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On Demons and Oracles

Alastair A. Abbott,[1],
Cristian S. Calude,[1] Karl Svozil[2]

[1]University of Auckland, NZ
[2]Vienna University of Technology, Austria

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Alastair A. Abbott,[1] Cristian S. Calude,[1] Karl Svozil[2]
[1]Department of Computer Science, University of Auckland
Private Bag 92019, Auckland, New Zealand
[2]Institut für Theoretische Physik, Vienna University of Technology,
Wiedner Hauptstraße 8-10/136 A-1040 Vienna, Austria
aabb009@aucklanduni.ac.nz,cristian@cs.auckland.ac.nz,svozil@tuwien.ac.at

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Abstract

This paper presents a personal, subjective overview of Church-Turing Thesis and some attempts to trespass the Turing barrier.

1 Turing's barrier

The concept of digital computation which emerged from the works of Church [1], Turing [2] and Gödel [3] is an important achievement of the last century. A large variety of mathematical models of computers and computations have been developed. Turing machines, lambda-calculus, combinatory logic, recursive functions, Markov algorithms, register machines are among the best known classical models. Newer models range from programming-oriented models including concurrent models like actor model and process calculi to quantum Turing machines, DNA computers, molecular computers, wetware computers and many others. A remarkable result was gradually proved: in spite of the apparent diversity, the computational capability of every model of computation is the same. All models are computationally equivalent. This strong mathematical evidence motivated a more general belief: the Turing model of computation is the right and most general concept for digital computation.

Turing [2] proved that Hilbert's Entscheidungs- problem—the decision problem for the predicate calculus[1]—is unsolvable by any Turing machine. Independently, Church [4] obtained the same negative result by using his lambda-calculus, so by proposing ([4], p. 356) to

> define the notion ... of an effectively calculable function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a lambda-definable function of positive integers)

he argued that Hilbert's Entscheidungsproblem *is unsolvable* (not only unsolvable by any lambda-definable function). Motivated by a similar identification proposed by Turing [2], and the (mathematical) equivalence between the sets of functions computed by Turing machines, lambda-definable functions and recursive functions, Kleene ([5], p. 232), introduced the Church-Turing Thesis:

[1]Find an effective method to determine whether an arbitrary formula of the predicate calculus system is provable in the system.

> A function of positive integers is effectively calculable only if recursive.

In the quest to give meaning to their negative solutions to Hilbert's Entscheidungsproblem, Turing and Church were interested in describing what humans could "in principle" compute, so originally the Church-Turing Thesis' main scope was purely mathematical. However, the Thesis itself is not a mathematical statement as one of the terms involved—the notion of effectively calculable function—is not mathematically defined. In particular, the Thesis cannot be (mathematically) proved: it needs empirical verification, which can continue for ever, or a plausible refutation. This fact was recognised almost immediately by Post ([6], p. 10), who objected to its presentation as a definition because it "blinds us to the need of its continual verification".

In time the scope of the Church-Turing Thesis shifted towards a more general goal, the ultimate limits of digital computation. In this new context the Church-Turing Thesis can be stated as[2]:

> Every function of positive integers which can be computed in a physical system is recursive, or equivalently, it can be computed by a Turing machine.

In this form the Church-Turing Thesis is a statement about what can be computed in a system of physical laws. Two components are involved: i) the mathematical component determines the dynamics of evolution of physical states into others and the relation between inputs and outputs, and ii) the physical component determines which dynamics can be performed in the given system of physical laws. Once we fix a system of physical laws described mathematically, the corresponding Church-Turing Thesis becomes a well-posed mathematical question which can be mathematically investigated [7]: it can be proved, disproved, or proved undecidable. Gödel, who was initially unconvinced by Church's argumentation [4], but changed his mind after reading Turing's paper [2], suggested the idea of an axiomatic approach for the notion of "effective calculability" meant to capture its generally accepted properties. In this spirit Gandy [8] proposed a programme where physical laws (like bounded velocity and finite density of information) are used to 'prove' the Church-Turing Thesis; this approach was extended to quantum theory by Arrighi and Dowek [9]. The Church-Turing Thesis has morphed into a class of Church-Turing Theses, each depending on the underlying system of physical laws; in some cases it is true, in some false (see, for example, Smith [10]).

The physical determination of the Church-Turing Thesis was rightly pointed out by Deutsch [11]:

> The reason why we find it possible to construct, say, electronic calculators, and indeed why we can perform mental arithmetic, cannot be found in mathematics or logic. The reason is that the laws of physics "happen" to permit the existence of physical models for the operations of arithmetic such as addition, subtraction and multiplication.

However, omitting the mathematical component

> Computers are physical objects, and computations are physical processes. What computers can or cannot compute is determined by the laws of physics and not by pure mathematics. (Deutsch [12], p. 98)

is wrong. The laws of physics can determine what dynamics can be performed in a given system of physical laws, but not what such dynamics "compute": this a mathematical issue. According to Timpson [13]

[2]Sometimes this is called the Physical Church-Turing Thesis.

> We must recognise that their [mathematical determinants] place is *prior* to that of physical determinants.

Here is an example. Consider an undecidable problem, say the halting problem[3]. In every system of physical laws the halting problem will be undecidable because of Turing's proof. This proof and its conclusion—the undecidability of the halting problem—tell us nothing about the system of physical laws, as no possible dynamics can be a solution to the problem: this is a mathematical fact true in any system of physical laws. Of course, the halting problem can be solved by other types of "machines": such a "solution"—obtained by mathematical or physical means—does not challenge the validity of Turing's result which concerns only the mathematical concept of Turing machine.

2 The land of hypercomputation

Hypercomputation studies models of computations in the hope of breaking the Turing barrier. By placing precise physical constraints on computations, hypercomputation contributes to the program of continuous verification of the Church-Turing Thesis suggested by Post.

The possibility of executing infinitely many "operations" in a finite amount of time is the core of many proposals. This idea is not new: Zeno's analysis of motion paved the way for the accelerated Turing machines featured in Section 3.

In 1939 Turing [14] introduced the seminal notion of *oracle* Turing machine, a standard machine having access to an infinite sequence of bits—the oracle—coding answers to as many questions, and made this machine compute with finite approximations of the infinite oracle. If the oracle is computable the resulting computation is equivalent to a standard computation, but in case the oracle is incomputable the machine trespasses the Turing barrier. In the expert hands of recursion-theorists (see Cooper and Odifreddi [15] for an understandable glimpse) oracle Turing machines haven been used to scrutinise the land of incomputable. The "crucial question", in the words of the mathematician M. Davis, is:

> Are there real physical processes that can be harnessed to do the work of Turing 'oracles'?

Davis gives an unequivocally negative answer (see [16, 17]). This issue will be re-visited in Section 5.

Rewind to 1970: In a footnote to [18] (p. 143) the logician G. Kreisel makes an astonishing suggestion: a collision problem related to the 3–body problem[4] could be regarded as "an analogue computation of a non-recursive function", so an instance of hypercomputation. This possibility gets a new dimension with Xia's [19] construction of no-collisions singularities in small Newtonian systems. Harnessing the incomputability identified in different physical systems (see [20, 21, 15]) becomes a possible source of hypercomputation.

Hypercomputation models have been constructed using neural networks [22], quantum mechanics ([23, 24, 25]), relativity theory ([26, 27, 28, 29]), inductive Turing machines [30] and many other ideas (see [31, 30] for overviews).

[3] Does there exist a Turing machine capable of deciding whether an arbitrary Turing machine halts on a given input?

[4] The problem of predicting the motion of a group of celestial objects that interact with each other gravitationally.

3 A case study: accelerated Turing machines

Centuries ago the ancient greek philosophers worried about the implications of an infinite divisibility of space and time. Zeno of Elea pointed out that motion itself would unexist, since the slightest finite movement would require an infinity of actions [32, 33, 34]. Subsequently, differential calculus suggests to formally overcome these issues by taking the finite differential quotient of spatial and temporal change.

The revival of these ancient ideas came with computation. Weyl (see also Blake [35, p. 651] and Russell [36, p. 144]) noted the potentiality to "complete" infinite computations in finite proper physical time as follows ([37], pp. 41-42; see also Ref. [38], p. 20):

> The impossibility of conceiving the continuum as rigid being cannot be formulated more concisely than by Zeno's well-known paradox of the race between Achilles and the tortoise. The remark that the successive partial sums $1 - \frac{1}{2^n}$ $(n = 1, 2, 3, \cdots)$ of the series
> $$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$
> do not increase beyond all bounds but converge to 1, by which one nowadays thinks to dispose of the paradox, is certainly relevant and elucidating. Yet, if the segment of length 1 really consists of infinitely many subsegments of lengths 1/2, 1/4, 1/8, . . ., as of 'chopped-off' wholes, then it is incompatible with the character of the infinite as the 'incompletable' [. . . and] there is no reason why a machine should not be capable of completing an infinite sequence of distinct acts of decision within a finite amount of time; by supplying the first result after 1/2 minute, the second after another 1/4 minute, the third 1/8 minute later than the second, etc. In this way it would be possible [. . .] to achieve a traversal of all natural numbers and thereby a sure yes-or-no decision regarding any existential question about natural numbers!

Such a device—termed an accelerated Turing machine, reflecting that the rate of computation accelerates over it's computational period—goes beyond the Church-Turing barrier. Questions such as the halting problem can be solved as the infinity of computational steps performed by a non-halting computation are performed within a finite period of time. This power comes at an interesting cost however: for such machines to hypercompute they must by necessity use an infinite amount of space [56].

As noted earlier, the mathematical model of an accelerated machine undergoing these infinite dynamics must be supplemented by a search for physical implementations. A growing number of research articles documents the increasing interest in the subject; both from a computer science perspective [39, 40], as well as from a more physical point of view [41, 42, 43, 44]. Several proposals exist to harness infinite divisibility of space and time, and to utilise physical processes for the construction of such infinity machines. Some of these physically inspired proposals involve, for example, investigating the state of a lamp with ever decreasing switching cycles, and several [45, 46, 47, 34, 44]. ultrarelativistic methods put observers in "fast orbits" to exploit relativistic differences in time or throw them toward black holes where space-time behaves strangely [26, 48, 49, 50, 51, 52, 22, 53, 54, 28, 29]. Further more proposals embed automata densely in physical space-time [55].

4 Statistical physics

Hypercomputation is also relevant in statistical physics—where information plays a key role—and can add to the dialogue on Maxwell's infamous paradoxical demon [57]. Here, the ability to hypercompute yields the potential to improve, if not reverse, energy dissipation.

Maxwell's demon is supposedly capable of separating lower-energy from higher-energy particles in an isolated box divided into two chambers. Traditionally, the demon accomplishes this by controlling a shutter in the dividing barrier, only opening it for particles with suitable velocity. In this way a temperature gradient might be produced which would, at least in the long run, contradict the second law of thermodynamics (in the form that no process exists whose sole purpose is the transformation of heat into work).

The contemporary "exorcism" of the demon and resolution of the paradox seems to presuppose that nature behaves reversibly—that is, the evolution of physical microstates is one-to-one. Thus, every separation or contraction of microstates in one region of phase space has to be accompanied by an equal compensating amount of mixing or expansion. In particular, any sorting action of the demon, associated with a decrease of entropy of the rest of the system, should be compensated by an increase of information in the demon's "mind" memory which is at least as large as the entropy decrease caused by the demon. One elegant way of demonstration [58, pp. 927-929] is by a one-molecule Szilárd engine [59, pp. 843–844] whose phase space volume is contracted by a factor of two by the demon's action. However, in order to complete the cycle, one bit of the demon's memory has to be reset—an irreversible, two-to-one transition causing an increase of entropy at least as big as $k_B \log 2$ [60].

But what if the demon's memory and computational capacity is, at least in principle, unbounded? This case corresponds to certain types of hypercomputability: here the contraction in physical configuration or phase space could go on forever at the price of consuming more and more memory of the demon, thereby realising a sort of *Hilbert's Hotel* scenario.

Furthermore, if the demon's capacity to compress information is unbounded, would it be possible to "compute" the (classical incomputable) algorithmic information content of the information acquired? Present literature [61, 62, 63] postulates that only the optimal compression yields optimally small compensation on the demon's side; other less optimal compressions result in an overall increase in entropy. Thus, in order to attain optimal performance, hypercomputational abilities of the demon must be assumed [64, Section 8.2].

5 Quantum oracles

Progress in the natural sciences during the 20th century was marked by two distinct departures from classical thought: Einstein's theories of relativity and the theory of quantum mechanics. Given the exploration of possibilities of hypercomputation based on relativity theory mentioned in Section 4, and that quantum mechanics if anything presents an even greater departure from classical omniscience, it is perhaps a little surprising that the possibility of hypercomputation by exploiting quantum mechanics has been explored so little. Research into the use of Quantum mechanics for computation has primarily explored the alternative computational model of quantum computing [65], but since its conception by Deutsch [11] it has been viewed as yet another model equivalent in computational power to Turing machines [66].

There are, however, possibilities of attaining hypercomputational power through the more subtle (although perhaps more direct) approach of considering quantum oracles instead of quantum computational models.

In contrast to quantum computation which utilises the simple and well known unitary dynamics of the Schrödinger equation, the construction of the quantum oracle exploits the heart of non-classicality in quantum mechanics: the measurement process. This is outside the reach of the Schrödinger equation and, while there are extremely strong results on the impossibility of a classical description of measurement which lie at the centre of our argument, there are several competing interpretations of the ontological structure of quantum systems which alter the way measurement is viewed [67]. As such, the quantum oracle we describe relies on the validity of the assumptions we make in forming the mathematical model of the strange, counterintuitive features of quantum measurements.

In quantum mechanical theory even simple systems can exist in states that are superpositions of other states, for which any attempt to measure the state will yield one of the possible outcomes seemingly at random [68]. Formally, the theory only describes the probability distribution of this process; the fact that, after measurement, a subsequent measurement of the state will yield the same result seems to indicate that the measurement process irreversibly changes the state of the system at random. The nature of this "state collapse" is outside the theory and in the realm of "interpretations", of which many exist: the standard Copenhagen interpretation(s) [67], the de Broglie-Bohm theory [69], the many-worlds interpretation of Everett [70], and many more exotic ones—among them the claim that measurements can be "undone" [71, 72, 73, 74, 75, 76, 77, 78, 79].

A natural interpretation of these state of affairs, and one argued early on by Einstein, Podolsky and Rosen (EPR) in their seminal paper [80], is that

> the description of reality as given by a [quantum mechanical] wave function is not complete

in that the result of a measurement is not probabilistic but in fact determined by some unknown, yet pre-existing, "element of physical reality". However, the failures of a classical, deterministic viewpoint to account for the predictions of quantum mechanics are exemplified by the "no-go" results of Bell [81] and Kochen and Specker [82]. Bell's results show the impossibility of any hidden variable theory to reproduce the statistical predictions of quantum mechanics under the assumption of locality. However, of more interest is the result of Kochen and Specker applicable to individual quanta. The Kochen-Specker Theorem proves that it is impossible to assign pre-existing values to the outcomes of measurements under the conditions of i) *value indefiniteness*: all observables, even those which are not compatible (cannot be simultaneously measured) have definite values corresponding the result of a measurement of them; and ii) *non-contextuality*: the value corresponding to the result of a measurement does not depend on which other compatible measurement are made alongside of it.

In a bid to maintain realism, Bell proposed that [83]

> the result of an observation may reasonably depend not only on the state of the system ... but also on the complete disposition of the apparatus.

Attempts to give complete, contextual, interpretations for quantum theory exist, such as the de Broglie-Bohm theory [69], and while such interpretations reproduce certain quantum mechanical predictions, they must by necessity embrace non-locality and remain distinctly non-classical and counterintuitive.

The many-worlds interpretation [70] offers a rather different way out. In this theory, measurement corresponds to the measuring apparatus becoming "entangled" with the state being measured, hence it makes no sense to even speak of the unique result of a measurement.

At the other end of the spectrum, Born opted [84] to

> give up determinism in the world of atoms.

This served as the basis for the Copenhagen interpretation [67], due largely to Born and Heisenberg amongst others, and has been the predominant view since.

If we interpret the Kochen-Specker Theorem as an evidence for Born's proposal to "give up determinism in the world of atoms" (which subtly requires the acceptance of measurement as a real process, ruling out the many worlds type scenario), then we can construct a simple device acting as an incomputable oracle. The Kochen-Specker Theorem, however, does not give us a straight choice between contextual and indeterministic realities. Even if we choose to reject the notion of a contextual reality, the theorem does not exclude the possible of partial-determinism; we may need to give it up only for some observables. However, the proof of the theorem is intrinsically co-ordinate free and only concerns itself with orthogonality relationships between observables. If no direction in Hilbert space is privileged by the proof, then how can one argue in favour of this partially-deterministic regime? Rather than being fundamental to the state, the asymmetry could manifest itself during the measurement process. Put bluntly, one can conceive of a *demon* possessing the observer and ensuring only those observables with definite values are ever measured; conversely, a *demon* could inhabit the state ensuring the observable we choose to measure is assigned a definite value, while those we do not measure are allowed to be indefinite. Such "super-deterministic" loopholes are known to exist in, and would invalidate tests of Bell inequalities [85, 86]. If we are to use a quantum system as an oracle, we must refuse to accept such demons from existing and conspiring to make our output be due to predetermined elements of reality. We hence choose to consider a complete departure from classical omniscience and only allow those observables in which the state was prepared as an eigenstate of to have definite values; elsewhere we have complete value indefiniteness.

Before we can construct our oracle we must make one final connection between value definiteness and computability by returning to EPR. Specifically, we ask what does it mean to be able to assign a definite value to a measurement outcome? According to EPR [80],

> if, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality [hidden variable] corresponding to this physical quantity.

A definite value exists exactly when there is a value allowing us to predict exactly the result of a measurement. Thus, if we repeat the state preparation and measurement process *ad infinitum* and the sequence produced by the concatenation of measurement outcomes is computable, then *every* measurement can be predicted with certainty and was thus of a value definite observable. This final assumption makes the important connection between computability and the classical notion of determinism that quantum mechanics appears to have abandoned.

From here it can be argued that, if each measurement is of a value indefinite observable (i.e. not of the observable which the state is in an eigenstate of) then the infinite sequence of results considered above must be incomputable. If it were computable, this would mean the measured observables were all value definite, contradicting the assertion of value indefiniteness everywhere.

Under these assumptions we are hence guaranteed that such a device would produce an incomputable sequence and act as an oracle. Such devices have in fact been considered for the use of random number generation [87], so perhaps Davis' verdict on the existence of physical process that can be harnessed as Turing oracles was a little premature; one of the most plausible physical interpretation of the conundrums presented by quantum mechanics allows us to do just that.

It is useful to compare and contrast this envisaged quantum oracle to a hypothetical realisation of a probabilistic Turing machine—a Turing machine which chooses transitions probabilistically from some predefined computable probability distribution. If we consider a trivial such devise which, regardless of the input, accepts with probability one-half, and rejects also with probability one-half, and consider the infinite sequence generated by running this machine on inputs $1, 2, \dots$ (where "accepting" on input i means the ith bit is 1), is this device different in any real respect to our quantum oracle? Naïvely it would seem not: both devices act as oracles where the ith bit is 1 with probability one-half. However, the sequences produced by this probabilistic oracle are only uniformly distributed—we cannot rule out the crucial probability-zero possibility of a computable sequence being produced. This probabilistic oracle would hence, in practice, be a Turing oracle with probability-one. While this may appear to contradict the well known Turing-equivalence of probabilistic Turing machines [88], we note the crucial distinction that this device does not formally compute any sequence at all—we are simply envisaging a single output of an infinite run of it being used as an oracle. The existence of a physical realisation of such a probabilistic Turing machine is, of course, as difficult a problem to solve as that of a Turing oracle; we simply note that such a device is not the same as our proposed quantum oracle which is stronger in that it is unable to produce any computable output.

While our quantum oracles behave as oracles in the Turing sense, we know of no way of saying more about the set which they are an oracle for. The most important open is: *what is the computational power of a Turing machine working with a quantum oracle?* We believe that such an oracle cannot solve the halting problem, and it is an open problem to determine what problems such devices can solve.

6 The known, the unknown, and the unknowable

The body of every subject can be divided into three parts: the known, the unknown, and the unknowable. In time, the unknown shrinks, with some facts migrating to the known and the unknowable parts. The unknowable is the most problematic part as to prove that a condition is impossible one has to show that it implies a contradiction or an absurdity. A limit implies an impossibility, but the converse implication is false. Impossibilities are provable, hence objective; limits tend to be subjective and temporal. In contrast to mathematics where the triad is sharp and its poles—the known and the unknowable—are rather stable, the division fluctuates in science. Mathematical limits, like Gödel's incompleteness theorem or incomputability results, cannot be automatically transferred to physics. Hypercomputation is a subject at the intersection of mathematics, computer science and various particular sciences, physics, chemistry, biology, so here impossibility and limits are difficult to obtain and tend to be temporal.

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