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Asat Arslanov
University of Auckland, New Zealand

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Abstract

We study here the degree-theoretic structure of set-theoretical splittings of recursively enumerable (r.e.) sets into differences of r.e. sets. As a corollary we deduce that the ordering of **wtt**–degrees of unsolvability of differences of r.e. sets is not a distributive semilattice and is not elementarily equivalent to the ordering of r.e. **wtt**–degrees of unsolvability.

Keywords: Recursively enumerable sets, degrees of unsolvability, weak truth table reducibility.

1 Introduction and Notation

We review here the main notation and notions which will be used in this paper. All other notation and notions can be found in [27] and [26]. *Recursively enumerable* (*r.e.*) sets are the sets for which there exist Turing machines that effectively enumerate them. The set of all natural numbers is denoted by ω. A set $A \subseteq \omega$ is called *d–r.e.* (*difference of r.e. sets*) if there are r.e. sets of natural numbers $A_1, A_2 \subseteq \omega$ such that $A = A_1 - A_2$.

Let be $\{W_e\}_{e\in\omega}$ and $\{\varphi_e\}_{e\in\omega}$ be, respectively, the standard enumerations of recursively enumerable sets and partial recursive functions. We will denote by capital Greek letters Φ, Ψ, Γ partial recursive functionals (Turing reductions/Turing computations), and by capital Latin letters sets of natural numbers and their corresponding characteristic functions. For sets A and B, put $A \oplus B = \{2x : x \in A\} \bigcup \{2x+1 : x \in B\}$. A recursive enumeration of an infinite r.e. set is denoted by $\{A_s\}_{s\in\omega}$, where $|A_{s+1} - A_s| = 1$ and $a_s = A_{s+1} - A_s$; for a finite set X, $|X|$ denotes the cardinality of X. The same notation will be used for a recursive approximation of a d–r.e. set A with the property that for all x $|\{s : A_s(x) \neq A_{s+1}(x)\}| \leq 2$. Here A_s is the finite part of the set A enumerated at stage s. Denote by $\Phi_{e,s}(A_s, x)\downarrow$ the fact that the partial recursive (p.r.) functional with oracle A_s converges in s stages on the input x; $\Phi_{e,s}(A_s, x)\uparrow$ denotes divergence (i.e. there is no outcome of computation) at stage s. The function $\lambda x, y\langle x, y\rangle$ denotes a pairing of $\omega \times \omega$, i.e. a recursive bijection from $\omega \times \omega$ onto ω. Using this mapping one inductively gets computable coding of all n–tuples of numbers. The restriction of the set/function A to the initial segment of length $k+1$ is denoted by $A\lceil k+1 = \{x \in A : x \leq k\}$. For sets $A, B \subseteq \omega$, A is *Turing reducible* (*T–reducible*) to B, $A \leq_T B$, if there is an $e \in \omega$ such that for all x, $\Phi_e(B; x) = A(x)$. The use–function for $\Phi_e(A, x)$ is defined as follows:

$$use(\Phi_e(A, x)) = \begin{cases} \mu y[\Phi_e(A\lceil y+1; x)\downarrow = \Phi_e(A; x)\downarrow], \\ \textbf{undefined}, \text{ otherwise.} \end{cases}$$

Here we use the standard μ notation for the minimization operator. As usual we assume that the use–function has the following property that for every e, s, A, x if $\Phi_{e,s}(A_s; x)\downarrow$ then $e, x, use(\Phi_{e,s}(A_s; x)) < s$. The set A is *weakly truth table reducible* to B, $A \leq_{wtt} B$, if there exist $e_0, e_1 \in \omega$ such that for all x , $\Phi_{e_0}(B; x) = A(x)$, and for all x, $\phi_{e_1}(x)\downarrow$ and $use(\Phi_{e_0}(B; x)) \leq \phi_{e_1}(x)$, that is, A is Turing reducible to B and the use–function of the Turing reduction is majorised by some total recursive function. We use

[*]Computer Science Department, The University of Auckland, New Zealand, email: aars01@cs.auckland.ac.nz.

here wtt–functionals defined as follows. Let $\{(\Phi_e, \phi_e)\}_{e\in\omega}$ be some enumeration of all pairs of partial recursive (p.r.) functionals and p.r. functions. Then define

$$\widehat{\Phi}_e(A;x) = \begin{cases} \Phi_e(A;x)\downarrow \text{ and } use(\Phi_e(A;x)) \leq \phi_e(x)\downarrow, \\ \textbf{undefined}, \text{ otherwise.} \end{cases}$$

The $\widehat{\Phi}_{e,s}(A;x)$–computation of the wtt–functional, executed in s stages, is defined analogously. It is clear that $A \leq_{wtt} B$ is equivalent to $\widehat{\Phi}_e(B) = A$, for some e. From now on we will be omitting the superscript symbols and that of the stage s when from the context it will be clear that we deal with wtt–functionals and computations at stage s. We will say that the set A *wtt* $-$ $(T-)$ *computes* the set B if $B \leq_{wtt} A$ $(B \leq_T A)$.

Equivalence classes induced by these reducibility relations are called **T**– (**wtt**–) *degrees of unsolvability.* The **T**–degree (sometimes called *Turing* degree) of A is denoted by the corresponding bold Latin letter **a** or $deg(A)$, and the **wtt**–degree of A by the corresponding bold capital Latin letter. A degree of unsolvability is called recursively enumerable (d–r.e.) if it contains an r.e. (d–r.e.) set.

There exists another equivalent way to define r.e. and d–r.e. sets which is by recursive approximation to their characteristic functions with at most one and two changes in the approximation, respectively: for a given set A we start by guessing that x is not in A and we may change our guess about the membership of x in A at most once in the r.e. case and twice in the d–r.e case, namely when we enumerate x into A and when we extract it from A. If to allow the approximation to change more often this approach leads to the definition of a more general and natural concept of a *n–recursively enumerable* set which includes the definitions for the r.e. and d–r.e. sets as particular cases.

A set $A \subseteq \omega$ is called *n–recursively enumerable (n–r.e.)* if there is a recursive function f such that for all x:

1. $A(x) = \lim_s f(x,s)$,
2. $f(x,0) = 0$,
3. $|\{s : f(x,s) \neq f(x,s+1)\}| \leq n$.

Then the class of all r.e. sets coincides with the class of 1–r.e. sets, and the class of differences of r.e. sets coincides with the class of 2–r.e. sets. Also, let us notice the well known fact (see [18]) that a set A is n–r.e. if and only if it can be represented as a boolean combination $(A_1 - A_2) \cup \ldots \cup (A_{n-1} - A_n)$ of n r.e. sets $A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1$.

The Ershov's hierarchy of recursively approximated sets was first introduced and studied by Putnam (see [25]) and Ershov (see [18]). The Turing degrees of n–r.e. sets were first studied by Cooper and Lachlan (see [19]). It was shown by Cooper (see [5]) that the n–r.e. sets form a proper degree hierarchy below $\mathbf{0}'$, the degree of Halting Problem, that is, there are, for each $n \geq 1$, $(n+1)$–r.e. sets of the Turing degree which doesn't contain n–r.e. sets.

The set of all n–r.e. **wtt**– and Turing degrees is denoted by $D_{n,wtt}$ and D_n, respectively. Denote by $\mathbf{D_{n,wtt}} \stackrel{\text{def}}{=} \langle D_{n,wtt}; \leq, \bigcup, \bigcap \rangle$ the partial ordering of n–r.e. **wtt**–degrees. In $\mathbf{D_{n,wtt}}$ one can naturally define the operation of least upper bound and the partial operation of greatest lower bound. An n–r.e. **wtt**–degree **A** is called *branching* if there are n–r.e. **wtt**–degrees **B** and **C** different from **A** such that **A** is the infimum of **B** and **C**, and **A** is *nonbranching* otherwise.

2 One Example of the Difference Splitting of R.E. Set

Weak truth table reducibility (wtt–reducibility) has been studied in the theory of recursive functions for a long time (it was introduced by Friedberg and Rogers, see [20]) and turned out to be an important concept of investigation of the lattice of r.e. sets and the algebraic structure of partial ordering of r.e. Turing degrees (see [2, 3, 4, 8, 9, 10, 11, 13, 16, 17, 14, 15, 23, 24, 22, 28]). This notion is useful in effective algebra, where, for example, it was used by Downey and Remmel in their solution of the classification problem of the algorithmic complexity of r.e. bases of r.e. vector spaces (see [15, 14]). Actually they proved that r.e. **wtt**–degrees which are below (in the ordering induced by wtt–reducibility) than **wtt**–degree of the given vector space V are exactly **wtt**–degrees of r.e. bases of this space V.

In this paper we study the degree–theoretic structure (under *wtt*–reducibility) of *d*–r.e. splittings of r.e. sets.

Definition 2.1. *By the* **difference (r.e.) splitting** *of r.e. set A we call two d–r.e. (r.e.) sets D_1 and D_2 such that $D_1 \bigcup D_2 = A$ and $D_1 \bigcap D_2 = \emptyset$, where $\bigcup$ and $\bigcap$ – standard set–theoretic operations.*

The degree–theoretic structure of difference splittings of r.e. sets has crucial differences from the degree–theoretic structure of r.e. splittings of r.e. sets. For example, for every r.e. splitting A_1, A_2 of any given r.e. set A, $A_i \leq_T A, i = 1, 2$ and $A_1 \oplus A_2 \equiv_T A$, while as we will show in the next statement there are difference splittings with exactly opposite properties.

Theorem 2.2. *There exists such a difference splitting D_1, D_2 of an r.e. set A so that $D_i \not\leq_T A$ and $deg(A) = deg(D_1 \oplus A) \cap deg(D_2 \oplus A)$.*

Proof. We will construct r.e. set A and, simultaneously, a splitting of A into two sets D_0, D_1, such that the following list of requirements is satisfied:

$$\begin{array}{ll} \mathcal{R}_{\langle e,i\rangle}: & D_i \neq \Phi_e(A) \text{ where } i = 0, 1; \\ \mathcal{N}_e: & \Phi_e(A \oplus D_0) = \Phi_e(A \oplus D_1) = f \text{ total function } \Longrightarrow f \leq_T A\,; \end{array}$$

Let us describe the strategies meeting these requirements. For the requirement $\mathcal{R}_{\langle e,i\rangle}$: numbers $x_{\langle e,i\rangle}$ that we will be using for the diagonalization strategy are taken from the $[\langle e, i\rangle]$–*section* of ω, i.e. from the set $\{\langle x, z\rangle : \langle x, z\rangle \in \omega$ and $z = \langle e, i\rangle\}$.

1. Wait for a stage s such that $\Phi_{e,s}(A; s) \downarrow= 0$.

2. Enumerate the number $x_{\langle e,i\rangle}$ into the set D_i and, thereby, into A. Restrict from further enumerations with priority $\langle e, i\rangle$ the interval $A_{s+1}\lceil\ use(\Phi(A; x_{\langle e,i\rangle}))$. Then we get the inequality

$$\Phi_{e,s}(A_s; x_{\langle e,i\rangle}) \neq D_{i,s+1}(x_{\langle e,i\rangle}) \downarrow= 1.$$

3. If for some later stage $t > s$, $\Phi_{e,t}(A_t; x_{\langle e,i\rangle}) \downarrow\neq 0$, then enumerate $x_{\langle e,i\rangle}$ into D_{1-i}, and again restrict $A_t\lceil use(\Phi_{e,t}(A_t; x_{\langle e,i\rangle}))$, thereby obtaining the final inequality.

It is clear to see that the strategy for the one such requirement imply only finite injuries to the strategies of lower priority. To satisfy the requirements $\mathcal{N}_e$ we are using minimal pair strategy (e.g., see [27, Chapter 9]). This strategy consists in the dropping its restraint at an e–expansionary stage of the construction to allow to the possible computation injury only one side of the oracle computations in the hypothesis of the $\mathcal{N}_e$–requirement and then in the restricting of the other side of oracle computations between e–expansionary stages by reimposing the restraint on the enumeration of numbers for the sake of a lower priority $\mathcal{R}_{\langle e,i\rangle}$–requirement. But in our case, because the set A belongs to oracles of both sides of computations there could be injuries of both sides. Besides this, the injuries of the same kind are possible because of transferring of the numbers from the set D_i to the set D_{1-i} to satisfy both the global requirement of set–theoretic splitting and some $\mathcal{R}_{\langle e,i\rangle}$. In the cases when at some stage s both sides of the computations on some number x and with oracles $A \oplus D_0$ and $A \oplus D_1$ are injured we will construct functional $f = \Theta_e(A)$ by enumerating as a marker for x the number which is greater than all numbers used until stage s of the construction, for example, $\langle e, x, s + 1\rangle$, into the set A. It is clear to see that all strategies cohere with each other and all requirements are satisfied. □

One theorem of D.Kaddah (see [21]) asserts that there exist nonrecursive r.e. **T**–degrees which are nonbranching in *d*–r.e. **T**–degrees. It implies the impossibility of extending the property pointed out in the previous statement to the all non T–complete r.e. sets. But the question remains: does every non *wtt*–complete r.e. set A could be split to the two differences of r.e. sets which are not *wtt*–computable in A, so, that the infimum of the **wtt**–degrees of relativizations of these sets with respect to A would be equal to the **wtt**–degree of A ? We are going to answer affirmatively to the question in the other paper.

To the present time there were found a number of properties which are possessed simultaneously by the all semilattices $\mathbf{D_{n,wtt}}, \mathbf{n} < \omega$. For example, there were proved next theorems:

1. (Ladner–Sasso, [22]) Density and splitting hold simultaneously in the r.e. **wtt**–degrees, i.e. the following statement

$$(\forall\ \mathbf{A},\mathbf{B})(\mathbf{A}<\mathbf{B}\implies(\exists\ \mathbf{C_0},\mathbf{C_1})(\mathbf{A}<\mathbf{C_0},\mathbf{C_1}<\mathbf{B}\ \wedge\ \mathbf{C_0}\bigcup\mathbf{C_1}=\mathbf{B}))$$

holds true in the algebraic structure $\mathbf{D_{1,wtt}}$.

1′. (see [1]) For a given $n\geq 2, n\in\omega$, density and splitting hold true simultaneously in $\mathbf{D_{n,wtt}}$.

2. (Ladner–Sasso, [22]) Anticupping property holds for every nonrecursive r.e. **wtt**–degree, i.e. the statement

$$(\forall\,\mathbf{A})(\mathbf{A}>\mathbf{0}\implies(\exists\mathbf{B}<\mathbf{A})(\forall\mathbf{C})(\mathbf{B}\bigcup\mathbf{C}\geq\mathbf{A}\Rightarrow\mathbf{B}\geq\mathbf{A}))$$

holds true in $\mathbf{D_{1,wtt}}$.

2′. (Downey, [9]) Strong anticupping property holds for every nonrecursive r.e. **wtt**–degree, i.e. (notation in the statement shows that first two quantifiers range through $\mathbf{D_1}$ and third one ranges through $\mathbf{D}(\leq\mathbf{0'})$)

$$(\forall\ \mathbf{A}\ r.e.)(\,\mathbf{A}>\mathbf{0}\implies(\exists\ \mathbf{B}\ r.e.\ <\mathbf{A})(\forall\mathbf{C}\ \Delta_0^2)(\mathbf{B}\bigcup\mathbf{C}\geq\mathbf{A}\Rightarrow\mathbf{B}\geq\mathbf{A}))$$

3. (Cohen, [6]) Every non *wtt*–complete r.e. **wtt**–degree is branching both in r.e. **wtt**–degrees and in n–r.e. **wtt**–degrees, for any $n\geq 2$.

Let's notice that the first two statements are among the most interesting structural properties (e.g. density/nondensity for partial orderings) that prove elementary non–equivalence of the partial orderings of r.e **T**–degrees and d–r.e. **T**–degrees (see, for example, [7]). All these facts point out to the existence of a great similarity in the structure of these partial orderings of **wtt**–degrees. Nevertheless, in the next paragraph it is proved that $\mathbf{D_{n,wtt}}$ is nondistributive semilattice, while it was shown by Lachlan (see [28]) that the partial ordering of r.e. **wtt**–degrees forms distributive semilattice.

3 On the Embedding of the Nondistributive Lattice Into d–R.E. wtt–Degrees

Theorem 3.1. *For every non* **wtt**–*complete set A there exist such an r.e. set E and such a difference splitting D_1, D_2 of the set E so that* **wtt**–*degrees of the sets $E\oplus A, D_1\oplus A, D_2\oplus A, D_1\oplus D_2\oplus A, A$ constitute the lattice–theoretic embedding of the modular lattice $\mathbf{M_5}$ into the upper semilattice of n–r.e.* **wtt**–*degrees, for any fixed $n\geq 2$.*

Proof. We will construct an r.e. set E and its difference splitting D_0, D_1 satisfying to the following infinite list of requirements: one global set–theoretic requirement $\mathcal{P}$:

$$\begin{cases} x\in A_{s+1}\backslash A_s\longrightarrow x\in D_{0,s+1}\text{ or }D_{1,s+1},\\ x\in D_{i,s}\backslash D_{i,s+1}\longrightarrow x\in D_{1-i,s+1}\backslash D_{1-i,s};\end{cases}$$

$$\begin{array}{rl} \mathcal{P}_e: & E\neq\Phi_e(A)\ ;\\ \mathcal{N}_e: & \Phi_e(D_0\oplus A)=\Phi_e(D_0\oplus A)=f\ \text{ total function }\implies f\leq_{wtt}A\,;\\ \mathcal{NP}_{\langle e,i\rangle}: & \Phi_e(E\oplus A)=\Phi_e(D_i\oplus A)=f\ \text{ total function }\implies f\leq_{wtt}A,\\ & \text{where } i=0,1;\end{array}$$

Lemma 3.2. *The* **wtt**–*degrees of the sets $A, E\oplus A, D_0\oplus A, D_1\oplus A, D_0\oplus D_1\oplus A$ that satisfy to the above list of requirements $\mathcal{P},\mathcal{P}_e,\mathcal{N}_e,\mathcal{NP}_e, e\in\omega$ constitute a lattice–theoretic embedding of the lattice $\mathbf{M_5}$ into the upper semilattice $\mathbf{D_{n,wtt}}$, for any fixed $n\geq 2$.*

Proof. 1. $E \oplus A \leq_{wtt} D_0 \oplus D_1 \oplus A$, because $E = D_0 \bigcup D_1$ and $D_0 \bigcap D_1 = \emptyset$. It is clear that $D_i \oplus E \leq_{wtt} D_0 \oplus D_1$.

2. $D_0 \oplus D_1 \leq_{wtt} D_i \oplus E$. It is sufficient to show that $D_i \leq_{wtt} D_{1-i} \oplus E$. The computation of $D_i(x)$: the query to the oracle $E : 2x+1 \in E$? If the answer is a positive one then the question follows to the oracle $D_{1-i} : 2x \in D_{1-i}$? If again we get a positive answer then $x \notin D_i$; if the answer is a negative one then $x \in D_i$. If $2x+1 \notin E$ then it is obvious that $x \notin D_i$. Thus $D_0 \oplus D_1 \oplus A \equiv_{wtt} D_i \oplus E \oplus A, i = 0, 1$.

3. Certainly, no one $D_i \oplus A, i = 0, 1$, *wtt*–computes the set $E \oplus A$ since otherwise $\mathcal{NP}_e$–requirements would imply $E \leq_{wtt} A$.

4. At the same time $E \oplus A$ doesn't *wtt*–compute no one $D_i \oplus A, i = 0, 1$: if $D_i \oplus A \leq_{wtt} E \oplus A$, then $D_i \oplus E \oplus A \leq_{wtt} E \oplus A \leq_{wtt} D_{1-i} \oplus E \oplus A$; but $D_{1-i} \oplus E \oplus A \not\leq_{wtt} D_i \oplus E \oplus A$ if to suppose that $D_{1-i} \oplus A \not\leq_{wtt} E \oplus A$, contradiction with 2. If to assume that $E \oplus A$ computes both sets D_i, that is $D_i \oplus A \leq_{wtt} E \oplus A, i = 0, 1$, then it would follow from the satisfaction of the $\mathcal{N}$–requirements that

$$D_0 \oplus A \leq_{wtt} D_0 \oplus A, E \oplus A \Longrightarrow D_0 \oplus A \leq_{wtt} A$$

and

$$D_1 \oplus A \leq_{wtt} D_1 \oplus A, E \oplus A \Longrightarrow D_1 \oplus A \leq_{wtt} A \Longrightarrow E \leq_{wtt} A,$$

since $E \leq_{wtt} D_0 \oplus D_1$, what is a contradiction with the conditions $\mathcal{R}_e, e \in \omega$. □

The requirements $\mathcal{P}_e$ will be satisfied by the modified Friedberg–Muchnik strategy; the requirements $\mathcal{N}_e$ — by the modified minimal pair strategy. Let us describe the main module of the strategy for $\mathcal{NP}$–requirements. It will consists of two strategies — the standard strategy of minimal pair and the variant of Downey's strategy from the Diamond theorem ([12]). It could be the result of the joint work of the Friedberg–Muchnik strategy and the attempt to satisfy the set–theoretic requirement about the splitting of the constructing set E that we should enumerate numbers simultaneously into E and and the one R_i for some i, thereby possibly destroying simultaneously both computations of the p.r. functionals $\Phi_e(E \oplus A; [l(\langle e, i\rangle, s) - 1])$ and $\Phi_e(R_i \oplus A; [l(\langle e, i\rangle, s) - 1]$ for some requirement $\mathcal{NP}_{\langle e,i\rangle}$. Let, for example, $x \in E_{p+1} \backslash E_p$ and $x \in D_{i,p+1} \backslash D_{i,p}$ and $x < r(\langle e, i\rangle, p+1)$. Then at the first $\langle e, i\rangle$–expansionary stage, if it exists at all, $s+1 > p+1$, there could be the change of computations of both p.r. functionals for $\mathcal{NP}$–requirement with the increase of the length of agreement between them, that is, for some $y < l(\langle e, i\rangle, ls(\langle e, i\rangle, p+1)) : \Phi_{e,s+1}(E \oplus A; y) \neq \Phi_{e,ls(\langle e,i\rangle,p+1)}(E \oplus A; y)$ and $\Phi_{e,s+1}(R_i \oplus A; y) \neq \Phi_{e,ls(\langle e,i\rangle,p+1)}(R_i \oplus A; y)$. In this case the strategy for the requirement $\mathcal{NP}_{\langle e,i\rangle}$ becomes active and achieves an inequality at the stage $s+1$ by the transferring the number x from the set R_i into the R_{1-i} thereby restoring its computation with oracle $R_i \oplus A$, that is,

$$\Phi_{e,s+1}(R_i \oplus A; y) = \Phi_{e,ls(\langle e,i\rangle,p+1)}(R_i \oplus A; y) =$$

$$= \Phi_{e,ls(\langle e,i\rangle,p+1)}(E \oplus A) \neq \Phi_{e,s+1}(E \oplus A; y).$$

To preserve inequality we are not going to change oracle $E \oplus A$ at the initial segment of length $\varphi(y)+1$.

Using the techniques of the priority method, all the above mentioned strategies easily cohere with each other with the one exception, which we will consider separately: it is when some $\mathcal{NP}_{\langle e,i\rangle}$–strategy α with finite outcome is situated on the tree of strategies below some $\mathcal{N}_e$–strategy or $\mathcal{NP}_{\langle e,i\rangle}$–strategy with an infinite outcome, that is, $\widehat{\beta\langle 0\rangle} \subseteq \alpha$. Let us suppose that at some stage $s+1$ the following situation holds for some $x < l(e, s+1) :$ $x \in E_{s+1} \backslash E_s$ and $x \in D_{i,s+1}$ and

$$\Phi_{e,s+1}(R_0 \oplus A; x) = \Phi_{e,s+1}(D_1 \oplus A; x) = q$$

and at all e–expansionary stages infinite outcome of the requirement $\mathcal{N}_e$ depends on x remaining in D_i. If then at some stage $t+1$ $\mathcal{NP}$–strategy α becomes active with this number x: $x \in D_{1-i,t+1} \backslash D_{1-i,t}$ and $x \notin D_{i,t+1}$, then the corresponding $\mathcal{N}$–strategy β could be injured by the changes to both oracles. Therefore at the next e–expansionary stage $u+1$ we should check if the computations of p.r. functionals in the requirement $\mathcal{N}_e$ are different ones: $\Phi_{e,u+1}(D_i \oplus A; x) = ?\, q$, and, if so, then we construct *wtt*–reduction $\Phi(A) = f$.

In the construction we are using the tree of strategies denoted by $\{0,1\}^{<\omega}$, where, as usual, the infinite outcome of strategy is denoted by 0, and the finite one by 1. The tree node α with $|\alpha| = 3e$ corresponds to the requirement $\mathcal{P}_e$, for the one with $|\alpha| = 3e+1$ – requirement $\mathcal{NP}_e$ and to the one with $|\alpha| = 3e+2$ – requirement $\mathcal{N}_e$. For α corresponding to $\mathcal{P}_e, \mathcal{N}_e$ and $\mathcal{NP}_e$ we are using the following auxiliary length of agreement functions:

$$lp(\alpha, s) = max\{x : (\forall y < x)(\Phi_{e,s}(A_s; y) = E_s(y))\};$$

$$l(\alpha, s) = max\{x : (\forall y < x)(\Phi_{e,s}(R_{0,s} \oplus A_s; y) = \Phi_{e,s}(R_{1,s} \oplus A_s; y))\};$$

$$ml(\alpha, s) = max\{l(\alpha, t) : t < s \text{ and } t \text{ is } \alpha - \text{stage}\};$$

$$L(\alpha, s) = max\{x : (\forall y < x)(\Phi_{e,s}(E_s \oplus A_s; y) = \Phi_{e,s}(R_{i,s} \oplus A_s; y))\};$$

$$M(\alpha, s) = max\{L(\alpha, t) : t < s \text{ and } t \text{ is } \alpha - \text{stage}\};$$

$$ls(\alpha, s) = max\{0, t : \ t < s \ \wedge \ l(\alpha, t) > ml(\alpha, t)\};$$

We recall that the stage $s+1$ is called α–expansionary one if it is an α–stage and $l(\alpha, s+1) > ml(\alpha, s+1)$; here under l and ml we mean the length of agreement functions for corresponding α. For every strategy α we fix some enumeration of the creative set K at the α–expansionary stages.

Construction. At stage 0 all the strategies are initialized, i.e. in the state when all parameters (if they are assigned) and computations are declared undefined.

Stage $s+1$. Approximation to the so called true path f (see [27, Chapter 14]) $\delta_{s+1} : \ |\delta_{s+1}| \leq s$. Let $\delta_{s+1}\lceil 0 = \emptyset$. Let we already have defined $\delta_{s+1}\lceil(n) = \alpha$. Now we define $\delta_{s+1}(n)$ by following the stated below conditions.

If $|\alpha| = 3e$, for some e, then execute corresponding action.

1. The strategy α doesn't have assigned number. If stage $k+1$ – is an α–expansionary stage then assign the number $x_\alpha \overset{\text{def}}{=} \langle c(\alpha), x_{k+1}\rangle$ as a *witness* of the strategy; here $x_{k+1} \in K_{k+1}$ and $c(\alpha)$ is a code of the node α in some fixed numbering of all finite binary sequences. Initialize all $\xi > \alpha$ and finish the stage.

2. For some witness $x_\alpha : \Phi_{e,s+1}(A_{s+1}; x_\alpha) \downarrow= 0$ and $E_{s+1}(x_\alpha) = 1$. Then let $\delta_{s+1}\lceil(n) = 0$.

3. For some witness x_α, $\Phi_{e,s+1}(A_{s+1}; x_\alpha) \downarrow= 0$ and $E_{s+1}(x_\alpha) = 0$. Then let $x_\alpha \in E_{s+1} \backslash E_s$. Initialize all $\xi > \alpha$ and finish the stage.

4. For some assigned witness $x_\alpha \Phi_{e,s+1}(A; x_\alpha) \uparrow$. Then let $\delta_{s+1}(n) = 1$.

If $|\alpha| = 3e+1$ and for some $e : e = \langle i, sg(j)\rangle$, where

$$sg(x) = \begin{cases} 1, x \geq 1; \\ 0, x = 0. \end{cases}$$

1. Stage $s+1$ is not α–expansionary. Then $\delta_{s+1}(n) = 1$.

2. Some strategy $\beta(P'_e) : \widehat{\alpha\langle 0\rangle} \subseteq \beta$ executed at some preceding α–expansionary stage $u+1$ point 3 with the witness x_β and for some $y < l(\alpha, u+1)$:

$$\Phi_{i,s+1}(E \oplus A; y) \neq \Phi_{i,u+1}(E \oplus A; y)$$

and

$$\Phi_{i,s+1}(R_{sg(j)} \oplus A; y) \neq \Phi_{i,u+1}(R_{sg(j)} \oplus A; y).$$

Then enumerate the number x_β from the set $R_{sg(j)}$ into $R_{1-sg(j)}$. Initialize all $\xi > \alpha$ and finish the stage.

3. In the case opposite to the previous two define $\delta_{s+1}(n) = 0$.

If $|\alpha| = 3e + 2$:

1. Stage $s + 1$ is not α–expansionary. Then $\delta_{s+1}(n) = 1$.

2. Some strategy $\beta(P_{e'}) : \widehat{\langle\alpha\rangle} \subseteq \beta''$ fulfilled at stage $ls(\alpha, ls(\alpha, s + 1))$ point 3 with witness x_β, some strategy $\beta'(\mathcal{NP}_{e''}) : \widehat{\alpha\langle 0\rangle} \subseteq \widehat{\beta'\langle 0\rangle} \subseteq \beta''$ fulfilled point 2 at stage $ls(\alpha, s)$, and for some $y < ls(\alpha, s + 1)$, where $s + 1$ is the k–th α–expansionary stage:

$$\Phi_{e,s+1}(R_0 \oplus A; y) \neq \Phi_{e,ls(\alpha,s+1)}(D_0 \oplus A; y) \text{ and}$$
$$\Phi_{e,s+1}(D_1 \oplus A; y) \neq \Phi_{e,s+1}(D_1 \oplus A; y).$$

Then enumerate the number $\langle e, y, k, \varphi_e(y)\rangle$ in A_{s+1}. Initialize all $\xi > \alpha$ and finish the stage.

3. In the case opposite to the preceding two cases define $\delta_{s+1}(n) = 0$.

Initialize all $\xi : \alpha <_L \xi$.

The end of stage $s + 1$.

The *true path* f is defined by induction as follows: $f\lceil 0 = \emptyset$. If $f\lceil n$ is defined then

$$f(n) = \mu\{k : k \in \{0, 1\} \ \& \forall s\, \exists t > s\, f\widehat{\lceil n\langle k\rangle} \subseteq \delta_t\}.$$

Now let us show that the function $\lambda n f(n)$ is defined everywhere and the strategy $f\lceil n$ satisfies the corresponding requirement.

Lemma 3.3. *For all positive integers n, $f\lceil n$ does exist and contributes at most finitely many times to construction). If $f\lceil n = \alpha$ is defined and is $\mathcal{N}$– or $\mathcal{NP}$–strategy with finite outcome, or $\mathcal{P}$–strategy, then the corresponding requirement is satisfied.*

Proof. By definition $f\lceil 0 = \emptyset$. Induction step: assume that Lemma holds for $\alpha = f\lceil m$, for $m < n$ and fix the least α–stage s after which α will never be initialized.

If $|\alpha| = 3e + 1$. Let's suppose that $\lim_s L(\alpha, s + 1) = \infty$ since otherwise the statement is obvious. Let's suppose that α acts after stage s; let $t+1$ be the least such stage. Then some $\mathcal{P}$–strategy β acted at the preceding α–expansionary stage and at the first after s α–expansionary stage $t+1$ for some $y < L(\alpha, t+1)$: $\Phi_{i,t+1}(E \oplus A; y) \neq \Phi_{i,t+1}(E \oplus A; y)$. Then the strategy $\alpha(\mathcal{NP})$ acts by enumerating the number x_β from $D_{sg(j)}$ into $D_{1-sg(j)}$ and restores the oracle $(D_{sg(j),t+1} \oplus A_{t+1})\lceil \underline{\varphi_e(y)} = (D_{sg(j),s+1} \oplus A_{s+1})\lceil \varphi_e(y)$ thereby receiving the inequality at stage $t + 1$. In that case $\delta\lceil(n + 1) = \widehat{\alpha\langle 1\rangle}, \mathcal{NP}_e$ is met and $\alpha(\mathcal{NP})$ will not be injured and doesn't act anymore.

If $|\alpha| = 3e$, that is, α is $\mathcal{P}$–strategy. Let's suppose that the corresponding requirement is not satisfied, i.e. $\lim_s lp(\alpha, s) = \infty$. This means that for some number z holds true the following statement:

$$(\forall x \in K)(x > z \longrightarrow (x \in K \longleftrightarrow (\exists t)\langle c(\alpha), x\rangle \in E_{t+1} \backslash E_t)) \longleftrightarrow$$
$$(\exists s > t)(a_s < \phi_e(\langle c(\alpha), x\rangle) \text{ and } a_s \in A_{s+1} \backslash A_s) \Longrightarrow K \leq_{wtt} A.$$

Hence there is such stage u at which α executes the point 3, i.e. $E(x_\alpha) = 1 \neq 0 = \Phi_e(A; x_\alpha) \downarrow$, and after which, by the assumption, the higher priority strategies don't act anymore and α initializes all $\xi > \alpha$ at stage u. Therefore for every α–stage $v > u$ α is in the state 4 and $\delta\lceil(n + 1) = \widehat{\alpha\langle 0\rangle}$. The variant when $|\alpha| = 3e + 2$ is also obvious. □

Lemma 3.4. *Let $\widehat{\alpha 0} \subset f$ for $|\alpha| = 3e + 1, 3e + 2$. Then the requirements $\mathcal{N}_e$ and $\mathcal{NP}_e$ are satisfied.*

Proof. By the preceding lemma we can fix the least α–stage $\underset{\smile}{s}$ such that α will neither be initialized nor be active one after stage s since otherwise it would be that $\widehat{\alpha 1} \subset f$. By the condition $lim_s\, l(\alpha, s) = \infty$. Let's fix arbitrary $x \in \omega$. Let s is the least stage $s > s_0 : s - \alpha$–expansionary and $l(\alpha, s) > x$, and

$$A_s \lceil \langle e, x, 2\phi_e(x), \phi_e(x)\rangle = A\lceil \langle e, x, 2\phi_e(x), \phi_e(x)\rangle.$$

Let $\Phi_{e,s_1}(D_{0,s_1} \oplus A_{s_1}; x) = \Phi_{e,s_1}(D_{1,s_1} \oplus A_{s_1}; x) = p$ and let $s_1 < s_2 < \ldots < s_n < \ldots$ are α–expansionary stage greater than s_1. Then

$$(\forall n)[\Phi_{e,s_n}(D_{0,s_n} \oplus A_{s_n}; y) = \Phi_{e,s_n}(D_{1,s_n} \oplus A_{s_n}; y) = p]$$

and $\Phi_e(D_i \oplus A; y) = p$, $i = 0, 1$. Notice that the numbers enumerated into A and $D_i, i = 0, 1$, could injure only one side of the equation, because the changes of both sides are coded into A and there exist only $2\varphi_e(x)$ changes in A which could make such injuries. □

□

Corollary 3.5. *For every incomplete r.e.* $\mathbf{wtt}$*–degree* $\mathbf{A}$ *there exists lattice theoretic embedding preserving null of the modular non-distributive lattice* $\mathbf{M_5}$ *into* $\mathbf{D_{2,wtt}}(\geq \mathbf{A})$.

Corollary 3.6. *For every incomplete r.e.* $\mathbf{wtt}$*–degree* $\mathbf{A}$ *the partial ordering* $\mathbf{D_{2,wtt}}(\geq \mathbf{A})$ *doesn't form distributive semilattice.*

Corollary 3.7. *For all positive integers* $n \geq 2$ *and for every incomplete r.e.* $\mathbf{wtt}$*–degree* $\mathbf{A}$ *the partial orderings* $\mathbf{D_{n,wtt}}(\geq \mathbf{A})$ *and* $\mathbf{D_{1,wtt}}(\geq \mathbf{A})$ *are not elementarily equivalent.*

The question remains if the structures $\mathbf{D_{n,wtt}}$ are all pairwise elementarily inequivalent for $n \geq 1$. The existence of many results which hold true simultaneously for all these structures with $n \geq 2$ suggests the following interesting conjecture: all the partial orderings $\mathbf{D_{n,wtt}}$ for $n \geq 2$ are pairwise elementarily equivalent.

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