseteq^* W$ for some n. However, this fails when $\overline{W} = B$, where B is the set obtained in Prop. 5.1(i). Instead, we need to make use of the fact that we are given a c.e. index for a computable set and also one for its complement. This will be apparent in the proof below.

Theorem 5.5. *There is a co-c.e. ultrafilter base F.*

Proof. We adapt the construction from the proof of the main result in [11], which states that there is an r-maximal set A such that the index set $\mathrm{Cof}_A = \{e\colon W_e \cup A =^* \omega\}$ is Σ^0_3-complete. Our proof can also be viewed as a variation on the proof of Theorem 3.8 in the setting of co-c.e. sets. We remark that by standard methods, one can extend the present construction to include permitting below a given high c.e. set.

We build a co-c.e. tower F by providing uniformly co-c.e. sets F_e for $e \in \omega$ that form a descending sequence with $F_e \supset_\infty F_{e+1}$. . We agree that whenever we remove x from F_e at a stage s, we also remove it from all F_i for $i > e$. Furthermore, no element is ever removed from F_0, so $F_0 = \omega$.

Let $\langle (V_{e,0}, V_{e,1}) \rangle_{e\in\omega}$ be an effective listing of all pairs of disjoint c.e. sets as defined in the proof of Theorem 3.8. The construction will ensure that the following requirements are met.

$$M_e\colon F_e \setminus F_{e+1} \text{ is infinite.}$$

$$P_e\colon V_{e,0} \cup V_{e,1} = \omega \Rightarrow F_{e+1} \subseteq^* V_{e,0} \vee F_{e+1} \subseteq^* V_{e,1}.$$

This suffices to establish that F is an ultrafilter base.

The tree of strategies is $T = \{0,1,2\}^{<\infty}$. Each string $\alpha \in T$ of length e is associated with M_e and also with P_e. We write $\alpha\colon M_e$ and $\alpha\colon P_e$ to indicate that we view α as a strategy of the respective type.

Streaming. For each string $\alpha \in T$ with $|\alpha| = e$, at each stage of the construction, we have a set S_α, thought of as a stream of numbers used by α. Each time α is initialized, S_α is is removed from F_{e+1}, and S_α is reset to be empty. Also, S_α is enlarged only at stages at which α appears to be on the true path. We will verify the following properties:

(1) $S_\emptyset = \omega$;
(2) if α is not the empty node then S_α is a subset of S_{α^-} (where α^- is the immediate predecessor of α);
(3) at every stage, $S_\gamma \cap S_\beta = \emptyset$ for incomparable strings γ and β;
(4) at the time a number x first enters S_α, x is in F_{e+1}; and
(5) if α is along the true path of the construction then S_α is an infinite computable set.

Note that S_α can be thought of as a set that is d.c.e. uniformly in α. The set S_α is finite if α is to the left of the true path of the construction; S_α is an infinite computable set if S_α is along the true path; and S_α is empty if α is to the right of the true path.

The intuitive strategy $\alpha\colon P_e$. Only strategies associated with a string of length $\leq e$ can remove numbers from F_e. A strategy $\alpha\colon P_e$ thins out S_α by removing some of its elements from F_{e+1}. It regards the set of remaining numbers as its private version of F_{e+1}. It has to make sure that no strategies β to its right remove numbers from F_{e+1} that it wants to keep. On the other hand, it can only process a number x once it knows whether x is in $V_{e,0}$ or $V_{e,1}$. The solution to this conflict is to reserve a number x, which by initialization withholds it from any action of such a β. It then waits until all numbers $\leq x$ are in $V_{e,0} \cup V_{e,1}$. If that never happens for some reserved x, then α is satisfied finitarily with eventual outcome 2. Otherwise, it will eventually process x: If $x \in V_{e,0}$, it continues its attempt to build F_{e+1} inside $V_{e,0}$; else it builds F_{e+1} inside $V_{e,1}$. It takes outcome 0 or 1, respectively, according to which case applies. Each time the apparent outcome is 0, the content of the output stream based on the assumption that the true outcome is 1 is removed from F_{e+1}. So if 0 is the true outcome, then indeed $F_{e+1} \subseteq^* V_{e,0}$.

The strategy $\alpha\colon M_e$ simply removes every other element of S_α from F_{e+1}. Then $\alpha : P_e$ actually only works with the stream of remaining numbers. There is no further interaction between the two types of strategies. (Note here that making F_{e+1} smaller is to the advantage of P_e.)

Construction.
Stage 0. Let δ_0 be the empty string. Let $F_e = \omega$ for each e.
Stage $s+1$. Let $S_{\emptyset,s} = [0, s)$.
Substage $e < s$. We suppose that $\alpha = \delta_{s+1} \upharpoonright e$ and S_α have been defined.

As mentioned above, the strategy $\alpha\colon M_e$ removes every other element of S_α from F_{e+1}. We let $\widetilde{S}_\alpha$ denote the set of remaining numbers. More precisely, if at the current stage we have $S_\alpha = \{r_0 < \ldots < r_k\}$ and r_k is new, then it puts r_k into $\widetilde{S}_\alpha$ iff k is even.

The strategy $\alpha\colon P_e$ picks the first applicable case below.
Case 1: Each reserved number of α has been processed: If there is a number x from $\widetilde{S}_\alpha$ greater than α's last reserved number (if any) and greater than s_0, pick x least and *reserve* it. Initialize $\alpha\widehat{\ }2$. Let $\alpha\widehat{\ }2$ be eligible to act next.

Note that if Case 1 doesn't apply then α has a reserved, unprocessed number x.
Case 2: $[0, x] \subseteq V_{e,0} \cup V_{e,1}$ and $x \in V_{e,0}$: Let s_0 be the greatest stage $< s$ at which α was initialized. Add x to $S_{\alpha\widehat{\ }0}$ and remove from F_{e+1} all numbers in the interval (s_0, x) which are not in $S_{\alpha\widehat{\ }0}$. Declare that α has *processed* x. Let $\alpha\widehat{\ }0$ be eligible to act next.
Case 3: $[0, x] \subseteq V_{e,0} \cup V_{e,1}$ and $x \in V_{e,1}$: Let s_0 be the greatest stage $< s$ at which α was initialized or $\alpha\widehat{\ }0$ was eligible to act. Add x to $S_{\alpha\widehat{\ }1}$ and remove from F_{e+1} all numbers in the interval (s_0, x) which are not in $S_{\alpha\widehat{\ }1}$. Declare that α has *processed* x. Let $\alpha\widehat{\ }1$ be eligible to act next.
Case 4: Otherwise: Let s_0 be the greatest stage $< s$ at which α was initialized or $\alpha\widehat{\ }2$ was eligible to act. Let $S_{\alpha\widehat{\ }2} = \widetilde{S}_\alpha \cap (s_0, s)$. Let $\alpha\widehat{\ }2$ be eligible to act next.
We define $\delta_{s+1}(e) = i$ where $\alpha\widehat{\ }i$, $0 \leq i \leq 2$, has been declared eligible to act next.

Verification. By construction and our convention above, F_e is co-c.e., and $F_e \supseteq F_{e+1}$ for each e.

Let $g \in 2^\omega$ denote the true path, namely, the leftmost path in $\{0, 1, 2\}^\omega$ such that $\forall e \exists^\infty s\, [g \upharpoonright e \preceq \delta_s]$. In the following, given e, let $\alpha = g \upharpoonright e$, and let s_α be the largest stage s such that α is initialized at stage s. We verify a number of claims.

Claim 5.6. *The "streaming properties" (1)-(5) hold.*

(1) and (2) hold by construction.

(3) Assume this fails for incomparable γ and β, so $x \in S_\gamma \cap S_\beta$ at stage s. We may as well assume that $\gamma = \alpha\widehat{\ }i$ and $\beta = \alpha\widehat{\ }k$ where $i < k$. By construction, $k \leq 1$ is not possible, so $k = 2$. Since $x \in S_{\alpha}\widehat{\ }i$ and $i \leq 1$, x was reserved by α at some stage $t \leq s$. So x can never enter $S_{\alpha\widehat{\ }2}$ by the initialization of $\alpha\widehat{\ }2$ when x was reserved.

(4) is true by construction.

(5) holds by the definition of the true path and because S_α is enumerated in increasing fashion at stages $\geq s_\alpha$.

Claim 5.7. *$F_e =^* S_\alpha$.*

The claim is verified by induction on e. It holds for $e = 0$ because $F_0 = S_\emptyset = \omega$. Suppose the claim is true for e. To verify it for $e+1$, let $\gamma = g \restriction (e+1)$, and let s_γ be the largest stage s such that γ is initialized at stage s.

First, we verify that $F_{e+1} \subseteq^* S_\gamma$. Suppose $x \in F_{e+1}$. Then $x \in F_e$, so inductively $x \in S_\alpha$ for almost all such x. By construction, any element x that isn't promoted to S_γ is also removed from F_{e+1} unless x is the last element α reserves. However, in that case, necessarily $\gamma = \alpha\widehat{\ }2$, so this leads to at most one new element in $F_{e+1} \setminus S_\gamma$.

Next, we verify that $S_\gamma \subseteq^* F_{e+1}$. Suppose $x \in S_\gamma$. Then $x \in S_\alpha$, so inductively $x \in F_e$ for almost all such x. At stage $s \geq s_\gamma$, an element x of S_α cannot be removed from F_{e+1} by a strategy $\beta >_L \alpha$ because $S_\beta \cap S_\alpha = \emptyset$ by (3) as verified above and since β can only remove elements from S_β. So x can only be removed by $\alpha\colon M_e$ or $\alpha\colon P_e$.

If $\alpha\colon M_e$ removes x from F_{e+1}, then $x \notin \widetilde{S}_\alpha$, contradiction. So, by construction, the only way x can be removed from F_{e+1} is by a strategy $\alpha\colon P_e$, which for a sufficiently large x means that x is not promoted to S_γ, either.

Claim 5.8. *Each requirement M_e is met, namely, $F_e \setminus F_{e+1}$ is infinite.*

To see this, recall that $\alpha = g \restriction e$. By the foregoing claim, the action of $\alpha : M_e$ removes infinitely many elements of $S_\alpha \subseteq F_e$ from F_{e+1}.

Claim 5.9. *Each requirement P_e is met.*

Suppose the hypothesis of P_e holds. Then every number that α reserves is eventually processed. So either $g(e) = 0$, in which case $F_{e+1} \subseteq^* V_{e,0}$ by Claim 5.7, or $g(e) = 1$, in which case $F_{e+1} \subseteq^* V_{e,1}$, also by Claim 5.7. □

6. Ultrafilter bases for other Boolean algebras

As mentioned in the introduction, we have set up our framework to apply to general countable Boolean algebras rather than merely the Boolean algebra of the computable sets, but mainly with subsequent research in mind. In this last section of our paper, we provide two results on more general Boolean algebras of sets.

Recall that $K(x)$ denotes the prefix-free complexity of a string x, and that a set $A \subseteq \omega$ is K-trivial if $\exists c \forall n\, K(A \restriction n) \leq K(0^n) + c$. For more background on K-trivial sets, see, e.g., Nies [13, Ch. 5] or [5]. Note that by combining results of various authors, the K-trivial degrees form a Turing ideal in the Δ^0_2-degrees (see, e.g., Nies [13, Sections 5.2, 5.4]).

Theorem 6.1. *There is a Δ^0_2-ultrafilter base for the Boolean algebra of the K-trivial sets.*

Proof. Kučera and Slaman [10] noted that there is a Δ^0_2-function h that dominates all functions that are partial computable in some K-trivial set. We use h in a variation of the proof of Theorem 3.8.

Let $\langle V_{e,0}, V_{e,1}\rangle_{e\in\omega}$ be a uniform listing of the K-trivials and their complements given by wtt-reductions to $\emptyset'$; such a listing exists by Downey, Hirschfeldt, Nies and Stephan [4] (also see [13, Theorem 5.3.28]). Let $T = \{0,1\}^{<\infty}$.

For each $\alpha \in T$, we define a (possibly finite) K-trivial set S_α. Let $S_\emptyset = \omega$. Suppose we have defined the set $S_\alpha = \{r_0 < r_1 < \ldots\}$. Let $\widetilde{S}_\alpha$ contain the numbers of the form r_{2i}. Let $S_{\alpha\frown k} = \widetilde{S}_\alpha \cap V_{e,k}$ for $e = |\alpha|$ and $k = 0, 1$. (Note that all these sets are K-trivial since the K-trivials form a Turing ideal.)

Uniformly recursively in $\emptyset'$, we build sets F_e, given by nondecreasing unbounded sequences of numbers $a^e_0 \le a^e_1 \le \ldots$. Suppose we have defined a^e_{k-1}. Let $\alpha \in T$ be the leftmost string of length e such that S_α has at least $k+1$ elements less than $h(k)$. If α exists let a^e_k be the k-th element of S_α, unless this is less than a^e_{k-1}, in which case we let $a^e_k = a^e_{k-1}$.

Let $g \in 2^\omega$ denote the leftmost path in $\{0,1\}^\omega$ such that for every e the set $S_{g\restriction e}$ is infinite. Fix e and let $\alpha = g \restriction e$. Let $p(k)$ be the $(k+1)$-st element of S_α. Since h dominates the function p, eventually in the definition of F_e we will always pick α. Hence $F_e =^* S_\alpha$. In particular, F_e is K-trivial. Also, the sequences $\langle a^e_k\rangle_{k\in\omega}$ are unbounded for each e, so F is Δ^0_2. Clearly, if S_α is infinite then $S_\alpha \supset_\infty S_\beta$ for $\alpha \prec \beta$. So $F_{e+1} \subset_\infty F_e$.

To verify that F is an ultrafilter base for the K-trivials, let R be a K-trivial set. Pick e such that $R = V_{e,0}$ and $\overline{R} = V_{e,1}$. If $g(e) = 0$ then $S_{g\restriction e+1} \subseteq^* V_{e,0}$, and hence $F_{e+1} \subseteq^* R$. Otherwise $S_{g\restriction e+1} \subseteq^* V_{e,1}$, and hence $F_{e+1} \subseteq^* \overline{R}$. $\square$

Finally we consider a Boolean algebra of sets in the subrecursive setting. By the techniques of Jockusch and Stephan [9] we have the following

Theorem 6.2. *An oracle C computes an ultrafilter base for the primitive recursive sets iff C' is of PA degree relative to $\emptyset'$.*

Proof. $\Rightarrow$: Suppose C computes an ultrafilter base F for the primitive recursive sets. Let $g \le_{\mathrm{T}} F$ be a function associated with F as in Definition 1.2. Then the range S of g is p-cohesive in the sense of [9], namely, S is cohesive for the primitive recursive sets. Hence S' and therefore C' is PA relative to $\emptyset'$ by [9, Theorem 2.1].

$\Leftarrow$: We follow the proof of [9, Theorem 2.1], making the necessary modifications. Let $\langle A_i\rangle_{i\in\omega}$ be a uniformly recursive list of all the primitive recursive sets. We call i a primitive recursive index for A_i (index, in brief). By hypothesis on C, there is a function $g \le_{\mathrm{T}} C'$ such that

$$\begin{aligned} |A_i \cap A_n| < |A_i \cap \overline{A}_n| \quad &\Rightarrow \quad g(i,n) = 0 \\ |A_i \cap \overline{A}_n| < |A_i \cap A_n| \quad &\Rightarrow \quad g(i,n) = 1 \end{aligned}$$

(because the conditions on the left are both Σ^0_2, and so C' computes a separating set for them).

We inductively define a C'-computable sequence of indices $\langle e_n\rangle_{n\in\omega}$. Let e_0 be an index for ω. If e_n has been defined and $A_{e_n} = \{r_0 < r_1 < \ldots\}$ (possibly finite), let e'_n be an index, uniformly obtained from e_n, such that $A_{e'_n} = \{r_0, r_2, \ldots\}$. Now let

$A_{e_{n+1}} = A_{e'_n} \cap \overline{A}_n$ if $g(e'_n, n) = 0$, and

$A_{e_{n+1}} = A_{e'_n} \cap A_n$ if $g(e'_n, n) = 1$.

By induction on n, one verifies that A_{e_n} is infinite and $A_{e_{n+1}} \subset_\infty A_{e_n}$. Since $g \le_{\mathrm{T}} C'$, the numbers e_n have a uniformly C-computable approximation $\langle e_{n,x} \rangle_{x \in \omega}$.

Let the ultrafilter base $F \le_{\mathrm{T}} C$ be given by $F_n(x) = A_{e_{n,x}}(x)$. Then $F_n =^* A_{e_n}$ is primitive recursive. Since $F_{n+1} \subseteq^* \overline{A}_n$ or $F_{n+1} \subseteq^* A_n$ for each n, the set F is an ultrafilter base for the primitive recursive sets. $\square$

References

[1] Andreas R. Blass. Combinatorial cardinal characteristics of the continuum. In: Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*, volume 1, pages 395–489. Springer, Dordrecht Heidelberg London New York, 2010.

[2] Jörg Brendle, Andrew D. Brooke-Taylor, Keng Meng Ng, and André O. Nies. An analogy between cardinal characteristics and highness properties of oracles. In: *Proceedings of the 13th Asian Logic Conference: Guangzhou, China*, pages 1–28. World Scientific, 2013. http://arxiv.org/abs/1404.2839.

[3] Jörg Brendle, Vera V. Fischer, and Yurii D. Khomskii. Definable maximal independent families. *Proceedings of the American Mathematical Society*, 147(8):3547–3557, 2019.

[4] Rodney G. Downey and Denis R. Hirschfeldt and André O. Nies, and Frank Stephan. Trivial reals. In *Proceedings of the 7th and 8th Asian Logic Conferences*, pages 103–131, Singapore, 2003. Singapore University Press.

[5] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic randomness and complexity*. Springer-Verlag, Berlin, 2010. 855 pages.

[6] André O. Nies (editor). Logic Blog 2019. Available at http://arxiv.org/abs/2003.03361, 2019.

[7] Noam Greenberg, Rutger Kuyper, and Daniel D. Turetsky. Cardinal invariants, non-lowness classes, and Weihrauch reducibility. *Computability*, 8(3-4):305–346, 2019.

[8] Carl G. Jockusch, Jr. Degrees in which the recursive sets are uniformly recursive. *Canad. J. Math.*, 24:1092–1099, 1972.

[9] Carl G. Jockusch, Jr. and Frank Stephan. A cohesive set which is not high. *Math. Log. Quart.*, 39:515–530, 1993.

[10] Antonín Kučera and Theodore A. Slaman. Low upper bounds of ideals. *J. Symbolic Logic*, 74:517–534, 2009.

[11] Steffen Lempp, André O. Nies, and D. Reed Solomon. On the filter of computably enumerable supersets of an r-maximal set. *Archive for Mathematical Logic*, 40(6):415–423, 2001.

[12] Maryanthe Malliaris and Saharon Shelah. General topology meets model theory, on ?? and ?? *Proceedings of the National Academy of Sciences*, 110(33):13300–13305, 2013.

[13] André O. Nies. *Computability and Randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009. 444 pages. Paperback version 2011.

[14] Nicholas A. Rupprecht. *Effective correspondents to cardinal characteristics in Cichoń's diagram*. PhD thesis, University of Michigan, 2010.

[15] Nicholas A. Rupprecht. Relativized Schnorr tests with universal behavior. *Arch. Math. Logic*, 49(5):555–570, 2010.

[16] Theodore A. Slaman and Robert M. Solovay. When oracles do not help. In *Proceedings of the fourth annual workshop on Computational learning theory*, pages 379–383. Morgan Kaufmann Publishers Inc., 1991.

[17] Robert I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic, Omega Series. Springer–Verlag, Heidelberg, 1987.

[18] David Soukup. Two infinite quantities and their surprising relationship. *arXiv preprint arXiv:1803.04331*, 2018.

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