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Abstract

We introduce a new notion of a cluster of infinite two player games between players 0 and 1. This is an infinite collection of games whose game trees can be composed into a graph which is similar to a tree except that the graph might not have the initial node. For each node of the graph there is an ancesstor node. We call this graph the arena of the cluster. For a game cluster we introduce a notion of a winner for the whole cluster. This notion is weaker than the requirement to win every game of the cluster. Any two player game can be viewed as a game cluster consisting of all its residual games [3, 18]. We extend the restricted memory determinacy (RMD) theorem of Gurevich-Harrington (GH), [3] to game clusters. We think that the notion of a game cluster improves the modeling power of two player games used to give semantics for concurrent processes [10, 11].

1 Introduction

In 1982 Yuri Gurevich and Leo Harington [3] published their celebtrated "short proof" of Rabin's decision method for $S2S$ [15]. As a basic vehicle for this proof they introduced a new kind of game determinacy for two person games. Following [18], [19], [20] we call this kind of game determinacy, restricted memory determinacy (RMD). The Gurevich-Harrington

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RMD is based on restricting the winning strategies to strategies respecting a certain equivalence relation over the game tree (arena). The Gurevich-Harrington proof of RMD as well as the Buchi's notion of state strategies [1], [2] inspired A. Nerode, A. Yakhnis, V. Yakhnis, and others to apply two player games to develop semantics of concurrent (shared memory and distributed) computations [10], [11] as well as models for behaviour of real-time control systems [9]. A number of researchers continued the work of Gurevich and Harrington by giving more detailed proofs (Monk, [5], McNaughton, [7]), providing extensions (Yakhnis-Yakhnis [18], [19], Zeitman, [20]), and applying the Gurevich-Harri ngton RMD theorem for modal logics of programs (Jutla, [4], Stuart, [16]). More general two player games are used by Y. Moschovakis to give a novel semantics for concurrent processes (see [6] and his subsequent papers).

This paper uses ideas of the original Gurevich-Harrington paper [3] in order to develop an extension of Gurevich-Harrington game-theoretic methods. We think that game-theoretic ideas developed by Gurevich-Harrington can be applied in a wide range of areas: logic, theory of concurrent and parallel computations, logic programming, real-time computing sytsems, artificial intelligence, robotics, operating systems design and verification, and hybrid systems theory, etc. Therefore, a goal of our extension is to expand potential applications of GH games.

Gurevich-Harrington games have several features. Each game generates a structure, a tree or a graph [Gurevich - Harrington [3], McNaughton [7], Zeitman [20],Yakhnis - Yakhnis[18] [19]], which has a fixed element, called **initial position**. Each play of the game begins from this **initial position**. For example, in games occuring on trees these elements are the roots of trees. The structures generated by games are **strongly locally finite**, that is the number of neighbours of every element is bounded by an $n \in \omega$. In addition, each player of the game has a finite alphabet from which the player picks elements and makes moves. This is **finiteness** of the game alphabet in [Gurevich - Harrington]. Each move of any player is **identified** with the choice of a letter from the alphabet. We omit all the above restrictions in our games. Namely, our game structures need not have initial elements. The game structures are not supposed to be locally finite. Each player of our games potentially has infinitely many choices to make moves.

One of our other intensions is to model proccesses which can be characterized as proccesses with **unknown past**. An example of a such proccess is a human–computer interaction: a user (computer) beginning to interact with a computer (user) does not neccessarily know the past history of the computer (user). We would like to point out that it is not a new idea to ivestigate proccessses with unknown past. For example, automata-theoretic treatment of procedures with unknown past has also been developed in [Nivat-Perrin [12], Perrin-Shupp [13], Semenov [14]]. The approach taken in these papers is motivated by problems from ergodic theory and symbolic dynamics [Nivat-Perrin [12], Perrin-Shupp [13]]. Another example is that investigations in modal and temporal logics with past tense temoral operators [17]. We also hope that our generalization of Gurevich–Harrington games is appropriate to develop a game-theoretic approach for investigating proccesses with unknown past.

The Gurevich-Harrington strategies with restricted memory have an important property. These strategies do not rely on the knowledge of a starting position of a play. On the other hand, as we already mentioned the Gurevich-Harrington RMD theorem refers to games which have the standard beginning position, the root, for all the plays of the game. A natural question arises: how to generalize the notion of a two player game in such a way as to include an idea of a game where plays are permitted to begin arbitrarily far in the past? Of course, we would like to preserve the "restricted memory" property for the winning strategies in the new games.

As we mentioned Gurevich-Harrington games give semantics for concurrent processes. When a play (concurrent process) is in progress the information about the start of the play (process) and a substantial part of its past is not always available. Therefore, we think that the notion of a game cluster improves the modeling power of two player games used to give semantics to concurrent processes.

Game Clusters. To achieve the above goals we introduce a new notion of game clusters. A game cluster is a collection of games between 0 and 1 which have the following property. There is a single directed graph called "left ray tree" (LRT) such that:

1. Every two nodes of the graph have a common ancestor.

2. Every two nodes which are not ancestors of each other have no common decendant.

3. A node of the graph may have an infinite number of ancestors. We call

a left ray the collection of all ancestors of a node.

The LRT graphs are similar to trees except they may possess infinite left rays. We require that for every game of the cluster its game tree is a subgraph of the graph LRT, and the LRT graph is the union of such subgraphs. Next, we require that if a graph node belongs to any two game trees, the same player makes his moves from this node in the corresponding games. Finally, we require that the winning sets for the games of the cluster must be related in the following way. There is a set W of nonextendible paths through the graph such that for every game Γ of the cluster each path from the winning set of Γ is extendible to a path from W. If a node p is the root of Γ we denote by W_p the winning set of Γ.

We then introduce the notion of winner for the game cluster. The graph LRT has its nodes naturally partitioned between players 0 and 1 as the nodes at which the corresponding players make moves. Therefore there is a notion of a strategy for each player which the player can use in every game of the cluster. We say that the player 0 wins the game cluster if there is a strategy for player 0 and a left ray in the LRT graph such that this strategy wins every game of the cluster whose root belongs to the left ray. We say that the player 1 wins the game cluster if the player has a strategy such that for every left ray there is a game of the cluster with its root on the left ray such that this strategy wins the game. Thus we can present a game cluster as a triple $\Gamma = (\mathcal{A}, W, 0)$, where $\mathcal{A}$ is an LRT graph. This means that the games of the cluster can be presented in the Gurevich-Harrington form as triples $\Gamma_p = (\mathcal{A}_p, W_p, 0)$, where $\mathcal{A}_p$ is the arena of a game from the cluster and $p \in \mathcal{A}$. This completes the description of a game cluster.

We restrict the winning sets W to the Gurevich-Harrington form. I.e. we fix a finite collection of subsets of the LRT graph and for each such subset C consider the set $[C]$ of all the nonextendible paths through the graph which meet C infinitely often. The set W has a Gurevich-Harrington form if it is a boolean combination of sets $[C]$. We call the finite collection of sets C the collection of colors [20].

We would like to explain briefly how to generalize the Gurevich-Harrington Latest Appearance Record (LAR), their restricted memory equivalence and and the notion of a strategy respecting this equivalence for game clusters. The difficulty is that the arena of a cluster might not have a root and Gurevich-Harrington definition of LAR does not apply directly. We bypass the difficulty by definig a new equivalence relation E. This equivalence is based on LARs with respect to nodes on a fixed left ray in the arena of the game cluster. The same left ray is used to define an equivalence E over the arena, which is an extension of the Gurevich-Harrington RMD equivalence. In particular, xEy implies that the games from the cluster with roots at x and y are isomorphic structures in a suitable language. This replaces the Gurevich-Harrington notion of coincidence of residual games. Let Γ_x and Γ_y be the structures just mentioned. We say that a strategy f for player 0 over arena strictly respects E if xEy implies that the structures (Γ_x, f_x) and (Γ_y, f_y) are isomorphic, where f_z is a unary predicate which is true on all nodes consistent [18] with f after z.

In the next two sections we give exact definitions for the above notions and state our RMD theorem as Theorem 3.1.

2 Games, Arenas, and Strategies

Our games occur on the arenas defined as follows.

Definition 2.1 *A partially ordered set $\mathcal{A}$ is an* **arena** *if it satisfies the following axioms (every set possessing only one minimal element is linearly ordered, every two tree nodes have a common ancestor below each of them, and the partial order is discrete):*

1. $\forall xyz(x \leq y \bigwedge z \leq y \rightarrow (x \leq z \bigvee z \leq x)$.

2. $\forall xy \exists z(z \leq x \bigwedge z \leq y)$.

3. $\forall xy(x \leq y \rightarrow \exists n \exists^n z(x < z < y))$.

Let T_0 and T_1 be subsets of A such that:

1. $T_0 \bigcup T_1 = T$.

2. $T_0 \bigcap T_1 = \emptyset$.

3. For any $x \in T$, if $x \in T_\epsilon$ then $Suc(x) \subset T_{1-\epsilon}$, where $\epsilon - 1 = 0$ if $\epsilon = 1$, and $\epsilon - 1 = 1$ if $\epsilon = 0$; and $Suc(x) = \{y | x < y \& \neg \exists z(x < z < y)\}$.

We say that the set T_ϵ is the **set of nodes** for the player ϵ.

If η is a path in an arena $\mathcal{A}$ and $p \in \eta$, then the **left ray** from p is the set $l(\eta_p) = \{y | y \in \eta \& y \geq p\}$.

Definition 2.2 *1. A game cluster is a triple $\Gamma = (\mathcal{A}, W, 0)$, where $\mathcal{A}$ is an arena, W is a set of paths in $\mathcal{A}$, and 0 is a player.*

2. If the arena $\mathcal{A}$ has no minimal elements then the game Γ is called **a game cluster with unbounded finite past**.

A strategy for the player ϵ is a many valued function from $T(\epsilon)$ to $T(\epsilon - 1)$ such that for every $x \in T(\epsilon)$, $f(x) \in Suc(x)$.

Definition 2.3 *Let $\Gamma = (\mathcal{A}, W, 0)$ be a game cluster with unbounded finite past.*

1. *The player 0* **wins** *the game cluster Γ if there exits a strategy f for 0 and a left ray α such that for any node $p \in \alpha$, f wins Γ_p.*

2. *The player 1* **wins** *the game cluster Γ if there exits a strategy g for 1 such that for any left ray α there exists $p \in \alpha$ such that g wins Γ_p.*

Note that our definition of winner is unsimmetrical. This is caused by a possible absence of an initial node in the arena of the game cluster. However, for two player games our definition is equivalent to the standard definition of a winner.

3 Restricted Memory Determinacy Theorem

Let $\Gamma = (A, W, 0)$ be a game cluster, and let $x \in A$. We define a model corresponding to a residual game Γ_x as follows:

1. The domain of the model is $A_x = \{y | x \leq y \in A\}$.

2. Predicates $(T_0)_x, (T_1)_x$ are defined by $A \bigcap T_0$ and $A \bigcap T_1$.

3. For each $\alpha \in W$, we define a unary predicate $A_x \bigcap l\alpha$.

We denote this model by Γ_x as we do the residual game.

Definition 3.1 *An equivalence relation η on A is a congruence if the following two properties hold:*

1. *If* $(x, y) \in \eta$, *then models* Γ_x *and* Γ_y *are isomorphic.*

2. *If* $(x, y) \in \eta$, $x \leq z$, *then for any isomorphism* $\beta : \Gamma_x \to \Gamma_y$ *the pair* $(z, \beta(z))$ *belongs to* η.

Definition 3.2 *A strategy* f *strictly respects a congruence* η *if for all* $(x, y) \in \eta$, *every isomorphism between the models* Γ_x *and* Γ_y *is an isomorphism between the models* (Γ_x, f_x) *and* (Γ_y, f_y), *where* f_x *is defined in the introduction.*

Let A be an arena of a game cluster, S be a finite set of "colors", $C = (C_s, s \in S)$ be a list of subsets of the arena colored by a corresponding color s. We permit the members of C to intersect. We define the notions of display and the latest appearance record as follows. We linearly order the set of colors S. Call any word over S in which every color occurs only once a **display**. Denote by $Display(S)$ the set of all displays.

Definition 3.3 *For any two nodes* $x, y \in A, x \leq y$ *define the latest appearance record of colors (LAR) as follows. Let* $d \in Display(S)$. *We define a function* $LAR(x, d, y)$:

$$LAR(x, d, x) = Delete(d \cdot l(x)),$$

where $\cdot$ *denotes a concatenation of words, and* $l(x)$ *is a word whose letters are all of the colors from* $\{s \in S \mid x \in C_s\}$ *written in their linear order in* S *from the least to the largest. Here Delete is an operation that deletes from two concatenated words the letters of the first word that also appear in the second word. Suppose* $LAR(x, d, t)$ *is defined, then for every* $y \in Suc(t)$

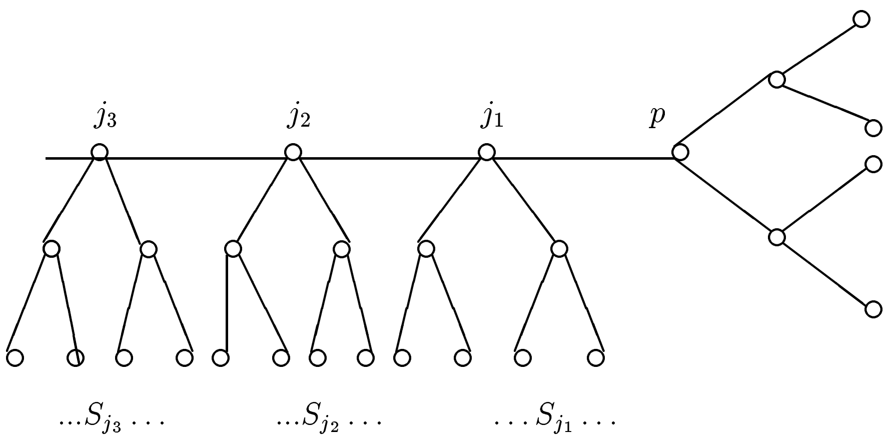
$$LAR(x, d, y) = Delete(LAR(x, d, t) \cdot l(y)).$$

We cover any arena by a disjoint collection of trees called **sectors** and define a congruence over the arena by means of this cover as follows.

Fix a node $p \in A$. Consider a left ray $\xi \subseteq A$ which ends at p. For every $j \in \xi$ define the sectors $Sect_j$ as follows:

$$Sect_j = \{x \in A \mid x \geq j \wedge \forall j' \in \xi (j < j' \to \neg(j' \leq x \bigvee x \leq j')\}.$$

Thus we have divided the arena into infinitely many sectors. Fig.1 presents this definition of sectors in the case when each node of the LRT has exactly tow immideate successors:



For every node p as above and any display $d \in Display(S)$, we define a congruence over the arena A as follows. First abbreviate $ELAR$ as a binary relation generated over the arena by the sectors and LAR:

$$ELAR(x,y) \Leftrightarrow ELAR_1(x,y) \bigvee \exists jj'(x \in Sect_j \& y \in Sect_{j'} \& j \neq j' \rightarrow LAR(j,d,x) = LAR(j'd,y)).$$

Define a congruence by

$$E^{d,p} = \{(x,y) \mid x,y \in A \wedge (A,C)_x \cong (A,C)_y \wedge ELAR(x,y)\}.$$

We abbreviate by $BooL(C) = Bool([C^s], s \in S)$ the collection of all the sets of paths from $\mathcal{A}$ which are boolean combinations of the sets $([C^s], s \in S)$.

Theorem 3.1 *Consider an arena A, a finite set of colors S, a list C of subsets of A colored by a corresponding color, a set of paths $W \in Bool(C)$ and a game cluster $\Gamma = (A, W, 0)$. Fix a node $p \in A$, a display $d \in Didsplay(S)$, and a congruence $E^{d,p}$. Then one of the players $\epsilon \in \{0,1\}$ wins the game cluster Γ and has a winning strat egy which strictly respects $E^{d,p}$.*

A partial case of this theorem is the Gurevich - Harrington's **determinacy theorem** [3]. A full proof of this theorem will appear in a more general form in [8].

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