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Continued Fractions of Transcendental Numbers

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Continued fractions of transcendental numbers

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Abstract. *We establish new combinatorial transcendence criteria for continued fraction expansions. Let* $\alpha = [0; a_1, a_2, \ldots]$ *be an algebraic number of degree at least three. One of our criteria implies that the sequence of partial quotients* $(a_\ell)_{\ell \ge 1}$ *of* α *cannot be generated by a finite automaton, and that the complexity function of* $(a_\ell)_{\ell \ge 1}$ *cannot increase too slowly.*

1. Introduction and results

A well-known open question in Diophantine approximation asks whether the continued fraction expansion of an irrational algebraic number α either is ultimately periodic (this is the case if, and only if, α is a quadratic irrational), or it contains arbitrarily large partial quotients. As a preliminary step towards its resolution, several transcendence criteria for continued fraction expansions have been established recently [1, 4, 5, 9] (we refer the reader to these papers for references to earlier works, which include [20, 14, 12]) by means of a deep tool from Diophantine approximation, namely the Schmidt Subspace Theorem. In the present note, we show how a slight modification of their proofs allows us to considerably improve two of these criteria. We begin by pointing out two important consequences of one of our new criteria. Thus, we solve two problems addressed and discussed in [1] and we establish for continued fraction expansions of algebraic numbers the analogues of the results of [3] on the expansions of algebraic numbers to an integer base.

Throughout this note, $\mathcal{A}$ denotes a finite or infinite set, called the alphabet. We identify a sequence $\mathbf{a} = (a_\ell)_{\ell \ge 1}$ of elements from $\mathcal{A}$ with the infinite word $a_1 a_2 \ldots a_\ell \ldots$, as well denoted by $\mathbf{a}$. This should not cause any confusion.

For $n \ge 1$, we denote by $p(n, \mathbf{a})$ the number of distinct blocks of n consecutive letters occurring in the word $\mathbf{a}$. The function $n \mapsto p(n, \mathbf{a})$ is called the complexity function of $\mathbf{a}$. A well-known result of Morse and Hedlund [21, 22] asserts that $p(n, \mathbf{a}) \ge n + 1$ for $n \ge 1$, unless $\mathbf{a}$ is ultimately periodic (in which case there exists a constant C such that $p(n, \mathbf{a}) \le C$ for $n \ge 1$).

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Our first result asserts that the complexity function of the sequence of partial quotients of an algebraic number of degree at least three cannot increase too slowly.

Theorem 1. *Let* $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ *be a sequence of positive integers which is not ultimately periodic. If the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots]$$

is algebraic, then

$$\lim_{n\to+\infty} \frac{p(n,\mathbf{a})}{n} = +\infty. \tag{1.1}$$

Theorem 1 improves Theorem 7 from [12] and Theorem 4 from [1], where

$$\lim_{n\to+\infty} p(n,\mathbf{a}) - n = +\infty$$

was proved instead of (1.1). This gives a positive answer to Problem 3 of that paper (we have chosen here a different formulation).

An infinite sequence $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ is an automatic sequence if it can be generated by a finite automaton, that is, if there exists an integer $k \geq 2$ such that a_ℓ is a finite-state function of the representation of ℓ in base k, for every $\ell \geq 1$. We refer the reader to [13] for a more precise definition and examples of automatic sequences. Let $b \geq 2$ be an integer. In 1968, Cobham [18] asked whether a real number whose b-ary expansion can be generated by a finite automaton is always either rational or transcendental. A positive answer to Cobham's question was recently given in [3]. We addressed in [1] the analogous question for continued fraction expansions. Since the complexity function of every automatic sequence $\mathbf{a}$ satisfies $p(n,\mathbf{a}) = O(n)$ (this was proved by Cobham [19] in 1972), Theorem 1 implies straightforwardly a negative answer to Problem 1 of [1].

Theorem 2. *The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a finite automaton.*

The proofs of Theorems 1 and 2 rest ultimately on a combinatorial transcendence criterion established by means of the Schmidt Subspace Theorem. This is as well the case for the similar results about expansions of irrational algebraic numbers to an integer base, see [3, 10].

Before stating our criteria, we introduce some notation. The length of a word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. For any positive integer k, we write W^k for the word $W \ldots W$ (k times repeated concatenation of the word W). More generally, for any positive real number x, we denote by W^x the word $W^{[x]}W'$, where W' is the prefix of W of length $\lceil (x - [x])|W| \rceil$. Here, and in all what follows, $[y]$ and $\lceil y \rceil$ denote, respectively, the integer part and the upper integer part of the real number y. We denote the mirror image of a finite word $W := a_1 \ldots a_\ell$ by $\overline{W} := a_\ell \ldots a_1$. In particular, W is a palindrome if, and only if, $W = \overline{W}$.

Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of elements from $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition $(*)$ if $\mathbf{a}$ is not ultimately periodic and if there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that:

(i) For every $n \geq 1$, either the word $W_n U_n V_n U_n$ or the word $W_n U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$;

(ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded from above;

(iii) The sequence $(|W_n|/|U_n|)_{n\geq 1}$ is bounded from above;

(iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Equivalently, the word $\mathbf{a}$ satisfies Condition (*) if there exists a positive real number ε such that, for arbitrarily large integers N, the prefix $a_1 a_2 \ldots a_N$ of $\mathbf{a}$ contains two disjoint occurrences of a word of length $[\varepsilon N]$ or it contains a word W of length $[\varepsilon N]$ and its mirror image $\overline{W}$, provided that W and $\overline{W}$ are disjoint.

We summarize our two new combinatorial transcendence criteria in the following theorem.

Theorem 3. *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. If $\mathbf{a}$ satisfies Condition $(*)$, then α is transcendental.

Theorem 3 is the combination of two transcendence criteria, a first one for stammering continued fractions (see Theorem 5; the terminology 'stammering' means that in (i) the word $W_n U_n V_n U_n$ is a prefix of the word $\mathbf{a}$ for infinitely many n) and a second one for quasi-palindromic continued fractions (see Theorem 6; the terminology 'quasi-palindromic' means that in (i) the word $W_n U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$ for infinitely many n). The condition that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ has to be bounded is not very restrictive, since it is satisfied by almost all real numbers (in the sense of the Lebesgue measure). Furthermore, it is clearly satisfied when $(a_\ell)_{\ell\geq 1}$ is bounded. However, this condition can be removed if $\mathbf{a}$ begins with arbitrarily large squares $U_n U_n$ (Theorem 2.1 from [9]) or with arbitrarily large palindromes $U_n \overline{U}_n$ (Theorem 2.1 from [5]).

Theorem 3 encompasses all the combinatorial transcendence criteria for continued fraction expansions established in [1, 4, 5, 9] under the assumption that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded.

Let $\mathbf{a}$ be a sequence of positive integers. If there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that $\mathbf{a}$ satisfies (i) to (iv) above and if, furthermore, $|W_n| < |U_n|$ for $n \geq 1$ (this is a crude simplification, one needs in fact a stronger assumption), then the transcendence of $[0; a_1, a_2, \ldots]$ was already proved in [1, 9, 5]. The novelty in Theorem 3 is that we allow $|W_n|$ to be large, provided however that the quotients $|W_n|/|U_n|$ remain bounded independently of n. This is crucial for the proofs of Theorems 1 and 2.

At present, we do not know any transcendence criterion involving palindromes for expansions to integer bases (see, however, [2]).

We end this section with an application of Theorem 5 to quasi-periodic continued fractions.

Theorem 4. *Consider the quasi-periodic continued fraction*

$$\alpha = [0; a_1, \ldots, a_{n_0-1}, \underbrace{a_{n_0}, \ldots, a_{n_0+r_0-1}}_{\lambda_0}, \underbrace{a_{n_1}, \ldots, a_{n_1+r_1-1}}_{\lambda_1}, \ldots],$$

where the notation implies that $n_{k+1} = n_k + \lambda_k r_k$ and the λ's indicate the number of times a block of partial quotients is repeated. Let $(p_\ell/q_\ell)_{\ell\geq1}$ denote the sequence of convergents to α. Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq1}$ is bounded. If the sequence $(a_\ell)_{\ell\geq1}$ is not ultimately periodic and

$$\liminf_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} > 1, \tag{1.2}$$

then the real number α is transcendental.

Theorem 4 improves Theorem 3.4 from [4], where, instead of the assumption (1.2), the stronger condition $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k > 2$ was needed.

Throughout the present note, the constants implied in $\ll$ are absolute.

2. Transcendence criterion for stammering continued fractions

In this section, we use the same notation as in [1] and we establish the part of Theorem 3 dealing with stammering continued fractions. Let $\mathbf{a} = (a_\ell)_{\ell\geq1}$ be a sequence of elements from $\mathcal{A}$. Let w and w' be non-negative real numbers with $w > 1$. We say that $\mathbf{a}$ satisfies Condition $(**)_{w,w'}$ if $\mathbf{a}$ is not ultimately periodic and if there exist two sequences of finite words $(U_n)_{n\geq1}$, $(V_n)_{n\geq1}$ such that:

(i) For every $n \geq 1$, the word $U_n V_n^w$ is a prefix of the word $\mathbf{a}$;

(ii) The sequence $(|U_n|/|V_n|)_{n\geq1}$ is bounded from above by w';

(iii) The sequence $(|V_n|)_{n\geq1}$ is increasing.

Theorem 5. *Let $\mathbf{a} = (a_\ell)_{\ell\geq1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq1}$ is bounded. If there exist non-negative real numbers w and w' with $w > 1$ such that $\mathbf{a}$ satisfies Condition $(**)_{w,w'}$, then α is transcendental.

Theorem 5 improves Theorem 2 from [1] and Theorem 3.1 from [9], where the assumption $w > ((2\log M/\log m) - 1)w' + 1$ was required, with $M = \limsup_{\ell\to+\infty} q_\ell^{1/\ell}$ and $m = \liminf_{\ell\to+\infty} q_\ell^{1/\ell}$. Furthermore, it contains Theorem 3.2 from [4].

The fact that Theorem 5 implies the stammering part of Theorem 3 is easy to see. Indeed, let $\mathbf{a} = a_1 a_2 \ldots$ be an infinite word over $\mathbf{Z}_{\geq1}$. Assume that there exist three sequences of finite words $(U_n)_{n\geq1}$, $(V_n)_{n\geq1}$ and $(W_n)_{n\geq1}$ satisfying $(ii), (iii)$ and (iv) of Condition $(*)$ and such that $\mathbf{a}$ begins with $W_n U_n V_n U_n$ for $n \geq 1$. Then, by (ii) of Condition

$(*)$, there exists $w > 1$ such that $\mathbf{a}$ begins with $W_n(U_nV_n)^w$ for $n \geq 1$. It then follows from (iii) of Condition $(*)$ and Theorem 5 that $[0; a_1, a_2, \ldots]$ is transcendental if the assumption on the growth of the sequence of partial quotients is satisfied. Conversely, the stammering part of Theorem 3 clearly implies Theorem 5.

Theorem 5 is the exact analogue of the combinatorial transcendence criterion for expansions to integer bases proved in [10].

Proof.

Since this proof is very close to that of Theorem 2 in [1], we do not write it completely. The new ingredient is estimate (2.6) below.

Assume that the real numbers w and w' are fixed, as well as the sequences $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ occurring in the definition of Condition $(**)_{w,w'}$.

Set $r_n = |U_n|$ and $s_n = |V_n|$, for $n \geq 1$. We assume that the real number $\alpha := [0; a_1, a_2, \ldots]$ is algebraic of degree at least three. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to α. Recall that the theory of continued fraction implies that

$$|q_\ell \alpha - p_\ell| < q_{\ell+1}^{-1}, \quad \text{for } \ell \geq 1, \tag{2.1}$$

and

$$q_{\ell+h} \geq q_\ell (\sqrt{2})^{h-1}, \quad \text{for } h, \ell \geq 1. \tag{2.2}$$

We observe that α admits infinitely many good quadratic approximants obtained by truncating its continued fraction expansion and completing by periodicity. Precisely, for every positive integer n, we define the sequence $(b_k^{(n)})_{k\geq 1}$ by

$$\begin{aligned} b_h^{(n)} &= a_h \quad \text{for } 1 \leq h \leq r_n + s_n, \\ b_{r_n+h+js_n}^{(n)} &= a_{r_n+h} \quad \text{for } 1 \leq k \leq s_n \text{ and } j \geq 0. \end{aligned}$$

The sequence $(b_k^{(n)})_{k\geq 1}$ is ultimately periodic, with preperiod U_n and with period V_n. Set

$$\alpha_n = [0; b_1^{(n)}, b_2^{(n)}, \ldots, b_k^{(n)}, \ldots]$$

and note that, since the first $r_n + [ws_n]$ partial quotients of α and of α_n are the same, we have

$$|\alpha - \alpha_n| \leq q_{r_n+[ws_n]}^{-2}. \tag{2.3}$$

Furthermore, α_n is root of the quadratic polynomial

$$\begin{aligned} P_n(X) := &(q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})X^2 \\ &- (q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1} + p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1})X \\ &+ (p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}). \end{aligned}$$

By (2.1), we have

$$\begin{aligned} &|(q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})\alpha - (q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1})| \\ &\leq q_{r_n-1}|q_{r_n+s_n}\alpha - p_{r_n+s_n}| + q_{r_n}|q_{r_n+s_n-1}\alpha - p_{r_n+s_n-1}| \\ &\ll q_{r_n}\, q_{r_n+s_n}^{-1} \end{aligned} \tag{2.4}$$

and, likewise,

$$|(q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})\alpha - (p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1})| \\ \ll q_{r_n}^{-1} q_{r_n+s_n}. \tag{2.5}$$

Using (2.3), (2.4), and (2.5), we then get

$$\begin{aligned}
|P_n(\alpha)| &= |P_n(\alpha) - P_n(\alpha_n)| \\
&= |(q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})(\alpha-\alpha_n)^2 \\
&\quad - (q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1} + p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1})(\alpha-\alpha_n)| \\
&= |\alpha-\alpha_n| \cdot |(q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})\alpha - (q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1}) \\
&\quad + (q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})\alpha - (p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1}) \\
&\quad + (q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})(\alpha_n-\alpha)| \\
&\ll |\alpha-\alpha_n| \cdot \left(q_{r_n} q_{r_n+s_n}^{-1} + q_{r_n}^{-1} q_{r_n+s_n} + q_{r_n}q_{r_n+s_n}|\alpha-\alpha_n|\right) \\
&\ll |\alpha-\alpha_n| q_{r_n}^{-1} q_{r_n+s_n} \\
&\ll q_{r_n}^{-1} q_{r_n+s_n} q_{r_n+[ws_n]}^{-2}.
\end{aligned} \tag{2.6}$$

This estimate, more precise than (16) from [1] (namely, we gain a factor $q_{r_n}^{-2}$), is the main source for our improvement. Continuing exactly as in [1], we consider the four linearly independent linear forms:

$$\begin{aligned}
L_1(X_1,X_2,X_3,X_4) =& \alpha^2 X_1 - \alpha(X_2+X_3) + X_4, \\
L_2(X_1,X_2,X_3,X_4) =& \alpha X_1 - X_2, \\
L_3(X_1,X_2,X_3,X_4) =& \alpha X_1 - X_3, \\
L_4(X_1,X_2,X_3,X_4) =& X_1.
\end{aligned}$$

Evaluating them on the quadruple

$$\begin{aligned}
\underline{z}_n := (&q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1}, q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1}, \\
&p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1}, p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}),
\end{aligned}$$

it follows from (2.4), (2.5), (2.6), and (2.2) that

$$\begin{aligned}
\prod_{1\le j\le 4} |L_j(\underline{z}_n)| &\ll q_{r_n+s_n}^2 q_{r_n+[ws_n]}^{-2} \\
&\ll 2^{-(w-1)s_n} \\
&\ll (q_{r_n}q_{r_n+s_n})^{-\delta(w-1)s_n/(2r_n+s_n)},
\end{aligned}$$

if n is sufficiently large, where we have set

$$M = 1 + \limsup_{\ell\to+\infty} q_\ell^{1/\ell} \quad \text{and} \quad \delta = \frac{\log 2}{\log M}.$$

Thus, with $\eta = \delta(w-1)/(2w'+1)$, we see that

$$\prod_{1 \le j \le 4} |L_j(\underline{z}_n)| \ll (q_{r_n}\, q_{r_n+s_n})^{-\eta}$$

holds for any sufficiently large integer n. We have obtained exactly the same estimate as in [1], but under a weaker assumption.

Following the proof from [1], we apply a first time the Schmidt Subspace Theorem. It implies that the points $\underline{z}_n$ lie in a finite number of proper subspaces of $\mathbf{Q}^4$. As in [1], we deduce that there exists an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1} = p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1}$$

for n in $\mathcal{N}_1$. Thus, for n in $\mathcal{N}_1$, the polynomial $P_n(X)$ can simply be expressed as

$$\begin{aligned} P_n(X) := {}& (q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})X^2 \\ & - 2(q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1})X + (p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}). \end{aligned}$$

Consider now the three linearly independent linear forms:

$$\begin{aligned} L'_1(X_1, X_2, X_3) &= \alpha^2 X_1 - 2\alpha X_2 + X_3, \\ L'_2(X_1, X_2, X_3) &= \alpha X_1 - X_2, \\ L'_3(X_1, X_2, X_3) &= X_1. \end{aligned}$$

Evaluating them on the triple

$$\begin{aligned} \underline{z}'_n := (& q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1}, q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1}, \\ & p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}), \end{aligned}$$

for n in $\mathcal{N}_1$, it follows from (2.4) and (2.6) that

$$\prod_{1 \le j \le 3} |L'_j(\underline{z}'_n)| \ll q_{r_n}\, q_{r_n+s_n}\, q^{-2}_{r_n+[ws_n]} \ll (q_{r_n}\, q_{r_n+s_n})^{-\eta},$$

with the same η as above, if n is sufficiently large.

The rest of the proof remains unchanged and we refer the reader to [1]. □

3. Transcendence criterion for quasi-palindromic continued fractions

In this section, we use the same notation as in [5] and we establish the part of Theorem 3 dealing with quasi-palindromic continued fractions. Let $\mathbf{a} = (a_\ell)_{\ell \ge 1}$ be a sequence of elements from $\mathcal{A}$. Let w be a real number. We say that $\mathbf{a}$ satisfies Condition $(*)_w$ if $\mathbf{a}$ is not ultimately periodic and if there exist three sequences of finite words $(U_n)_{n\ge1}$, $(V_n)_{n\ge1}$ and $(W_n)_{n\ge1}$ such that:

(i) For every $n \ge 1$, the word $W_nU_nV_n\overline{U}_n$ is a prefix of the word $\mathbf{a}$;
(ii) The sequence $(|V_n|/|U_n|)_{n\ge1}$ is bounded from above;
(iii) The sequence $(|U_n|/|W_n|)_{n\ge1}$ is bounded from below by w;
(iv) The sequence $(|U_n|)_{n\ge1}$ is increasing.

It is understood that, if W_n is the empty word, then $|U_n|/|W_n|$ is infinite.

Theorem 6. *Let* $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ *be a sequence of positive integers. Let* $(p_\ell/q_\ell)_{\ell\geq 1}$ *denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ *is bounded. If there exists* $w > 0$ *such that* $\mathbf{a}$ *satisfies Condition* $(*)_w$*, then* α *is transcendental.*

Theorem 6 improves Theorem 2.4 from [5], where the assumption $w > (2\log M/\log m) - 1$ was required, with $M = \limsup_{\ell\to+\infty} q_\ell^{1/\ell}$ and $m = \liminf_{\ell\to+\infty} q_\ell^{1/\ell}$.

Proof.

We keep the notation of the proof of Theorem 2.4 from [5] and we explain which changes should be made in order to establish Theorem 6.

Assume that the real number w is fixed, as well as the sequences $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$. Set $r_n = |W_n|$, $s_n = |W_n U_n|$ and $t_n = |W_n U_n V_n \overline{U_n}|$, for $n \geq 1$. Assume that the real number $\alpha := [0; a_1, a_2, \ldots]$ is algebraic of degree at least three.

For $n \geq 1$, consider the rational number P_n/Q_n defined by

$$\frac{P_n}{Q_n} := [0; W_n U_n V_n \overline{U_n}\, \overline{W_n}]$$

and denote by P'_n/Q'_n the last convergent to P_n/Q_n which is different from P_n/Q_n. It has been proved in [5] that

$$|Q_n\alpha - P_n| < Q_n q_{t_n}^{-2}, \quad |Q'_n\alpha - P'_n| < Q_n q_{t_n}^{-2}, \tag{3.1}$$

$$|Q_n\alpha - Q'_n| < Q_n q_{s_n}^{-2}, \tag{3.2}$$

and

$$Q_n \leq 2 q_{r_n} q_{t_n} \leq 2 q_{s_n} q_{t_n}. \tag{3.3}$$

Inequality (3.2) is a consequence of the mirror formula

$$\frac{q_{\ell-1}}{q_\ell} = [0; a_\ell, a_{\ell-1}, \ldots, a_1], \quad \text{for } \ell \geq 1,$$

which is a key ingredient for the proof of the combinatorial transcendence criteria for quasi-palindromic continued fractions. Since

$$\begin{aligned}\alpha(Q_n\alpha - P_n) - (Q'_n\alpha - P'_n) &= \alpha Q_n\left(\alpha - \frac{P_n}{Q_n}\right) - Q'_n\left(\alpha - \frac{P'_n}{Q'_n}\right) \\ &= (\alpha Q_n - Q'_n)\left(\alpha - \frac{P_n}{Q_n}\right) + Q'_n\left(\frac{P'_n}{Q'_n} - \frac{P_n}{Q_n}\right),\end{aligned}$$

it follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned}|\alpha^2 Q_n - \alpha Q'_n - \alpha P_n + P'_n| &\ll Q_n q_{s_n}^{-2} q_{t_n}^{-2} + Q_n^{-1} \\ &\ll Q_n^{-1}.\end{aligned} \tag{3.4}$$

Instead of considering the four linearly independent linear forms with algebraic coefficients

$$\begin{aligned} L_1(X_1, X_2, X_3, X_4) =& \alpha X_1 - X_3, \\ L_2(X_1, X_2, X_3, X_4) =& \alpha X_2 - X_4, \\ L_3(X_1, X_2, X_3, X_4) =& \alpha X_1 - X_2, \\ L_4(X_1, X_2, X_3, X_4) =& X_2, \end{aligned}$$

as in [5], we introduce the linear form

$$L_5(X_1, X_2, X_3, X_4) = \alpha^2 X_1 - \alpha X_2 - \alpha X_3 + X_4,$$

and we deduce from (3.1), (3.2), (3.3) and (3.4) that

$$\prod_{2 \le j \le 5} |L_j(Q_n, Q'_n, P_n, P'_n)| \ll Q_n^2 \, q_{t_n}^{-2} \, q_{s_n}^{-2} \ll q_{r_n}^2 \, q_{s_n}^{-2}.$$

By (2.2) and (3.3), we have

$$q_{r_n}^2 \, q_{s_n}^{-2} \ll 2^{-|U_n|} \ll Q_n^{-\delta(s_n - r_n)/(r_n + t_n)},$$

if n is sufficiently large, where we have set

$$M = 1 + \limsup_{\ell \to +\infty} q_\ell^{1/\ell} \quad \text{and} \quad \delta = \frac{\log 2}{\log M}.$$

Consequently, since $\mathbf{a}$ satisfies Condition $(*)_w$, there exists $\varepsilon > 0$ such that

$$\prod_{2 \le j \le 5} |L_j(Q_n, Q'_n, P_n, P'_n)| \ll Q_n^{-\varepsilon},$$

for every sufficiently large n. We have obtained exactly the same estimate as in (6.18) from [5], but under a weaker assumption.

Following the proof from [5], we apply a first time the Schmidt Subspace Theorem. It implies that the points (Q_n, Q'_n, P_n, P'_n) lie in a finite number of proper subspaces of $\mathbf{Q}^4$. As in [5], we deduce that there exists an infinite set of distinct positive integers $\mathcal{N}_2$ such that $Q'_n = P_n$ for n in $\mathcal{N}_2$. Thus, for n in $\mathcal{N}_2$, we have

$$|\alpha^2 Q_n - 2\alpha Q'_n + P'_n| \ll Q_n^{-1}, \tag{3.5}$$

instead of (3.4). Consider now the three linearly independent linear forms:

$$\begin{aligned} L'_1(X_1, X_2, X_3) =& \alpha^2 X_1 - 2\alpha X_2 + X_3, \\ L'_2(X_1, X_2, X_3) =& \alpha X_2 - X_3, \\ L'_3(X_1, X_2, X_3) =& X_1. \end{aligned}$$

Evaluating them on the triple (Q_n, Q'_n, P'_n) for n in $\mathcal{N}_2$, it follows from (3.1), (3.3) and (3.5) that

$$\prod_{1\le j\le 3} |L'_j(Q_n, Q'_n, P'_n)| \ll Q_n q_{t_n}^{-2} \ll q_{r_n} q_{t_n}^{-1} \ll q_{r_n} q_{s_n}^{-1} \ll Q_n^{-\varepsilon/2},$$

with the same ε as above, if n is sufficiently large.

We then apply again the Schmidt Subspace Theorem and we continue as in the proof of Theorem 2.4 from [5]. We omit the details. □

4. Proofs of Theorems 1 and 4

Proof of Theorem 1.

Let $\mathbf{a}$ be as in the statement of the theorem. Assume that there is an integer $\kappa \ge 2$ such that the complexity function of $\mathbf{a}$ satisfies

$$p(n, \mathbf{a}) \le \kappa n \qquad \text{for infinitely many integers } n \ge 1. \tag{4.1}$$

This implies in particular that $\mathbf{a}$ is written over a finite alphabet, consequently the sequence $(q_\ell^{1/\ell})_{\ell\ge1}$ is bounded. In the proof of Theorem 1 from [3], it is shown that, under the assumption (4.1), the word $\mathbf{a}$ satisfies Condition $(**)_{1+1/\kappa,4\kappa}$. Consequently, Theorem 1 follows from Theorem 5. □

Proof of Theorem 4.

If the sequence $(r_k)_{k\ge0}$ is bounded, then Theorem 4 is Corollary 3.3 of [4]. Thus, we assume that $(r_k)_{k\ge0}$ is unbounded and we consider the infinite set $\mathcal{K}$ composed of the positive integers k such that $r_k > \max\{r_0, \ldots, r_{k-1}\}$. Let $\varepsilon > 0$ and k_0 be such that $\lambda_{k_0} > 2$ and $\lambda_{k+1} > (1+\varepsilon)\lambda_k$ for $k \ge k_0$. Let k be in $\mathcal{K}$ with $k > k_0$. Set

$$U_k = a_1 a_2 \ldots a_{n_k - 1}$$

and

$$V_k = (a_{n_k} \ldots a_{n_k + r_k - 1})^{[\lambda_k/2]}.$$

Observe that $\mathbf{a}$ begins with $U_k V_k^2$. Furthermore, setting

$$n'_0 = n_0 + \sum_{h=0}^{k_0-1} \lambda_h r_h,$$

we have

$$\begin{aligned} |U_k| &\le n'_0 + \sum_{h=k_0}^{k-1} \lambda_h r_h \\ &\le n'_0 + r_k \lambda_k \left(\frac{1}{1+\varepsilon} + \ldots + \frac{1}{(1+\varepsilon)^{k-k_0}} \right) \\ &\le n'_0 + r_k \lambda_k / \varepsilon \end{aligned}$$

and

$$|V_k| \geq (\lambda_k - 1) r_k / 2.$$

Consequently, there exists a positive real number w' such that the word $\mathbf{a} = a_1 a_2 \dots$ satisfies Condition $(**)_{2,w'}$. We conclude by applying Theorem 5. □

5. Concluding remarks

It seems that we are now able to get the analogues for continued fraction expansions to all the transcendence results established recently for expansions to an integer base and whose proofs ultimately rest on the Schmidt Subspace Theorem. For instance, combining the arguments of [11] with Theorem 3, it is easy to prove that if $1 \leq m < M$ are integers and $\mathbf{a} = a_1 a_2 \dots$ is a word over $\{m, M\}$ such that $[0; a_1, a_2, \dots, a_\ell, \dots]$ is algebraic, then there are arbitrarily large (finite) blocks W such that $W^{7/3}$ occurs in $\mathbf{a}$.

Further recent developments have shown that the use of quantitative versions of the Schmidt Subspace Theorem allows us often to strengthen or to complement results established by means of the qualitative Schmidt Subspace Theorem, see for instance the survey [16]. In particular, proceeding as in [6, 7, 8], it is very likely that we can get transcendence measures for automatic continued fractions and for transcendental real numbers whose sequence of partial quotients $\mathbf{a}$ is such that $n \mapsto p(n, \mathbf{a})/n$ is bounded.

Furthermore, proceeding as in [15] and in [17], we can prove that if $\mathbf{a} = a_1 a_2 \dots$ is a word over $\mathbf{Z}_{\geq 1}$ such that $\alpha = [0; a_1, a_2, \dots, a_\ell, \dots]$ is algebraic of degree at least three, then there exists $\delta > 0$ such that

$$\limsup_{n \to +\infty} \frac{p(n, \mathbf{a})}{n (\log n)^\delta} = +\infty,$$

and there exists an effectively computable positive constant M such that

$$p(n, \mathbf{a}) \geq \left(1 + \frac{1}{M}\right) n, \quad \text{for } n \geq 1.$$

We plan to return to these questions in a subsequent note.

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