



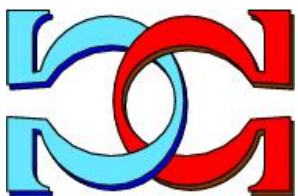
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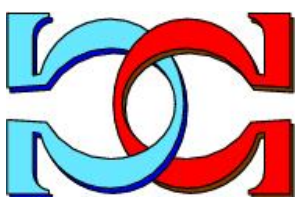
**A Correspondence Principle
for Exact Constructive
Dimension**



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A Correspondence Principle for Exact Constructive Dimension*

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Abstract

Exact constructive dimension as a generalisation of Lutz's [Lut00, Lut03] approach to constructive dimension was recently introduced in [Sta11]. It was shown that it is in the same way closely related to a priori complexity, a variant of Kolmogorov complexity, of infinite sequences as their constructive dimension is related to asymptotic Kolmogorov complexity.

The aim of the present paper is to extend this to the results of [Hit02, Hit05, Sta98] (see also [DH10, Section 13.6]) where it is shown that the asymptotic Kolmogorov complexity of infinite sequences in Σ_2^0 -definable sets is bounded by their Hausdorff dimension.

Using Hausdorff's original definition one obtains upper bounds on the a priori complexity functions of infinite sequences in Σ_2^0 -definable sets via the exact dimension of the sets.

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Lutz's [Lut00, Lut03] effectivisation of classical Hausdorff dimension led to the definition of constructive and computable dimensions of sets of infinite sequences. He put also the question of whether there is a correspondence principle stating that the constructive (or computable) dimension of sufficiently simple sets coincides with their Hausdorff dimension (cf. [Hit05]). A first positive answer for classical dimensions and sets definable by finite automata follows from the results of [Sta93, MS94], and for Σ_2^0 -definable sets positive answers were given in [Hit02, Hit05] and [Sta98].

In a recent paper [Sta11] the above mentioned results by Lutz and results by Ryabko were generalised from the case of 'usual' (classical) constructive and Hausdorff dimension to the case of exact dimension [GMW88, Hau18]. This concerns Lutz's martingale characterisation of Hausdorff dimension and Ryabko's [Rya84] (see

also [CH94]) determining of the dimension of the level sets of the constructive dimension (or asymptotic Kolmogorov complexity) of sets of infinite sequences.

Usually, the Hausdorff dimension (here also called classical Hausdorff dimension) of a set of reals is a real number α characterising a certain density or measure property of this set (see the textbooks [Edg08, Fal90] or [MS94]). If one looks to Hausdorff's original paper [Hau18], however, one finds that he defined the Hausdorff dimension to be a non-decreasing, right continuous function $h : (0, \infty) \rightarrow (0, \infty)$, nowadays called a *gauge function* [GMW88].

The paper [Sta11] provided a generalisation of the martingale characterisation of Hausdorff dimension and the determining of the dimension of the level sets to the case of exact dimension and to Kolmogorov complexity functions of infinite sequences.

In the papers [Hit02, Hit05] and [Sta98] (see also [DH10, Section 13.6]) a tight bound on the maximum asymptotic Kolmogorov complexity of sequences in Σ_2^0 -sets by its 'usual' Hausdorff dimension was presented and computable martingales successful on Σ_2^0 -sets with an exponent close to the Hausdorff dimension were constructed.

The purpose of the present paper is to generalise these results to a correspondence principle for the case of exact dimensions. This results also in a more precise bound on the maximum Kolmogorov complexity of sequences in Σ_2^0 -sets than the mere asymptotics given in the above mentioned papers.

The paper is organised as follows. After introducing some notation and some preliminaries on gauge functions and Hausdorff's original approach we present in Section 2 necessary results, mainly from [Sta11] on exact Hausdorff dimension, martingales and their effectivisation. Then Sections 3.1 and 3.2 show that the correspondence principles for constructive and computable dimensions hold for Σ_2^0 -definable sets of sequences and gauge functions satisfying some computability constraints. The proofs follow mainly the line of the proofs given in [Sta98] and are given in the appendix.

1 Notation and Preliminaries

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \dots\}$ we denote the set of natural numbers and by \mathbb{Q} the set of rational numbers. Let $X = \{0, 1, \dots, r-1\}$ be an alphabet of cardinality $|X| = r \geq 2$. By X^* we denote the set of finite words on X , including the *empty word* e , and X^ω is the set of infinite strings (ω -words) over X . Subsets of X^* will be referred to as *languages* and subsets of X^ω as *ω -languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$.

We denote by $|w|$ the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\eta)$ ($\eta \in X^* \cup X^\omega$) by $w \sqsubseteq \eta$, and $\eta \upharpoonright n$ is the n -length prefix of η provided $|\eta| \geq n$. The δ -*limit* of a language $V \subseteq X^*$ is the ω -language $V^\delta := \{\zeta : \zeta \in X^\omega \wedge |\mathbf{pref}(\zeta) \cap V| = \infty\}$. A language $W \subseteq X^*$ is referred to as *prefix-free* if $w \sqsubseteq v$ and $w, v \in W$ imply $w = v$.

For a computable domain \mathcal{D} , such as \mathbb{N} , \mathbb{Q} or X^* , we refer to a function $f : \mathcal{D} \rightarrow \mathbb{R}$ as *left computable* (or *approximable from below*) provided the set $\{(d, q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q < f(d)\}$ is computably enumerable. Accordingly, a function $f : \mathcal{D} \rightarrow \mathbb{R}$ is called *right computable* (or *approximable from above*) if the set $\{(d, q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q > f(d)\}$ is computably enumerable, and f is *computable* if f is right and left computable. In contrast to this we refer to a function $f : \mathcal{D} \rightarrow \mathbb{Q}$ as *computable* provided f returns the exact value $f(d) \in \mathbb{Q}$. Accordingly, a real number $\alpha \in \mathbb{R}$ is left computable, right computable or computable provided the constant function $c_\alpha(t) = \alpha$ is left computable, right computable or computable, respectively.

A *super-martingale* is a function $\mathcal{V} : X^* \rightarrow [0, \infty)$ which satisfies $\mathcal{V}(e) \leq 1$ and the super-martingale inequality

$$r \cdot \mathcal{V}(w) \geq \sum_{x \in X} \mathcal{V}(wx) \text{ for all } w \in X^*. \quad (1)$$

If Eq. (1) is satisfied with equality \mathcal{V} is called a *martingale*. Closely related with (super-)martingales are continuous (or cylindrical) (semi-)measures $\mu : X^* \rightarrow [0, 1]$ where $\mu(e) \leq 1$ and $\mu(w) \geq \sum_{x \in X} \mu(wx)$ for all $w \in X^*$.

1.1 Gauge functions and Hausdorff's original approach

A function $h : (0, \infty) \rightarrow (0, \infty)$ is referred to as a *gauge function* provided h is right continuous and non-decreasing.¹ If not stated otherwise, we will always assume that $\lim_{t \rightarrow 0} h(t) = 0$.

The h -dimensional outer measure of F on the space X^ω is given by

$$\mathcal{H}^h(F) := \liminf_{n \rightarrow \infty} \left\{ \sum_{v \in V} h(r^{-|v|}) : V \subseteq X^* \wedge F \subseteq V \cdot X^\omega \wedge \min_{v \in V} |v| \geq n \right\}. \quad (2)$$

If $\lim_{t \rightarrow 0} h(t) > 0$ then $\mathcal{H}^h(F) < \infty$ if and only if F is finite.

The usual α -dimensional Hausdorff measure \mathcal{H}^α is defined by gauge functions $h_\alpha(t) = t^\alpha, \alpha \in [0, 1]$, that is, $\mathcal{H}^\alpha = \mathcal{H}^{h_\alpha}$.

In this case the (usual or classical) Hausdorff dimension of a set $F \subseteq X^\omega$ is defined as

$$\dim_{\text{H}} F := \sup\{\alpha : \alpha = 0 \vee \mathcal{H}^\alpha(F) = \infty\} = \inf\{\alpha : \alpha \geq 0 \wedge \mathcal{H}^\alpha(F) = 0\}. \quad (3)$$

As we see from Eq. (2) for our purposes the behaviour of gauge function is of interest only in a small vicinity of 0. Moreover, in many cases we are not interested in the exact value of $\mathcal{H}^h(F)$ when $0 < \mathcal{H}^h(F) < \infty$. Thus we can often make use of scaling a gauge function and altering it in a range (ε, ∞) apart from 0.

The following properties of gauge functions h and the related measure \mathcal{H}^h are proved in the standard way.

Property 1 *Let h, h' be gauge functions.*

¹In fact, since we are only interested in the values $h(r^{-n}), n \in \mathbb{N}$, the requirement of right continuity is just to conform with the usual meaning (cf. [GMW88]).

1. If $c \cdot h(r^{-n}) \leq h'(r^{-n})$ for some $c > 0$, then $c \cdot \mathcal{H}^h(F) \leq \mathcal{H}^{h'}(F)$.
2. If $\lim_{n \rightarrow \infty} \frac{h(r^{-n})}{h'(r^{-n})} = 0$ then $\mathcal{H}^{h'}(F) < \infty$ implies $\mathcal{H}^h(F) = 0$, and $\mathcal{H}^h(F) > 0$ implies $\mathcal{H}^{h'}(F) = \infty$.

Here the first property implies a certain equivalence of gauge functions. In fact, if $c \cdot h \leq h'$ and $c \cdot h' \leq h$ in the sense of Property 1.1 then for all $F \subseteq X^\omega$ the measures $\mathcal{H}^h(F)$ and $\mathcal{H}^{h'}(F)$ are both zero, finite or infinite.

In the same way the second property gives a partial pre-order of gauge functions (see [GKP94, Section 9.1]). By analogy to the change-over-point $\dim_{\text{H}} F$ for $\mathcal{H}^\alpha(F)$ this partial pre-order yields a suitable notion of Hausdorff dimension in the range of arbitrary gauge functions.

Definition 1 We refer to a gauge function h as an *exact Hausdorff dimension function* for $F \subseteq X^\omega$ provided

$$\mathcal{H}^{h'}(F) = \begin{cases} \infty, & \text{if } \lim_{n \rightarrow \infty} \frac{h(r^{-n})}{h'(r^{-n})} = 0, \text{ and} \\ 0, & \text{if } \lim_{n \rightarrow \infty} \frac{h'(r^{-n})}{h(r^{-n})} = 0. \end{cases}$$

In fact, Hausdorff [Hau18] defined the dimension of a set F as an equivalence class of gauge functions $[h]$ such that $0 < \mathcal{H}^h(F) < \infty$. Property 1 shows that our definition covers this case.

Definition 1 is not as simple as the one of the classical Hausdorff dimension in Eq. (3), and it seems to be much more difficult to find the exact borderline, if it exists, between gauge functions with $\mathcal{H}^h(F) = 0$ and such with $\mathcal{H}^h(F) = \infty$.

2 Previous Results

2.1 Exact Hausdorff dimension and martingales

In this section we show a generalisation of Lutz's martingale characterisation of Hausdorff dimension to exact dimension.

Let $S_{c,h}[\mathcal{V}] := \{\xi : \xi \in X^\omega \wedge \limsup_{n \rightarrow \infty} \frac{\mathcal{V}(\xi[0..n])}{r^n \cdot h(r^{-n})} \geq c\}$, for a super-martingale $\mathcal{V} : X^* \rightarrow [0, \infty)$, a gauge function h and a value $c \in (0, \infty]$. In particular, $S_{\infty,h}[\mathcal{V}]$ is the set of all ω -words on which the super-martingale \mathcal{V} is *successful* w.r.t. the order function $f(n) = r^n \cdot h(r^{-n})$ in the sense of Schnorr [Sch71]. $S_{\infty,h}[\mathcal{V}]$ is also referred to as the *success set* of the super-martingale \mathcal{V} w.r.t. the order function $f(n) = r^n \cdot h(r^{-n})$.

Observe that $S_{c,h}[\mathcal{V}] \subseteq S_{c',h'}[\mathcal{V}]$ whenever $c, c' \in (0, \infty]$ and $\lim_{n \rightarrow \infty} \frac{h'(r^{-n})}{h(r^{-n})} = 0$.

Now we can generalise Lutz's result.

Theorem 1 ([Sta11, Theorem 1]) *Let $F \subseteq X^\omega$. Then a gauge function h is an exact Hausdorff dimension function for F if and only if*

1. *for all gauge functions h' with $\lim_{n \rightarrow \infty} \frac{h'(r^{-n})}{h(r^{-n})} = 0$ there is a super-martingale \mathcal{V} such that $F \subseteq S_{\infty,h'}[\mathcal{V}]$, and*
2. *for all gauge functions h'' with $\lim_{n \rightarrow \infty} \frac{h(r^{-n})}{h''(r^{-n})} = 0$ and all super-martingales \mathcal{V} it holds $F \not\subseteq S_{\infty,h''}[\mathcal{V}]$.*

2.2 Effectivisation of exact Hausdorff dimension

The constructive dimension is a variant of dimension defined analogously to Theorem 1 using only left computable super-martingales. For the usual family of gauge functions $h_\alpha(t) = t^\alpha$ it was introduced by Lutz [Lut00, Lut03] and resulted, similarly to \dim_{H} in a real number assigned to a subset $F \subseteq X^\omega$. In the case of left computable super-martingales the situation turned out to be even simpler than in the case of arbitrary super-martingales because the results of Levin [ZL70] and Schnorr [Sch71] show that there is an optimal left computable super-martingale \mathcal{U} , that is, every other left computable super-martingale \mathcal{V} satisfies $\mathcal{V}(w) \leq c_{\mathcal{V}} \cdot \mathcal{U}(w)$ for all $w \in X^*$ and some constant $c_{\mathcal{V}} > 0$ not depending on w . Thus we may define (cf. [Sta11])

Definition 2 Let $F \subseteq X^\omega$. We refer to $h : \mathbb{R} \rightarrow \mathbb{R}$ as an *exact constructive dimension function* for F provided $F \subseteq S_{\infty, h'}[\mathcal{U}]$ for all $h', \lim_{t \rightarrow 0} \frac{h(t)}{h'(t)} = 0$, and $F \not\subseteq S_{\infty, h''}[\mathcal{U}]$ for all $h'', \lim_{t \rightarrow 0} \frac{h''(t)}{h(t)} = 0$.

Originally, Levin [ZL70] showed that there is an optimal left computable continuous semi-measure \mathbf{M} on X^* . As usual, we call a function $\mu : X^* \rightarrow [0, \infty)$ a *continuous (or cylindrical) semi-measure* on X^* provided $\mu(e) \leq 1$ and $\mu(w) \geq \sum_{x \in X} \mu(wx)$ for all $w \in X^*$. One easily verifies that μ is a continuous semi-measure if and only if $\mathcal{V}(w) := r^{|w|} \cdot \mu(w)$ is a super-martingale. Thus we might use $\mathcal{U}_{\mathbf{M}}$ with $\mathcal{U}_{\mathbf{M}}(w) := r^{|w|} \cdot \mathbf{M}(w)$ as our optimal left computable super-martingale.

Closely related to Levin's optimal left computable semi-measure is the *a priori entropy (or complexity)* $\text{KA} : X^* \rightarrow \mathbb{N}$ defined by²

$$\text{KA}(w) := \lfloor -\log_r \mathbf{M}(w) \rfloor \quad (4)$$

The requirement $\text{KA}(w) \geq 0$ is one reason why we assumed $\mathbf{M}(e) \leq 1$.

The following theorem derives a bound for the set of sequences whose KA-complexity function is bounded.

Theorem 2 ([Sta11, Theorem 4]) *Let $-\infty < c < \infty$ and let h be a gauge function. Then there is a $c' > 0$ such that*

$$\{\xi : \text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log_r h(r^{-n}) + c\} \subseteq S_{c', h}[\mathcal{U}].$$

Conversely, if $\xi \in S_{c, h}[\mathcal{U}], c < \infty$, then from Eq. (4) one easily calculates $\text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log_r h(r^{-n}) + c''$ for some $c'' \in (0, \infty)$. Thus we obtain a complexity characterisation of the success sets of the universal super-martingale \mathcal{U} .

$$\bigcup_{c > 0} \{\xi : \text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log_r h(r^{-n}) + c\} = \bigcup_{c > 0} S_{c, h}[\mathcal{U}] \quad (5)$$

For gauge functions h' tending faster to 0 than h the following relations follow from $S_{c, h}[\mathcal{U}] \subseteq S_{\infty, h'}[\mathcal{U}]$.

²Here we follow the notation of [US96], in [DH10] a priori complexity was denoted by KM .

Corollary 1 *Let h, h' be gauge functions such that $\lim_{t \rightarrow 0} \frac{h'(t)}{h(t)} = 0$. Then*

1. $\{\xi : \exists c(\text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log_r h(r^{-n}) + c)\} \subseteq S_{\infty, h'}[\mathcal{U}]$, and
2. $\mathcal{H}^{h'}(\{\xi : \exists c(\text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log_r h(r^{-n}) + c)\}) = 0$.

3 The Results

In [Hit02, Hit05, Sta98] the correspondence principle could be stated for arbitrary (real) values of classical dimension. In the case of gauge functions the situation is more complicated. On the one hand because of the involved Definition 1, and, on the other hand, for the following reason (cf. also [Sta11, Remark 2]). Unlike the classical case where the computable (even the rational) numbers are dense in the reals, for gauge functions it holds that, if $\alpha \in (0, 1)$ is not a computable real, there is no computable function between $h_\alpha(t) = t^\alpha$ and $h_\alpha(t) = t^\alpha + \log_r \frac{1}{t}$.

First we mention the following general lower bound to the complexity function KA from [Mie08] together with Eq. (5) yields a tight estimate for gauge functions satisfying $F \not\subseteq S_{c, h''}[\mathcal{U}]$ for arbitrary $F \subseteq X^\omega$ (cf. Definition 2).

Theorem 3 ([Mie08]) *Let $F \subseteq X^\omega$, h be a gauge function and $\mathcal{H}^h(F) > 0$.*

Then for every $c > 0$ with $\mathcal{H}^h(F) > c \cdot \mathcal{U}(e)$ there is a $\xi \in F$ such that $\text{KA}(\xi[0..n]) \geq_{\text{a.e.}} -\log_r h(r^{-n}) - \log_r c$.

In order to obtain the announced upper bound, in view of Eq. (5) in the following two parts we show that for Σ_2^0 -definable subsets $F \subseteq X^\omega$ and gauge functions h satisfying some computability constraints there are left-computable super-martingales or computable martingales \mathcal{V} , respectively, such that $F \subseteq S_{\infty, h'}[\mathcal{V}]$ whenever $\mathcal{H}^h(F) = 0$ and $\lim_{t \rightarrow 0} \frac{h'(t)}{h(t)} = 0$.

3.1 Constructive Dimension

As in [Sta98] we ask now for an estimate of the condition $F \subseteq S_{c,h'}[\mathcal{U}]$ of Definition 2. The results use the following construction.

We start with an auxiliary lemma characterising subsets $F \subseteq X^\omega$ having null measure.

Lemma 1 ([Rei04]) *Let $F \subseteq X^\omega$ and h be a gauge function. Then $\mathcal{H}^h(F) = 0$ if and only if there is a language $V \subseteq X^*$ such that $F \subseteq V^\delta$ and $\sum_{v \in V} h(r^{-|v|}) < \infty$.*

The following theorem gives a constructive version of Lemma 1.

Theorem 4 *If $F \subseteq X^\omega$ is a Σ_2 -definable ω -language and h is a right computable gauge function such that $\mathcal{H}^h(F) = 0$ then there are a computable non-decreasing function $\bar{h} : \{r^{-i} : i \in \mathbb{N}\} \rightarrow \mathbb{Q}$ and a computable language $V \subseteq X^*$ satisfying*

1. $\bar{h}(r^{-n}) \geq h(r^{-n})$ for all $n \in \mathbb{N}$,
2. $F \subseteq V^\delta$ and $\sum_{v \in V} \bar{h}(r^{-|v|}) < \infty$.

Interpolating the computable function \bar{h} we obtain the following consequence.

Corollary 2 *If $F \subseteq X^\omega$ is a Σ_2 -definable ω -language and h is a right computable gauge function such that $\mathcal{H}^h(F) = 0$ then there is a computable non-decreasing function $\bar{h} : \mathbb{Q} \rightarrow \mathbb{Q}$ satisfying $\mathcal{H}^{\bar{h}}(F) = 0$ and $\bar{h}(t) \geq h(t)$ for $t \in \mathbb{Q} \cap (0,1)$.*

Our Theorem 4 yields the required upper bound for the prefix complexity KP, and hence also of the a priori complexity KA of an ω -word in F .

To this end we use the characterisation of KP via discrete semi-measures (cf. [DH10, US96]).³

³Here we follow also the notation of [US96], in [DH10] prefix complexity was denoted by K .

A mapping $\nu : X^* \rightarrow \mathbb{R}$ is referred to as a *discrete semi-measure* provided $\sum_{w \in X^*} \nu(w) < \infty$. It is known that there is an optimal left computable discrete semi-measure, that is, a left computable discrete semi-measure \mathbf{m} such that for every left computable discrete semi-measure ν there is a constant c_ν such that $\forall w (w \in X^* \rightarrow \nu(w) \leq c_\nu \cdot \mathbf{m}(w))$. This measure \mathbf{m} defines the prefix complexity (similarly as \mathbf{M} defines the a priori complexity KA) $\text{KP}(w) := \lfloor -\log_r \mathbf{m}(w) \rfloor$.

If $V \subseteq X^*$ is computably enumerable and $\bar{h} : \{r^{-n} > n \in \mathbb{N}\} \rightarrow \mathbb{R}$ is a left computable function such that $\sum_{v \in V} \bar{h}(r^{-|v|}) < \infty$ then

$$\nu(w) := \begin{cases} \bar{h}(r^{-|w|}), & \text{if } w \in V, \text{ and} \\ 0, & \text{otherwise} \end{cases} \quad (6)$$

defines a left computable discrete semi-measure. Thus Theorem 4 implies the following upper bound on the complexity functions of ω -words.

Lemma 2 *If $F \subseteq X^\omega$ is a Σ_2 -definable ω -language and h is a right computable gauge function such that $\mathcal{H}^h(F) = 0$ then*

$$\text{KP}(\xi[0..n]) \leq_{\text{i.o.}} -\log_r h(r^{-n}) + O(1) \text{ for all } \xi \in F, \text{ and}$$

$$\text{KA}(\xi[0..n]) \leq_{\text{i.o.}} -\log_r h(r^{-n}) + O(1) \text{ for all } \xi \in F.$$

The latter inequality follows from the former and $\text{KA}(w) \leq \text{KP}(w) + O(1)$ (see e.g. [DH10, US96]).

Finally, Lemma 2, Eq. (5) and Corollary 1 prove the following.

Theorem 5 *If $F \subseteq X^\omega$ is a union of Σ_2 -definable sets and h is a right computable gauge function such that $\mathcal{H}^h(F) = 0$ then $F \subseteq S_{\infty, h'}[\mathcal{U}]$ for every gauge function h' such that $\lim_{t \rightarrow 0} \frac{h'(t)}{h(t)} = 0$.*

3.2 Computable Dimension

Computable dimension is based on computable super-martingales as constructive dimension was based on left computable super-martingales. In contrast to the latter, for the former there is no

universal computable super-martingale (cf. [DH10, Sch71]). Thus we define analogously to Theorem 1

Definition 3 We refer to a gauge function h as an *exact computable dimension function* for $F \subseteq X^\omega$ provided

1. for all gauge functions h' with $\lim_{n \rightarrow \infty} \frac{h'(r^{-n})}{h(r^{-n})} = 0$ there is a computable super-martingale \mathcal{V} such that $F \subseteq S_{\infty, h'}[\mathcal{V}]$, and
2. for all gauge functions h'' with $\lim_{n \rightarrow \infty} \frac{h(r^{-n})}{h''(r^{-n})} = 0$ and all computable super-martingales \mathcal{V} it holds $F \not\subseteq S_{\infty, h''}[\mathcal{V}]$.

As for the constructive case the second item is fulfilled provided $\mathcal{H}^h(F) > 0$. For Item 1 we prove that for computable gauge functions h and Σ_2^0 -definable sets $F \subseteq X^\omega$ with $\mathcal{H}^h(F) = 0$ there is a computable martingale \mathcal{V} such that $F \subseteq \bigcup_{c>0} S_{c, h}[\mathcal{V}]$.

In order to achieve our goal we introduce families of covering codes as in [Sta98]. For a prefix code $C \subseteq X^*$ we define its *minimal complementary code* as

$$\widehat{C} := (X \cup \mathbf{pref}(C) \cdot X) \setminus \mathbf{pref}(C).$$

If $C = \emptyset$ we have $\widehat{C} = X$, and if $C \neq \emptyset$ the set \widehat{C} consists of all words $w \cdot x \notin \mathbf{pref}(C)$ where $w \in \mathbf{pref}(C)$ and $x \in X$. It is readily seen that $C \cup \widehat{C}$ is a maximal prefix code, $C \cap \widehat{C} = \emptyset$, and $\mathbf{pref}(C \cup \widehat{C}) = \{e\} \cup \mathbf{pref}(C) \cup \widehat{C}$.

We call $\mathfrak{C} := (C_w)_{w \in X^*}$ a *family of covering codes* provided each C_w is a finite prefix code. Since then the set $C_w \cup \widehat{C}_w$ is a finite maximal prefix code, every word $u \in X^*$ has a uniquely specified \mathfrak{C} -factorisation $u = u_1 \cdots u_n \cdot u'$ where $u_{i+1} \in C_{u_1 \cdots u_i} \cup \widehat{C}_{u_1 \cdots u_i}$ for $i = 0, \dots, n-1$ ($u_1 \cdots u_i = e$, if $i = 0$) and $u' \in \mathbf{pref}(C_{u_1 \cdots u_n} \cup \widehat{C}_{u_1 \cdots u_n})$. Analogously, every $\zeta \in X^\omega$ has a uniquely specified \mathfrak{C} -factorisation $\zeta = u_1 \cdots u_i \cdots$ where $u_{i+1} \in C_{u_1 \cdots u_i} \cup \widehat{C}_{u_1 \cdots u_i}$ for $i = 1, \dots$.

In what follows we use martingales derived from prefix codes in the following manner.

Lemma 3 Let $h : \mathbb{R} \rightarrow \mathbb{R}$ a gauge function and $\emptyset \neq C \subseteq X^*$ be a prefix code satisfying $\sum_{v \in C} h(r^{-|v|}) < \infty$. Then there is a martingale $\mathcal{V}_C^{(h)} : X^* \rightarrow [0, \infty)$ such that

$$\mathcal{V}_C^{(h)}(w) = \begin{cases} \frac{r^{|w|} \cdot h(r^{-|w|})}{\sum_{v \in C} h(r^{-|v|}) + \sum_{u \in \widehat{C}} r^{-|u|}} & , \text{ for } w \in C, \text{ and} \\ \frac{1}{\sum_{v \in C} h(r^{-|v|}) + \sum_{u \in \widehat{C}} r^{-|u|}} & , \text{ for } w \in \widehat{C}. \end{cases} \quad (7)$$

Remark 1 If C is a finite prefix code and $h : \mathbb{Q} \rightarrow \mathbb{Q}$ is computable then $\mathcal{V}_C^{(h)}$ is a computable martingale.

For a gauge function $h : \mathbb{R} \rightarrow \mathbb{R}$ let $h_w(t) := \frac{h(r^{-|w|} \cdot t)}{h(t)}$ and let $\mathfrak{C} := (C_w)_{w \in X^*}$ be a family of covering codes.

Using the martingales $\mathcal{V}_{C_w}^{(h_w)}$ we define a new martingale $\mathcal{V}_{\mathfrak{C}}$ as follows:

For $u \in X^*$ consider the \mathfrak{C} -factorisation $u_1 \cdots u_n \cdot u'$, and put

$$\mathcal{V}_{\mathfrak{C}}^{(h)}(u) := \left(\prod_{i=0}^{n-1} \mathcal{V}_{C_{u_1 \cdots u_i}}^{(h_{u_1 \cdots u_i})}(u_{i+1}) \right) \cdot \mathcal{V}_{C_{u_1 \cdots u_n}}^{(h_{u_1 \cdots u_n})}(u'),$$

that is, $\mathcal{V}_{\mathfrak{C}}^{(h)}$ is in some sense the concatenation of the martingales $\mathcal{V}_{C_w}^{(h_w)}$. Observe that $\mathcal{V}_{\mathfrak{C}}^{(h)}$ is computable if only $h : \mathbb{R} \rightarrow \mathbb{R}$ is a computable function, the codes C_w are finite and the function which assigns to every w the corresponding code C_w is computable.

We have the following.

Lemma 4 Let $h : \mathbb{N} \rightarrow \mathbb{Q}$ be a gauge function and let $\mathfrak{C} = (C_w)_{w \in X^*}$ be a family of covering codes such that $\sum_{v \in C_w} \frac{h(r^{-|wv|})}{h(r^{-|v|})} \leq r^{-|w|}$ for all $w \in X^*$.

If the ω -word $\xi \in X^\omega$ has a \mathfrak{C} -factorisation $\xi = u_1 \cdots u_i \cdots$ such that for some $n_\xi \in \mathbb{N}$ and all $i \geq n_\xi$ the factors u_{i+1} belong to $C_{u_1 \cdots u_i}$. Then there is a constant $c_\xi > 0$ not depending on i for which

$$\mathcal{V}_{\mathfrak{C}}(u_1 \cdots u_i) \geq c_\xi \cdot r^{|u_1 \cdots u_i|} \cdot h(r^{-|u_1 \cdots u_i|}).$$

Now we derive the announced result.

Theorem 6 For every Σ_2 -definable ω -language $F \subseteq X^\omega$ and every computable gauge function $h : \mathbb{Q} \rightarrow \mathbb{R}$ such that $\mathcal{H}^h(F) = 0$ there is a computable martingale \mathcal{V} such that $F \subseteq \bigcup_{c>0} S_{c,h}[\mathcal{V}]$.

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A Proofs

A.1 Proof of Theorem 4

Proof. Let $h_n : \mathbb{Q} \rightarrow \mathbb{Q}$, $n \in \mathbb{N}$ be computable approximations of h such that $h_n(t) \geq h_{n+1}(t) \geq h(t)$ and $\lim_{n \rightarrow \infty} h_n(t) = h(t)$ for $t \in (0, 1) \cap \mathbb{Q}$. As it was explained above we may assume that $h(r^{-n}) \geq r^n$. Moreover, the functions h_n are assumed to be non-decreasing on the set $\{r^{-n} : n \in \mathbb{N}\}$.

Furthermore, let $(U_j)_{j \in \mathbb{N}}$ be an effective enumeration of all finite prefix codes over X such that $\sup\{|v| : v \in U_j\} \leq \sup\{|v| : v \in U_{j+1}\}$, and let $F \in \Sigma_2$ be given by $F = \bigcup_{k \in \mathbb{N}} X^\omega \setminus L_k \cdot X^\omega$ where $M_F := \{(w, k) : w \in L_k\}$ is a computable set and the family of languages $(L_k)_{k \in \mathbb{N}}$ satisfies $L_k := \bigcap_{i=0}^k L_i \cdot X^*$ (cf. [Sta98]).

Define the predicate

$$\mathbf{test}(k, j, n) := \Leftrightarrow \left((U_j \cup (L_k \cap X^n)) \cdot X^\omega = X^\omega \wedge \sum_{v \in U_j} h_n(r^{-|v|}) < r^{-k} \right).$$

Observe that $\mathbf{test}(k, j, n)$ is computable and if $\mathbf{test}(k, j, n)$ is true then the conditions $F \subseteq U_j \cdot X^\omega$ and $\forall v (v \in U_j \rightarrow k < |v|)$ are satisfied.

The first condition follows from $L_k \cdot X^\omega \cap F = \emptyset$ and the second one from $h_n(r^{-|v|}) > r^{-2|v|}$.

Now the following algorithm, when given M_F , computes a finite prefix code C_k and a value m_k satisfying the conditions $F \subseteq C_k \cdot X^\omega$ and $\sum_{v \in C_k} h_{m_k}(r^{-|v|}) < r^{-k}$:

Algorithm C_k

```

0   input     $k$ 
1            $n = 0$ 
2           repeat   $j = -1$ 
3               repeat   $j = j + 1$ 
4                   until   $\mathbf{test}(k, j, n) \vee (\sup\{|v| : v \in U_j\} > n)$ 
5                    $n = n + 1$ 
6           until   $\mathbf{test}(k, j, n)$ 
7   output   $C_k := U_j, m_k := n$ 

```

By construction we have $k < |v| \leq m_k$ for $v \in C_k$.

Informally, for every $n \geq 0$ our algorithm successively searches for a U_j satisfying the condition $\text{test}(k, j, n)$, more precisely, it searches until such a U_j is found or else all U_j having $\sup\{|v| : v \in U_j\} \leq n$ fail to satisfy $\text{test}(k, j, n)$.

In the latter case the value of n is increased (thus allowing for a larger maximum codeword length, a larger complementary ω -language $(L_k \cap X^n) \cdot X^\omega$ and a closer approximation h_{n+1} of the gauge function h) and the search starts anew. Consequently, the algorithm terminates if and only if there is a finite prefix code U such that $\sum_{v \in U} h_n(r^{-|v|}) < r^{-k}$ and $U \cdot X^\omega \cup (L_k \cap X^n) \cdot X^\omega = X^\omega$ for some $n \in \mathbb{N}$.

First we show that our algorithm always terminates. Observe that for every $\varepsilon > 0$ there is a $W \subseteq X^*$ such that $F \subseteq W \cdot X^\omega$ and $\sum_{w \in W} h(r^{-|w|}) < \frac{\varepsilon}{2}$.

Since $X^\omega \setminus L_k \cdot X^\omega$ is a closed subset of F , for $\varepsilon \leq r^{-k}$ we find a finite subset $W' \subseteq W$ such that $X^\omega \setminus L_k \cdot X^\omega \subseteq W' \cdot X^\omega$. Then $\sum_{w \in W} h(r^{-|w|}) < \frac{\varepsilon}{2}$ implies that $\sum_{w \in W'} h_n(r^{-|w|}) < \varepsilon$ for $n \geq n_{\varepsilon, k}$, say.

Consequently, there is a finite prefix code $U_j \subseteq W'$ satisfying $(U_j \cup L_k) \cdot X^\omega = X^\omega$ and thus $(U_j \cup (L_k \cap X^n)) \cdot X^\omega = X^\omega$ for $n \geq n'_{j, k}$. This shows that the predicate $\text{test}(k, j, n)$ is satisfied whenever $n \geq \max\{n_{r^{-k}, k}, n'_{j, k}\}$.

Now we define $V := \bigcup_{i \in \mathbb{N}} C_i$ and show that V meets the requirements of the theorem. We have $w \in V$ if and only if $\exists i (i < |w| \wedge w \in C_i)$. This predicate is computable, since $i < |w|$ bounds the quantifier $\exists i$ from above. Thus the language V is computable.

Next we show that $F \subseteq V^\delta$. If $\zeta \in F$ there is an i_ζ such that $\zeta \in X^\omega \setminus L_i \cdot X^\omega$ for all $i \geq i_\zeta$. Hence, for every $i \geq i_\zeta$ the ω -word ζ has a prefix $w_i \in C_i$. As it was observed above, $|w_i| > i$. Consequently, ζ has infinitely many prefixes in $V = \bigcup_{i \in \mathbb{N}} C_i$.

Finally, in order to define the function \bar{h} we let $\ell_i := \max\{m_k : k < i\}$. Clearly, the value ℓ_i can be computed from i . Define $\bar{h}(r^{-i}) := h_{\ell_i}(r^{-i})$. Then $h_{m_k}(t) \geq h(t)$ implies $\bar{h}(r^{-i}) \geq h(r^{-i})$ and $\ell_i \leq \ell_{i+1}$ shows that $\bar{h}(r^{-i}) \geq \bar{h}(r^{-(i+1)})$.

It remains to show that $\sum_{v \in V} \bar{h}(r^{-|v|}) < \infty$. Taking into account that $k < |v| \leq m_k$, for $v \in C_k$, we have $\bar{h}(r^{-|v|}) = h_{\ell_{|v|}}(r^{-|v|}) \leq h_{m_k}(r^{-|v|})$ for $v \in C_k$ and thus

$$\begin{aligned} \sum_{v \in V} \bar{h}(r^{-|v|}) &\leq \sum_{k \in \mathbb{N}} \sum_{v \in C_k} h_{m_k}(r^{-|v|}) \\ &\leq \sum_{k \in \mathbb{N}} r^{-k} < \infty. \end{aligned}$$

A.2 Proof of Lemma 2

Proof. We use the computable subset $V \subseteq X^*$ and the computable function \bar{h} defined in the proof of Theorem 4 and define the discrete semi-measure ν via Eq. (6). Then $\nu(w) \leq c \cdot \mathbf{m}(w)$, for all $w \in X^*$ and, consequently $\text{KP}(w) \leq -\log_r \bar{h}(r^{-|w|}) \leq -\log_r h(r^{-|w|})$, for $w \in V$. The assertion follows from $F \subseteq V^\delta$.

A.3 Proof of Lemma 3

Proof. Set $\Gamma := \sum_{v \in C} h(r^{-|v|}) + \sum_{u \in \widehat{C}} r^{-|u|}$, and define for $u \in \text{pref}(C \cup \widehat{C}) \setminus (C \cup \widehat{C})$ and $w \in C \cup \widehat{C}$, $v \in X^*$

$$\begin{aligned} \mathcal{V}_C^{(h)}(u) &:= \frac{r^{|u|}}{\Gamma} \cdot \left(\sum_{u \cdot w \in C} h(r^{-|u \cdot w|}) + \sum_{u \cdot w \in \widehat{C}} r^{-|u \cdot w|} \right) \\ \mathcal{V}_C^{(h)}(w \cdot v) &:= \mathcal{V}_C^{(h)}(w). \end{aligned}$$

Then $\mathcal{V}_C^{(h)}$ fulfils Eq. (7). We still have to show the property $\mathcal{V}_C^{(h)}(u) = \frac{1}{r} \sum_{x \in X} \mathcal{V}_C^{(h)}(ux)$. This identity is obvious if $u \in (C \cup \widehat{C}) \cdot X^*$.

Now, let $u \notin (C \cup \widehat{C}) \cdot X^*$, that is, $u \in \text{pref}(C \cup \widehat{C}) \setminus (C \cup \widehat{C})$. Then

$$\begin{aligned} \sum_{x \in X} \frac{\mathcal{V}_C^{(h)}(ux)}{r} &= \sum_{x \in X} \frac{r^{|ux|}}{r \cdot \Gamma} \cdot \left(\sum_{uxw \in C} h(r^{-|uxw|}) + \sum_{uxw \in \widehat{C}} r^{-|uxw|} \right) \\ &= \frac{r^{|u|}}{\Gamma} \cdot \sum_{x \in X} \left(\sum_{uxw \in C} h(r^{-|uxw|}) + \sum_{uxw \in \widehat{C}} r^{-|uxw|} \right), \end{aligned}$$

because for $u \in \text{pref}(C \cup \widehat{C}) \setminus (C \cup \widehat{C})$ the set $\{w : w \in C \cup \widehat{C} \wedge u \sqsubseteq w\}$ partitions into the sets $\{w : w \in C \cup \widehat{C} \wedge ux \sqsubseteq w\}$ ($x \in X$), and the required equation follows.

A.4 Proof of Lemma 4

Proof. Since \widehat{C}_w is a code, we have $\sum_{v \in \widehat{C}_w} r^{-|v|} \leq 1$, and from the assumption we obtain

$$\sum_{v \in C_w} h_w(r^{-|v|}) + \sum_{v \in \widehat{C}_w} r^{-|v|} \leq r^{-|w|} + 1.$$

Now $|u_i| \geq 1$ implies $|u_1 \cdots u_i| \geq i$, and the above Eq. (7) yields

$$\mathcal{V}_{C_{u_1 \cdots u_i}}^{(\alpha)}(u_{i+1}) \geq \begin{cases} \frac{1}{r^{-i} + 1} = \frac{r^i}{1 + r^i} & , \text{ if } i \leq n_\xi, \text{ and} \\ \frac{r^{|u_{i+1}|} \cdot h_w(r^{-|u_{i+1}|})}{r^{-i} + 1} & , \text{ if } i > n_\xi. \end{cases} \quad (8)$$

Put

$$c_\xi := \prod_{i=0}^{\infty} \frac{r^i}{1 + r^i} \cdot \prod_{i=0}^{n_\xi} r^{|u_{i+1}|} \cdot h_w(r^{-|u_{i+1}|}) = r^{|u_1 \cdots u_{n_\xi}|} \cdot h(r^{-|u_1 \cdots u_{n_\xi}|}) \cdot \prod_{i=0}^{\infty} \frac{r^i}{1 + r^i}.$$

Clearly, $c_\xi > 0$, and using Eq. (8) by induction on i the assertion is easily verified.

A.5 Proof of Theorem 6

Proof. We use computable approximations $h_n : \mathbb{Q} \rightarrow \mathbb{Q}$ of h such that $h_n(t) \leq h_{n+1}(t)$ and $h_n(t) \leq h(t) \leq (1 + r^{-n}) \cdot h_n(t)$ for $t \in (0, 1) \cap \mathbb{Q}$.

In virtue of Lemma 4 it suffices to construct a computable family of covering codes $\mathfrak{C} = (C_w)_{w \in X^*}$ such that the function which assigns to every w the corresponding finite prefix code C_w is computable.

To this end we modify the predicate **test** introduced in the proof of Theorem 4 as follows:

$$\begin{aligned} \mathbf{test}'(w, j, n) & :\Leftrightarrow \left(n \geq |w| \wedge (w \cdot U_j \cup (L_{|w|} \cap X^{|w|+n})) \cdot X^\omega \supseteq w \cdot X^\omega \right. \\ & \quad \left. \wedge \sum_{v \in U_j} \frac{(1 + r^{-n}) \cdot h_n(r^{-|wv|})}{h_n(r^{-|w|})} < r^{-|w|} \right). \end{aligned}$$

In the same way we modify the algorithm presented there.

Algorithm C_w

```

0   input   $w$ 
1        $n = 0$ 
2       repeat   $j = -1$ 
3           repeat   $j = j + 1$ 
4           until   $\text{test}'(w, j, n) \vee (\sup\{|v| : v \in U_j\} > n)$ 
5            $n = n + 1$ 
6       until   $\text{test}'(w, j, n)$ 
7   output  $C_w := U_j$ 

```

Similar to the proof of Theorem 4 this algorithm computes a prefix code C_w with $\sum_{v \in C_w} \frac{h(r^{-|wv|})}{h(r^{-|w|})} < r^{-|w|}$ and $w \cdot C_w \cdot X^\omega \supseteq w \cdot X^\omega \setminus L_{|w|} \cdot X^\omega$.

Next we show that under the hypotheses of the theorem the algorithm always terminates. We have $\mathcal{H}^h(F \cap w \cdot X^\omega) = 0$ for all $w \in X^*$. Thus for $w \in X^*$ and every $\varepsilon > 0$ there is a prefix-free language $W \subseteq X^*$ such that $F \cap w \cdot X^\omega \subseteq W \cdot X^\omega$ and $\sum_{v \in W} h(r^{-|v|}) < \varepsilon \cdot \frac{h_0(r^{-|w|})}{1+r^{-|w|}}$. As in the proof of Theorem 4, in view of $F \supseteq X^\omega \setminus L_{|w|} \cdot X^\omega$, there is a finite subset $W' \subseteq W$ such that $w \cdot X^\omega \setminus L_{|w|} \cdot X^\omega \subseteq W' \cdot X^\omega$. Consequently, for n large enough the condition $\text{test}'(w, j, n)$ will be satisfied for suitable j .

It remains to show that every $\xi \in F$ has a \mathfrak{C} -factorisation $\xi = u_1 \cdots u_i \cdots$ such that almost all factors u_{i+1} belong to the corresponding codes $C_{u_1 \cdots u_i}$.

Let $\xi \in F$. Then there is a $k \in \mathbb{N}$ such that $\xi \in X^\omega \setminus L_i \cdot X^\omega$ for all $i \geq k$. Consequently, $w \in \mathbf{pref}(\xi)$ implies $w \notin L_k = L_k \cdot X^*$, and according to the definition of \mathfrak{C} there is a $u \in C_w$ such that $w \cdot u \in \mathbf{pref}(\xi)$ whenever $|w| \geq k$.