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Łukasiewicz Logic and Weighted Logics over MV-Semirings

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Łukasiewicz Logic and Weighted Logics over MV-Semirings *

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Abstract

We connect Łukasiewicz logic, a well-established many-valued logic, with weighted logics, recently introduced by Droste and Gastin. We use this connection to show that for formal series with coefficients in semirings derived from MV-algebras, recognizability and definability in a fragment of second order Łukasiewicz logic coincide.

1 Introduction

Recently, Droste and Gastin introduced *weighted logics* in [7]. In weighted logics, formulas are interpreted in semirings. The connectives $\vee$ and $\wedge$ exactly reflect the semiring operations and no natural definition of negation is available. Basically, weighted logics are many-valued second order logics on words.

We compare weighted logics to traditional many-valued logics, especially Łukasiewicz logic. Łukasiewicz logic emerged in 1920 as three-valued logic and was soon generalized to the infinite set of truth values $[0, 1]$. Like other many-valued logics, Łukasiewicz logic was developed as generalization of two-valued logic extending the set of truth values while keeping as many as possible intuitive properties of the classical connectives ([11, 12, 14]). In addition to the classical connectives $\vee, \wedge$ and $\neg$, Łukasiewicz logic contains two connectives $\underline{\vee}$ (strong disjunction) and $\&$ (strong conjunction). Formulas of Łukasiewicz logic are interpreted in the standard MV-algebra $([0, 1], \oplus, \otimes, \neg, 0, 1)$ where $\neg x = 1 - x$ for all $x \in [0, 1]$, the truth function of $\underline{\vee}$ is the Łukasiewicz t-conorm $\oplus$ and the truth function of $\&$ is the Łukasiewicz t-norm $\otimes$. The connectives $\underline{\vee}$ and $\&$ satisfy the *normal condition* of many-valued logics [11], i.e. restricted to $\{0, 1\}$, the MV-algebra operations $\oplus$ and $\otimes$ coincide with the Boolean operations $\vee$ and $\wedge$, respectively. Intensive studies of Łukasiewicz logic resulted in decidability results, axiomatization, and proof theories for propositional and first-order Łukasiewicz logic. Until the recent approach of Běhounek and Cintula [2], there has not been much interest in higher order Łukasiewicz logic.

In [8], Gerla introduced semiring-reducts of MV-algebras. These semirings are commutative and idempotent. In [8, 6], automata and recognizable series over these semirings were defined and studied.

*Talk given at Weighted Automata: Theory and Applications (WATA 2006)

We define fragments $\mathsf{MSO}_{\vee}^{(Ł,W)}(A,\mathbb{X})$ of monadic second order Łukasiewicz logic appropriate for the definition of formal series. These fragments correspond to weighted logics over MV-semirings. Hence we can apply results from [7] to show that recognizability by automata over the MV-semiring $\mathbb{W}$ and definability in $\mathsf{MSO}_{\vee}^{(Ł,W)}(A,\mathbb{X})$ coincide.

The paper is organized as follows: After a recapitulation of some notions in Section 2, we introduce MV-algebras in Section 3. In Section 4 we show how to derive semirings from MV-algebras. Section 5 contains a very short overview over weighted automata and recognizable series over MV-semirings. Łukasiewicz logic is introduced in Section 6 and a fragment $\mathsf{MSO}_{\vee}^{(Ł,W)}(A,\mathbb{X})$ of Łukasiewicz logic, appropriate for the characterization of formal series is presented Section 7.

Our main result, the coincidence of weighted logics over MV-semirings and our fragment $\mathsf{MSO}_{\vee}^{(Ł,W)}(A,\mathbb{X})$ of Łukasiewicz logic is given in Section 8. In Section 9, we prove that a formal series is Ł-recognizable iff it is definable in $\mathsf{MSO}_{\vee}^{(Ł,W)}(A,\mathbb{X})$. This is a special case of a result in [7], but for MV-semirings, we present a simpler proof.

2 Preliminaries

As usual, $\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{R}$ denote the sets of natural numbers, integers, rational and real numbers. Two real numbers $a,b\in\mathbb{R}$ define the real interval $[a,b]=\{r\in\mathbb{R}\mid a\le r\le b\}$. For a finite set S, $|S|$ is the cardinality of S.

A *semiring* is a algebraic structure $\mathbb{K}=(K,+,\cdot,0_K,1_K)$ where $(K,+,0_K)$ is a commutative monoid, $(K,\cdot,1_K)$ is a monoid, 0_K is absorbing w.r.t. $\cdot$, and $\cdot$ distributes over $+$. A semiring $\mathbb{K}=(K,+,\cdot,0_K,1_K)$ is called commutative if $\cdot$ is commutative. $\mathbb{K}$ is idempotent iff $+$ is idempotent. Note that $\mathbb{K}$ is idempotent iff $1_K+1_K=1_K$ [10]. For idempotent semirings $\mathbb{K}$, the restriction $(\{0_K,1_K\},+,\cdot,0_W,1_W)$ is isomorphic to the Boolean algebra $(\{0,1\},\vee,\wedge,0,1)$. On idempotent semirings, the *natural ordering* $\le$ is defined by

$$x\le y \quad \text{iff} \quad x+y=y$$

If 1_W is a maximal w.r.t. this ordering then $x+1_W=1_W$ holds for every $x\in W$, i.e. 1_W is absorbing for $+$.

An algebraic structure is *locally finite* iff every finitely generated subset of its domain is finite.

For a finite alphabet (set of symbols) A, A^n denotes the set of all words $a_1\cdots a_n$ where $a_i\in A$ for all $i\in\{1,\dots,n\}$ and $A^*=\bigcup_{n\in\mathbb{N}}A^n$ where $A^0=\{\varepsilon\}$ with the empty word ε. The length of $w\in A^*$ is denoted by $|w|$ and $\mathsf{pos}(w)=\{0,\dots,|w|\}$ is the set of positions (next to letters) in w.

For a set W (of truth values) and any set A, a mapping $S:A\longrightarrow W$ is called *W-valued set* on A, and a mapping $S:A^n\longrightarrow W$ is called n-ary *W-valued relation* on A. For a set W and an alphabet A, a mapping $S:A^*\longrightarrow W$ is a *W-valued language* over A. If W is the domain of a semiring, a W-valued language $S:A^*\longrightarrow W$ is called *formal series*.

3 MV-algebras

MV-algebras were introduced by Chang as tool to prove the completeness of Łukasiewicz logic (see e.g. [11, 12, 5]), but they are also interesting research objects for algebraists.

Definition 3.1. An MV-algebra is a structure $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$ where

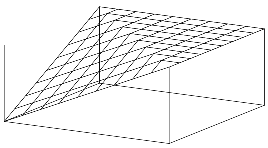
1. $(W, \oplus, 0_M)$ is a commutative monoid,
2. for all $x \in W : x \oplus 1_W = 1_W$,
3. $\neg 0_W = 1_W$ and $\neg 1_W = 0_W$,
4. for all $x, y \in W : \neg(\neg x \oplus \neg y) = x \otimes y$,
5. for all $x, y \in W : x \oplus (\neg x \otimes y) = y \oplus (\neg y \otimes x)$

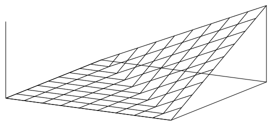
We give several examples for MV-algebras.

Example 3.1. For every $a, b \in \mathbb{R}$ where $a < b$, the structure $([a, b], \oplus, \otimes, \neg, a, b)$ where for all $x, y \in [a, b]$

$$\neg x = a + b - x$$
$$x \oplus y = \min\{b, x + y - a\} \qquad\qquad x \otimes y = \max\{a, x + y - b\}$$

is an MV-algebra. The pictures below show both functions.





For $a = 0$ and $b = 1$ in Example 3.1, we obtain the *standard MV-algebra*

$$[0, 1]^{\text{Ł}} = ([0, 1], \oplus, \otimes, \neg, 0, 1) \tag{1}$$

In $[0, 1]^{\text{Ł}}$, the functions $\otimes$ and $\oplus$ are called *Łukasiewicz t-norm* and *Łukasiewicz t-conorm*. The standard MV-algebra $[0, 1]^{\text{Ł}}$ is locally finite [14].

For every interval $[a, b] \subseteq \mathbb{R}$ the function

$$f : [0, 1] \longrightarrow [a, b] \qquad \text{where} \qquad f(x) = a + x(b - a)$$

is an isomorphism of $[0, 1]^{\text{Ł}}$ and the MV-algebra $([a, b], \oplus, \otimes, \neg, a, b)$.

Example 3.2.

1. the countable MV-algebra $([0, 1] \cap \mathbb{Q}, \oplus, \otimes, \neg, 0, 1)$,
2. for every $n \in \mathbb{N} \setminus \{0\}$, the finite MV-algebra $\left(\left\{\frac{i}{n} \mid i \in \{0, \ldots, n\}\right\}, \oplus, \otimes, \neg, 0, 1\right)$ (These are isomorphic to the finite MV-algebras $(\{0, \ldots, n\}, \oplus, \otimes, \neg, 0, n)$ induced by initial sequences of natural numbers.),
3. for $n = 1$, the MV-algebra defined in 2. is the Boolean algebra $(\{0, 1\}, \vee, \wedge, \neg, 0, 1)$

Since all MV-algebras in Example 3.2 are (isomorphic to) sub-MV-algebra of $[0, 1]^{\text{Ł}}$, they are locally finite.

Example 3.3. The set of all functions from a nonempty set S to the unit interval $[0, 1]$ is the domain of the MV-algebra $\left([0, 1]^S, \oplus, \otimes, \neg, 0, 1\right)$ where $\oplus, \otimes, \neg$ are the point-wise extensions of the standard MV-algebra operations and 0 and 1 are the constant functions mapping every element in S to 0 and 1, respectively.

Every MV-algebra $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$ has the following properties:

1. for all $x \in W : \neg\neg x = x$,
2. for all $x \in W : x \otimes 1_W = x$ and $x \otimes 0_W = 0_W$,
3. for all $x, y \in W : x \otimes (\neg x \oplus y) = y \otimes (\neg y \oplus x)$
4. $\neg$ is an isomorphism between $\mathbb{W}$ and the dual MV-algebra $(W, \otimes, \oplus, \neg, 1_W, 0_W)$.

The *natural ordering* $\leq$ on an MV-algebra $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$ is defined as follows:

$$\text{for all } x, y \in W: \qquad x \leq y \qquad \text{iff} \qquad \neg x \oplus y = 1_W \tag{2}$$

Then $(W, \leq, 0_W, 1_W)$ is a bounded lattice where for all $x, y \in W$, the lattice operations satisfy the equations

$$x \vee y = x \oplus (\neg x \otimes y) \qquad \text{and} \qquad x \wedge y = x \otimes (\neg x \oplus y) \tag{3}$$

If for an MV-algebra $\mathbb{W}$, the natural ordering is total then $\mathbb{W}$ is called *MV-chain*.

In all MV-algebras in Examples 3.1 and 3.2, the ordering defined by Equation 2 coincides with the natural ordering of real numbers. Hence these MV-algebras are MV-chains and the lattice operations with respect to this ordering are

$$\begin{aligned} x \vee y &= x \oplus (\neg x \otimes y) = \max\{x, y\} \\ x \wedge y &= x \otimes (\neg x \oplus y) = \min\{x, y\} \end{aligned} \tag{4}$$

Since $\oplus$ and $\otimes$ are associative, we can use the common abbreviations

$$\bigoplus_{i \in \{1,\ldots,n\}} a_i = a_i \oplus \cdots \oplus, a_n \qquad \text{and} \qquad \bigotimes_{i \in \{1,\ldots,n\}} a_i = a_i \otimes \cdots \otimes, a_n$$

and an easy computation implies for $[0,1]^{Ł}$

$$\bigoplus_{i \in \{1,\ldots,n\}} a_i = \min\left\{1, \sum_{i=1}^{n} a_i\right\} \qquad \bigotimes_{i \in \{1,\ldots,n\}} a_i = \max\left\{0, \sum_{i=1}^{n} a_i - n + 1\right\} \tag{5}$$

In general, the MV-algebra operations $\oplus$ and $\otimes$ are not idempotent and do not distribute over each other, but the following distributive laws hold in every MV-algebra

$$\begin{aligned} x \otimes (y \vee z) &= (x \otimes y) \vee (x \otimes z) \\ x \oplus (y \wedge z) &= (x \oplus y) \wedge (x \oplus z) \end{aligned} \tag{6}$$

In any MV-algebra $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$, if $\oplus$ or $\otimes$ are idempotent or distribute over each other then $\mathbb{W}$ is a Boolean algebra [14].

Remark 3.1. For every MV-algebra $\mathbb{W}$, the set $\{0_W, 1_W\}$ induces a sub-MV-algebra $(\{0_W, 1_W\}, \oplus, \otimes, \neg, 0_W, 1_W)$ of $\mathbb{W}$ that is isomorphic to the Boolean algebra $(\{0, 1\}, \vee, \wedge, 0, 1)$.

Chang's Completeness Theorem justifies the particular importance of the standard MV-algebra.

Theorem 3.1 ([4]). *An equation holds in every MV-algebra iff it holds in* $[0,1]^{Ł}$.

4 MV-semirings

As mentioned in Section 3, in an arbitrary MV-algebra $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$, the operations $\oplus$ and $\otimes$ do not distribute over each other. Hence usually the reduct $\mathbb{W} = (W, \oplus, \otimes, 0_W, 1_W)$ is not a semiring.

Gerla has shown in [8, 9, 6], that MV-algebras have semiring-reducts containing the lattice operations $\vee$ and $\wedge$.

Proposition 4.1 ([8]). *For every MV-algebra* $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$*, both structures*

$$\mathbb{W}_\vee = (W, \vee, \otimes, 0_W, 1_W) \qquad \text{and} \qquad \mathbb{W}_\wedge = (W, \wedge, \oplus, 1_W, 0_W)$$

are commutative semirings and the function $\neg$ *is an isomorphism between* $\mathbb{W}_\vee$ *and* $\mathbb{W}_\wedge$*.*

We will call semirings constructed in this way *MV-semirings*. By definition every MV-semiring is commutative and idempotent. In [8, 9, 6], weighted automata over these semirings were introduced.

Example 4.1. From the MV-algebras in Examples 3.1 and 3.2,
we derive the following semirings

- non-countable semirings $([a, b], \max, \otimes, a, b)$,
  especially $[0, 1]^{\text{Ł}}_\wedge = ([0, 1], \max, \otimes, 0, 1)$
- the countable semiring $([0, 1] \cap \mathbb{Q}, \max, \otimes, 0, 1)$
- for $n \in \mathbb{N} \setminus \{0\}$ the finite MV-semirings $\left(\left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right\}, \max, \otimes, 0, 1\right)$ and $(\{0, \dots, n\}, \max, \otimes, 0, n)$.
- the Boolean semiring $(\{0, 1\}, \max, \min, 0, 1)$

By definition, all MV-semirings are commutative and idempotent.

If an MV-algebra $\mathbb{W}$ is locally finite then the semirings $\mathbb{W}_\vee$ and $\mathbb{W}_\wedge$ are locally finite. Hence both semirings $[0, 1]^{\text{Ł}}_\vee$, $[0, 1]^{\text{Ł}}_\wedge$ and all their sub-semirings are locally finite.

5 MV- and Ł-recognizable series

According to the general definitions of weighted automata [16, 1], a finite $(\mathbb{W}, A)$-automaton $(Q, \alpha, \delta, \beta)$ is defined by a finite set Q of states, two vectors $\alpha, \beta : Q \longrightarrow W$, and a morphism $\delta : A^* \longrightarrow (Q^2 \longrightarrow W)$ from words to square matrices of order $|Q|$. Since δ is a morphism, it is uniquely defined by its restriction $\delta : A \longrightarrow (Q^2 \longrightarrow W)$ to letters.

For a word $w \in A^*$, the weight of a *run* $(p_0, \dots, p_{|w|}) \subseteq Q^{|w|+1}$ of the weighted automaton $\mathcal{A}$ on w is

$$v\left((p_0, \dots, p_{|w|}), w\right) = \alpha(p_0) \otimes \left(\bigotimes_{i \in \mathsf{pos}(w) \setminus \{0\}} \delta(w_i)(p_{i-1}, p_i) \right) \otimes \beta(p_{|w|}) \tag{7}$$

As usual, the *behavior* of the $(\mathbb{W}, A)$-automaton $\mathcal{A} = (Q, \alpha, \delta, \beta)$ is the formal series $\|\mathcal{A}\| : A^* \longrightarrow W$ that maps every $w \in A^*$ to

$$\|\mathcal{A}\|(w) = \bigvee_{\substack{(p_0, \dots, p_{|w|}) \\ \subseteq Q^{|w|+1}}} \left(\alpha(p_0) \otimes \bigotimes_{i \in \mathsf{pos}(w) \setminus \{0\}} \delta(w_i)(p_{i-1}, p_i) \otimes \beta(p_{|w|}) \right) \tag{8}$$

A formal series $S : A^* \longrightarrow W$ is *MV-recognizable* iff there is an MV-semiring $\mathbb{W}$ and a $(\mathbb{W}, A)$-automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = S$. A formal series $S : A^* \longrightarrow W$ is *L-recognizable* iff there is an $([0,1]^{\text{Ł}}_{\vee}, A)$-automaton $\mathcal{A}$ such that $\|\mathcal{A}\| = S$.

6 Łukasiewicz logic

Łukasiewicz logic is a well-investigated many-valued logic (see e.g. [11, 14, 12, 5]). There are several definitions for the syntax of Łukasiewicz logic that differ mostly in the sets of default connectives. Usually, the unary negation symbol $\neg$ and some of the binary connectives $\&$ (strong conjunction), $\veebar$ (strong disjunction), $\vee, \wedge$ (weak disjunction and conjunction) are used. Since all binary connectives are interpreted by commutative and associative operations, we will use the common abbreviations for finite sets $I = \{1, \ldots, n\}$:

$$\bigvee_{i \in I} \varphi_i = \varphi_1 \vee \cdots \vee \varphi_n \qquad \bigwedge_{i \in I} \varphi_i = \varphi_1 \wedge \cdots \wedge \varphi_n \qquad \mathop{\&}_{i \in I} \varphi_i = \varphi_1 \,\&\, \cdots \,\&\, \varphi_n$$

Usually, formulas in Łukasiewicz logic are interpreted in the standard MV-algebra $[0,1]^{\text{Ł}}$ but we will also use other MV-algebras $\mathbb{W}$. In many versions of Łukasiewicz logic, truth constants (syntactic representatives for truth values in the domain of $\mathbb{W}$) are part of the syntax.

In Łukasiewicz predicate logic, the quantifiers $\forall$ and $\exists$ are used. We will also use quantifiers $\forall_{\text{Ł}}$ and $\exists_{\text{Ł}}$ not present in standard Łukasiewicz logic. This extension is reasonable since $\forall_{\text{Ł}}$ is associated to the Łukasiewicz conjunction $\&$ like $\forall$ is associated to $\wedge$ in classical (and many-valued) logic.

Like in classical logic, atoms are constructed by a set Σ of relations symbols, each coming with its arity, and a set $\mathbb{X}$ of variables. Since we will define special monadic second order logics, the set $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$ contains *individual variables* (first order) in $\mathbb{X}_1$ and *set variables* (monadic second order) in $\mathbb{X}_2$, and atoms of the form $X(x)$ where $X \in \mathbb{X}_2$ and $x \in \mathbb{X}_1$ are allowed.

The set $\mathsf{atom}(\Sigma, \mathbb{X})$ of atoms in monadic second order Łukasiewicz logic is

$$\mathsf{atom}(\Sigma, \mathbb{X}) = \{p(x_1, \ldots, x_n) \mid p \in \Sigma, x_1, \ldots, x_n \in \mathbb{X}_1\} \cup \{X(x) \mid X \in \mathbb{X}_2, x \in \mathbb{X}_1\}$$

where p is an n-ary relation symbol.

Formulas in monadic second order Łukasiewicz logic $\mathsf{MSO}^{(\text{Ł},W)}(\Sigma, \mathbb{X})$ are constructed from atoms, connectives and quantifiers, as follows

$$\varphi ::= c \mid P \mid \neg\varphi \mid \varphi * \psi \mid Qx\varphi$$

where $c \in W$ is a truth value, P is an atom, $* \in \{\&, \veebar, \vee, \wedge\}$, $Q \in \{\forall, \exists, \forall_{\text{Ł}}, \exists_{\text{Ł}}\}$, and $x \in \mathbb{X}$.

The semantics of Łukasiewicz logic is a generalization of the classical semantics of predicate logic. For a set W of truth values, a *W-valued Σ-structure* $\mathcal{S} = (S, [\cdot]_S)$ is defined by

- the non-empty domain S and
- for every n-ary relation symbol p in Σ, a W-valued relation $[p]_S : S^n \longrightarrow W$.

An *assignment* for $\mathbb{X}$ in S maps first order variables to domain elements and monadic second order variables to unary W-relations:

$$\sigma : \mathbb{X}_1 \longrightarrow S \qquad \sigma : \mathbb{X}_2 \longrightarrow (S \longrightarrow W)$$

An *interpretation* is a pair $(\mathcal{S}, \sigma)$ of a W-valued Σ-structure $\mathcal{S}$ and an assignment σ. The (truth) value of atoms under an interpretation $(\mathcal{S}, \sigma)$ is defined as follows

$$\begin{aligned} [\![p(x_1, \dots, x_n)]\!]_{(\mathcal{S},\sigma)} &= [p]_S\,(\sigma(x_1), \dots, \sigma(x_n)) \\ [\![X(x)]\!]_{(\mathcal{S},\sigma)} &= \sigma(X)\,(\sigma(x)) \end{aligned} \tag{9}$$

Usually, formulas in Łukasiewicz logics are interpreted in the standard MV-algebra $[0,1]^{\text{Ł}}$ but we may use any other MV-algebra $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$ as well. The strong connectives $\&$, $\underline{\vee}$ and $\neg$ are interpreted by the MV-algebra operations $\oplus, \otimes$ and $\neg$, respectively. The value of a non-atomic formula $\varphi \in \mathsf{MSO}^{(\text{Ł},W)}(\Sigma, \mathbb{X})$ in an interpretation $(\mathcal{S}, \sigma)$ is defined as usual:

$$\begin{aligned} [\![\neg\varphi]\!]_{(\mathcal{S},\sigma)} &= \neg\,[\![\varphi]\!]_{(\mathcal{S},\sigma)} \\ [\![\varphi \,\underline{\vee}\, \psi]\!]_{(\mathcal{S},\sigma)} &= [\![\varphi]\!]_{(\mathcal{S},\sigma)} \oplus [\![\psi]\!]_{(\mathcal{S},\sigma)} & [\![\varphi \,\&\, \psi]\!]_{(\mathcal{S},\sigma)} &= [\![\varphi]\!]_{(\mathcal{S},\sigma)} \otimes [\![\psi]\!]_{(\mathcal{S},\sigma)} \\ [\![\varphi \vee \psi]\!]_{(\mathcal{S},\sigma)} &= [\![\varphi]\!]_{(\mathcal{S},\sigma)} \vee [\![\psi]\!]_{(\mathcal{S},\sigma)} & [\![\varphi \wedge \psi]\!]_{(\mathcal{S},\sigma)} &= [\![\varphi]\!]_{(\mathcal{S},\sigma)} \wedge [\![\psi]\!]_{(\mathcal{S},\sigma)} \\ [\![\exists x\varphi]\!]_{(\mathcal{S},\sigma)} &= \bigvee_{i \in I} [\![\varphi]\!]_{(\mathcal{S},\sigma[x\mapsto i])} & [\![\forall x\varphi]\!]_{(\mathcal{S},\sigma)} &= \bigwedge_{i \in I} [\![\varphi]\!]_{(\mathcal{S},\sigma[x\mapsto i])} \\ [\![\exists_{\text{Ł}} x\varphi]\!]_{(\mathcal{S},\sigma)} &= \bigoplus_{i \in I} [\![\varphi]\!]_{(\mathcal{S},\sigma[x\mapsto i])} & [\![\forall_{\text{Ł}} x\varphi]\!]_{(\mathcal{S},\sigma)} &= \bigotimes_{i \in I} [\![\varphi]\!]_{(\mathcal{S},\sigma[x\mapsto i])} \end{aligned} \tag{10}$$

for finite $S \cup W^S$, $I = S$ for $x \in \mathbb{X}_1$ and $I = W^S$ for $x \in \mathbb{X}_2$.

Every formula $\varphi \in \mathsf{MSO}^{(\text{Ł},W)}(\Sigma, \mathbb{X})$ defines a mapping from the set of all interpretations into the truth domain W. If $\varphi \in \mathsf{MSO}^{(\text{Ł},W)}(\Sigma, \mathbb{X})$ is a sentence, i.e. does not contain free variables, the assignment σ is irrelevant. Hence every sentence $\varphi \in \mathsf{MSO}^{(\text{Ł},W)}(\Sigma, \mathbb{X})$ defines a mapping from structures to truth values, i.e. a $\mathbb{W}$-valued language.

Remark 6.1. By Remark 3.1, the set $\{0_W, 1_W\}$ of truth values is closed under all truth functions for connectives in $\mathsf{MSO}^{(\text{Ł},W)}(\Sigma, \mathbb{X})$. Hence $\mathsf{MSO}^{(\text{Ł},W)}(\Sigma, \mathbb{X})$ satisfies the *normal condition* (see [11]) of many-valued logics, i.e. $\underline{\vee}$ and $\&$ coincide with $\vee$ and $\wedge$, respectively.

7 Łukasiewicz logic on words

To characterize recognizable formal series, a fragment of Łukasiewicz logic suffices. The fragment $\mathsf{MSO}^{(\text{Ł},W)}(A, \mathbb{X})$ is parameterized by an alphabet A and an MV-algebra $\mathbb{W}$. Since we want to characterize sets of words, we fix the usual signature that contains the binary relation symbol $\leq$ and the set $\{P_a \mid a \in A\}$ of unary letter predicates. For every truth value $c \in W$, a truth constant c is present in the syntax of $\mathsf{MSO}^{(\text{Ł},W)}(A, \mathbb{X})$.

Since we want do describe formal series, we are only interested in the truth value of formulas in *word structures*. Every word $w \in A^*$ defines the word structure $\underline{w} = (\mathsf{pos}(w), [\cdot]_w)$ where

$$[\leq]_w\,(i,j) \quad = \quad \begin{cases} 1_W & \text{iff } i \leq j \\ 0_W & \text{otherwise} \end{cases}$$

$$\text{for every } a \in A\text{:} \quad [P_a]_w\,(i) \quad = \quad \begin{cases} 1_W & \text{iff } i > 0 \text{ and } w_i = a \\ 0_W & \text{otherwise} \end{cases}$$

An assignment σ for $\underline{w}$ maps first order variables to positions in w and second order variables to (characteristic functions of) sets of positions in w:

$$\sigma : \mathbb{X}_1 \longrightarrow \mathsf{pos}(w) \qquad \sigma : \mathbb{X}_2 \longrightarrow (\mathsf{pos}(w) \longrightarrow \{0_W, 1_W\})$$

The value of atoms in an interpretation $(\underline{w}, \sigma)$ is calculated from $[\cdot]_{\underline{w}}$ and σ according to Equation 9:

$$\begin{aligned}
[\![x \leq y]\!]_{(\underline{w},\sigma)} &= \begin{cases} 1_W & \text{iff } \sigma(x) \leq \sigma(y) \\ 0_W & \text{otherwise} \end{cases} \\
[\![P_a(x)]\!]_{(\underline{w},\sigma)} &= \begin{cases} 1_W & \text{iff } \sigma(x) > 0 \text{ and } w_{\sigma(x)} = a \\ 0_W & \text{otherwise} \end{cases} \\
[\![X(x)]\!]_{(\underline{w},\sigma)} &= \sigma(X)(\sigma(x)) \in \{0_W, 1_W\}
\end{aligned} \tag{11}$$

Remark 7.1. Although $\underline{w}$ is a W-valued structure, all atoms are crisp, i.e. have truth values in $\{0_W, 1_W\}$ under every interpretation.

The semantics of $\mathsf{MSO}^{(\text{Ł},W)}(A, \mathbb{X})$-formulas is defined inductively according to the equations in (11) and (10). Every sentence $\varphi \in \mathsf{MSO}^{(\text{Ł},W)}(A, \mathbb{X})$ defines the W-valued language

$$S_\varphi : A^* \longrightarrow W \qquad \text{where} \qquad S_\varphi(w) = [\![\varphi]\!]_{\underline{w}} \tag{12}$$

Definition 7.1. For an alphabet A, a truth domain $\mathbb{W}$ and a logic language L, a W-valued language $S : A^* \longrightarrow W$ is *L-definable* iff there is a sentence $\varphi \in L$ such that $[\![\varphi]\!]_{\underline{w}} = S$.

To define formal series, we restrict our logic $\mathsf{MSO}^{(\text{Ł},W)}(A, \mathbb{X})$ to the fragments $\mathsf{MSO}^{(\text{Ł},W)}_{\vee}(A, \mathbb{X})$ where

- the negation symbol $\neg$ is applied to atoms only,
- $\vee$ and $\&$ are the only binary connectives,
- $\exists$ and $\forall_{\text{Ł}}$ are the only quantifiers.

and $\mathsf{MSO}^{(\text{Ł},W)}_{\wedge}(A, \mathbb{X})$ where

- the negation symbol $\neg$ is applied to atoms only,
- $\wedge$ and $\underline{\vee}$ are the only binary connectives,
- $\forall$ and $\exists_{\text{Ł}}$ are the only quantifiers.

Due to the restricted syntax, the truth value of a sentence $\varphi \in \mathsf{MSO}^{(\text{Ł},W)}_{\vee}(A, \mathbb{X})$ in a word structure can be calculated using only the MV-algebra operations $\vee$ and $\otimes$, i.e. the operations of the MV-semiring $\mathbb{W}_\vee$ defined in Proposition 4.1. By a similar argument, every formula in $\mathsf{MSO}^{(\text{Ł},W)}_{\wedge}(A, \mathbb{X})$ can be interpreted in the MV-semiring $\mathbb{W}_\wedge$. Therefore the following proposition is immediate.

Proposition 7.1. *Let A be an alphabet and $\mathbb{W} = (W, \oplus, \otimes, \neg, 0_W, 1_W)$*

1. *For every sentence $\varphi \in \mathsf{MSO}_{\vee}^{(L,W)}(A, \mathbb{X})$, the mapping*

$$S_\varphi : A^* \longrightarrow W \qquad \textit{where} \qquad S_\varphi(w) = [\![\varphi]\!]_{\underline{w}}$$

is a formal series over the semiring $\mathbb{W}_\vee$.

2. *For every sentence $\varphi \in \mathsf{MSO}_{\wedge}^{(L,W)}(A, \mathbb{X})$, the mapping*

$$S_\varphi : A^* \longrightarrow W \qquad \textit{where} \qquad S_\varphi(w) = [\![\varphi]\!]_{\underline{w}}$$

is a formal series over the semiring $\mathbb{W}_\wedge$.

Since $\neg$ is an isomorphism of the semirings $\mathbb{W}_\vee$ and $\mathbb{W}_\wedge$, we may restrict our attention to the fragment $\mathsf{MSO}_{\vee}^{(\text{Ł},W)}(A, \mathbb{X})$.

8 Weighted logics over MV-semirings

According to [7], a finite alphabet A and the semiring $\mathbb{W}$ define the weighted logic $\mathsf{MSO}(\mathbb{W}, A)$. Since weighted logics were introduced to allow the characterization of recognizable series by a logic formalism, the connectives in weighted logics reflect the semiring operations very closely. Hence for a general semiring $\mathbb{W}$, the meaning of connectives and quantifiers is sometimes counter-intuitive.

MV-algebra operations are designed as truth functions for a generalization of the classical connectives. Hence in weighted logics over MV-semirings, connectives inherit an intuitive meaning for the underlying MV-algebra.

Comparing the fragment $\mathsf{MSO}_{\vee}^{(\text{Ł},W)}(A, \mathbb{X})$ of Łukasiewicz logic to the weighted logic $\mathsf{MSO}(W, A)$, we notice a strong similarity. The set of relation symbols $\{\leq\} \cup \{P_a \mid a \in A\}$ is the same for both logics. Hence for a fixed set $\mathbb{X}$ of variables, the sets of atoms coincide in $\mathsf{MSO}(W, A)$ and $\mathsf{MSO}_{\vee}^{(\text{Ł},W)}(A, \mathbb{X})$.

Both logics $\mathsf{MSO}(\mathbb{W}, A)$ and $\mathsf{MSO}_{\vee}^{(\text{Ł},W)}(A, \mathbb{X})$ use different symbols for conjunction and generalization. This is necessary, because in Łukasiewicz logic, the meaning of $\wedge$ and $\forall$ is predefined and satisfies certain logic laws. In general, this predefined meaning does not coincide with the interpretation of $\wedge$ and $\forall$ in weighted logic over general semirings.

Another difference between weighted logic and our version of Łukasiewicz logic is merely philosophical and concerns the view to the semiring elements. In weighted logic, they are special atoms having a fixed meaning in every word interpretation. In Łukasiewicz logic, their syntactic representatives are truth constants (connectives of arity 0).

We present the semantics of truth constants, atoms and negated atoms in a weighted logic $\mathsf{MSO}(W, A)$ as defined in [7] in comparison to the semantics of the adequate fragment $\mathsf{MSO}_{\vee}^{(\text{Ł},W)}(A, \mathbb{X})$ of Łukasiewicz logic:

$$
\begin{array}{lll}
& \mathsf{MSO}(W,A) & \mathsf{MSO}_{\vee}^{(\text{Ł},W)}(A,\mathbb{X})
\end{array}
\tag{13}
$$

$$
\begin{aligned}
\llbracket c \rrbracket\,(w,\sigma) &= \ c & &= \llbracket c \rrbracket_{(\underline{w},\sigma)}\\
\llbracket x \leq y \rrbracket\,(w,\sigma) &= \begin{cases} 1 & \text{iff } \sigma(x) \leq \sigma(y)\\ 0 & \text{otherwise}\end{cases} & &= \llbracket x \leq y \rrbracket_{(\underline{w},\sigma)}\\
\llbracket P_a(x) \rrbracket\,(w,\sigma) &= \begin{cases} 1 & \text{iff } \sigma(x) > 0 \text{ and } w_{\sigma(x)} = a\\ 0 & \text{otherwise}\end{cases} & &= \llbracket P_a(x) \rrbracket_{(\underline{w},\sigma)}\\
\llbracket x \in X \rrbracket\,(w,\sigma) &= \begin{cases} 1 & \text{iff } \sigma(x) \in \sigma(X)\\ 0 & \text{otherwise}\end{cases} & &= \llbracket X(x) \rrbracket_{(\underline{w},\sigma)}\\
\llbracket \neg\varphi \rrbracket\,(w,\sigma) &= \begin{cases} 1 & \text{iff } \llbracket \varphi \rrbracket\,(w,\sigma) = 0\\ 0 & \text{otherwise}\end{cases} & &= \llbracket \neg\varphi \rrbracket_{(\underline{w},\sigma)}
\end{aligned}
$$

According to the definition of weighted logics in [7] and Equation (10), the semantics of non-atomic formulas is defined by

$$
\begin{array}{lll}
& \mathsf{MSO}(W,A) & \mathsf{MSO}_{\vee}^{(\text{Ł},W)}(A,\mathbb{X})
\end{array}
\tag{14}
$$

$$
\begin{aligned}
\llbracket \varphi \vee \psi \rrbracket\,(w,\sigma) &= \llbracket \varphi \rrbracket\,(w,\sigma) \vee \llbracket \psi \rrbracket\,(w,\sigma) = \llbracket \varphi \vee \psi \rrbracket_{(\underline{w},\sigma)}\\
\llbracket \varphi \wedge \psi \rrbracket\,(w,\sigma) &= \llbracket \varphi \rrbracket\,(w,\sigma) \otimes \llbracket \psi \rrbracket\,(w,\sigma) = \llbracket \varphi \,\&\, \psi \rrbracket_{(\underline{w},\sigma)}\\
\llbracket \exists x\varphi \rrbracket\,(w,\sigma) &= \bigvee_{i\in I} \llbracket \varphi \rrbracket\,(w,\sigma[x \mapsto i]) \quad = \llbracket \exists x\varphi \rrbracket_{(\underline{w},\sigma)}\\
\llbracket \forall x\varphi \rrbracket\,(w,\sigma) &= \bigotimes_{i\in I} \llbracket \varphi \rrbracket\,(w,\sigma[x \mapsto i]) \quad = \llbracket \forall_{\text{Ł}} x\varphi \rrbracket_{(\underline{w},\sigma)}
\end{aligned}
$$

where $I = \mathsf{pos}(w)$ for $x \in \mathbb{X}_1$ and $I = 2^{\mathsf{pos}(w)}$ for $x \in \mathbb{X}_2$.

Definition 8.1. The mapping $t : \mathsf{MSO}(W,A) \longrightarrow \mathsf{MSO}_{\vee}^{(\text{Ł},W)}(A,\mathbb{X})$ is defined by

$$
\begin{aligned}
t(\varphi) &= \varphi \quad \text{if } \varphi \text{ is an atom or truth constant}\\
t(\neg\varphi) &= \neg(t(\varphi))\\
t(\varphi \vee \psi) &= t(\varphi) \vee t(\psi)\\
t(\varphi \wedge \psi) &= t(\varphi) \,\&\, t(\psi)\\
t(\exists x\varphi) &= \exists x(t(\varphi))\\
t(\forall x\varphi) &= \forall_{\text{Ł}} x(t(\varphi))
\end{aligned}
$$

Note that the function t in Definition 8.1 is a bijection. An easy induction using the equations in (13) and (14) shows the following theorem.

Theorem 8.1. *For every word $w \in A$, every valuation σ and every formula $\varphi \in \mathsf{MSO}(\mathbb{W},A)$,*

$$\llbracket \varphi \rrbracket_{\mathcal{V}}\,(w,\sigma) = \llbracket t(\varphi) \rrbracket_{(\underline{w},\sigma)}$$

Restricted to sentences, we obtain

Corollary 8.1. *For every sentence* $\varphi \in \mathsf{MSO}(W, A)$*:* $[\![\varphi]\!] = S_{t(\varphi)}$.

This immediately implies

Corollary 8.2. *A formal series* $S : A^* \longrightarrow W$ *is definable in* $\mathsf{MSO}(W, A)$ *iff* S *is definable in* $\mathsf{MSO}_{\vee}^{(Ł,W)}(A, \mathbb{X})$.

Corollary 8.2 allows to apply all results about locally finite semirings from [7] to our fragment of Łukasiewicz logic. From

Theorem 8.2 ([7]). *For a locally finite commutative semiring* $\mathbb{K}$ *and an alphabet* A*, it is decidable*

1. *for* $\varphi, \psi \in \mathsf{MSO}(\mathbb{K}, A)$*, whether* $[\![\varphi]\!] = [\![\psi]\!]$
2. *for* $\varphi \in \mathsf{MSO}(\mathbb{K}, A)$*, whether* $0 \in [\![\varphi]\!]\,(A^*)$

we infer the following decidability result about our fragment of Łukasiewicz logic.

Corollary 8.3. *For every alphabet* A*, it is decidable*

1. *for* $\varphi, \psi \in \mathsf{MSO}_{\vee}^{(Ł,W)}(A, \mathbb{X})$*, whether* $S_\varphi = S_\psi$
2. *for* $\varphi \in \mathsf{MSO}_{\vee}^{(Ł,W)}(A, \mathbb{X})$*, whether* $0 \in S_\varphi\,(A^*)$

9 $\mathsf{MSO}_{\vee}^{(\mathbf{Ł},W)}(A, \mathsf{X})$-definable series

In [7], Droste and Gastin generalized the well-known theorem by Büchi and Elgot [13, 3] to the following theorem.

Theorem 9.1 ([7]). *Let* $\mathbb{W}$ *be a commutative semiring and* A *an alphabet. Then a series* $S : A^* \longrightarrow K$ *is recognizable iff* S *is definable in restricted* $\mathsf{MSO}(W, A)$*.*

In formulas from restricted $\mathsf{MSO}(W, A)$, the quantifier $\forall$ does not bind second order variables and all $\forall$-quantified formulas satisfy a certain semantic condition. For a definition of restricted $\mathsf{MSO}(W, A)$, see [7].

The proof of Theorem 9.1 in [7] is a generalization of the constructive proof of Büchi's and Elgot's theorem. For a weighted automaton $\mathcal{A} = (Q, \alpha, \delta, \beta)$ (w.l.o.g. we assume $Q = \{1, \ldots, n\}$), a sentence $\varphi_{\mathcal{A}} \in \mathsf{MSO}(\mathbb{W}, A)$ is constructed such that $\|\mathcal{A}\| = [\![\varphi_{\mathcal{A}}]\!]$.

The following sentence is an alternative to the formula in [7]. It is closer to the traditional style in [17, 15] and avoids the concept of unambiguous formulas used in [7].

$$\begin{aligned}
\varphi_{\mathcal{A}} &= \exists X_1 \ldots \exists X_n \,(\varphi_p \wedge \varphi_\alpha \wedge \varphi_\delta \wedge \varphi_\beta) \quad \text{where} \qquad (15)\\
\varphi_p &= \forall x \bigvee_{p \in Q} \left(X_p(x) \wedge \bigwedge_{q \in Q \setminus \{p\}} \neg X_q(x) \right)\\
\varphi_\alpha &= \forall x \left(\neg \mathsf{first}(x) \vee \left(\mathsf{first}(x) \wedge \bigvee_{q \in Q} (X_q(x) \wedge \alpha(q)) \right) \right)\\
\varphi_\delta &= \forall x \forall y \left(\neg S(x,y) \vee \left(S(x,y) \wedge \bigvee_{\substack{a \in A\\ p,q \in Q}} (X_p(x) \wedge X_q(y) \wedge P_a(y) \wedge \delta(a)(p,q)) \right) \right)\\
\varphi_\beta &= \forall x \left(\neg \mathsf{last}(x) \vee \left(\mathsf{last}(x) \wedge \bigvee_{q \in Q} (X_q(x) \wedge \beta(q)) \right) \right)
\end{aligned}$$

where, as usual, $\mathsf{first}, \mathsf{last}$ and S abbreviate the following formulas:

$$\begin{aligned}
\mathsf{first}(x) &\equiv \forall y \neg (y < x)\\
\mathsf{last}(x) &\equiv \forall y \neg (x < y)\\
S(x,y) &\equiv x < y \wedge \forall z\, (\neg(x < z) \vee \neg(z < y))
\end{aligned}$$

Theorem 9.2. *For every weighted automaton $\mathcal{A} = (Q, \alpha, \delta, \beta)$ over an arbitrary semiring and the sentence $\varphi_{\mathcal{A}}$ defined in Equation (15),*

$$\|\mathcal{A}\| = S_{\varphi_{\mathcal{A}}}$$

Proof. For every word $w \in A^*$, the value of $\varphi_{\mathcal{A}}$ on w is

$$[\![\varphi_{\mathcal{A}}]\!]_{\underline{w}} = \sum_{\sigma:\{X_q \mid q \in Q\} \to 2^{\mathsf{pos}(w)}} \Big([\![\varphi_p]\!]_{(\underline{w},\sigma)} \cdot [\![\varphi_\alpha]\!]_{(\underline{w},\sigma)} \cdot [\![\varphi_\delta]\!]_{(\underline{w},\sigma)} \cdot [\![\varphi_\beta]\!]_{(\underline{w},\sigma)} \Big) \tag{16}$$

We fix an interpretation $(\underline{w}, \sigma)$ where $\sigma : \{X_q \mid q \in Q\} \to 2^{\mathsf{pos}(w)}$ and determine the values of the subformulas φ_p, φ_α, φ_δ, and φ_β under this interpretation.

It is easy to check that the semantics of the formula φ_p coincides with the classical semantics of the formula $t(\varphi_p)$, i.e.

$$[\![\varphi_p]\!]_{(\underline{w},\sigma)} = \begin{cases} 1_W & \text{if } \{\sigma(X_q) \mid q \in Q\} \text{ is a partition of } \mathsf{pos}(w) \\ 0_W & \text{otherwise} \end{cases}$$

Since 0_W is absorbing for $\cdot$ and neutral for $+$, we may simplify Equation 16 to

$$[\![\varphi_{\mathcal{A}}]\!]_{\underline{w}} = \sum_{\substack{\sigma:\{X_q \mid q \in Q\} \longrightarrow 2^{\mathsf{pos}(w)} \\ \text{defines partition}}} \Big([\![\varphi_\alpha]\!]_{(\underline{w},\sigma)} \cdot [\![\varphi_\delta]\!]_{(\underline{w},\sigma)} \cdot [\![\varphi_\beta]\!]_{(\underline{w},\sigma)} \Big) \tag{17}$$

In the following computations of $[\![\varphi_\alpha]\!]_{(\underline{w},\sigma)}$, $[\![\varphi_\alpha]\!]_{(\underline{w},\sigma)}$, and $[\![\varphi_\delta]\!]_{(\underline{w},\sigma)}$, we assume σ to define a partition $\{\sigma(X_q) \mid q \in Q\}$ of $\mathsf{pos}(w)$. Then there is a unique sequence

$$(q_0, \ldots, q_{|w|}) \in Q^{|w|+1} \qquad \begin{array}{l} \text{such that for all } i \in \mathsf{pos}(w): \\ q_i \in Q \text{ is the unique state where } i \in \sigma(X_{q_i}) \end{array} \tag{18}$$

Since 0_W is absorbing for $\cdot$, we have

$$[\![X_q(x) \wedge \alpha(q)]\!]_{(\underline{w},\sigma)} = [\![X_q(x)]\!]_{(\underline{w},\sigma)} \cdot \alpha(q) = \begin{cases} \alpha(q) & \text{if } \sigma(x) \in \sigma(X_q) \\ 0_W & \text{otherwise} \end{cases}$$

Since 0_W is neutral for $+$ and the assignment σ defines a partition, we obtain

$$\begin{aligned}
&\left[\!\!\left[\neg \mathsf{first}(x) \vee \left(\mathsf{first}(x) \wedge \bigvee_{q \in Q} (X_q(x) \wedge \alpha(q)) \right) \right]\!\!\right]_{(\underline{w},\sigma[x \mapsto i])} \\
&= [\![\neg \mathsf{first}(x)]\!]_{(\underline{w},\sigma[x \mapsto i])} + \left([\![\mathsf{first}(x)]\!]_{(\underline{w},\sigma[x \mapsto i])} \cdot \sum_{q \in Q} \left([\![X_q(x) \wedge \alpha(q)]\!]_{(\underline{w},\sigma[x \mapsto 1])} \right) \right) \\
&= \begin{cases} \alpha(q_0) & \text{for } i = 0 \\ 1_W & \text{otherwise} \end{cases}
\end{aligned}$$

and since 1_W is neutral for $\cdot$,

$$\begin{aligned}
\llbracket \varphi_\alpha \rrbracket_{(\underline{w},\sigma)} &= \prod_{i \in \mathsf{pos}(w)} \left\llbracket \neg \mathsf{first}(x) \vee \left(\mathsf{first}(x) \wedge \bigvee_{q \in Q} (X_q(x) \wedge \alpha(q)) \right) \right\rrbracket_{(\underline{w},\sigma[x \mapsto i])} \\
&= \alpha(q_0)
\end{aligned} \tag{19}$$

An analogous computation results in

$$\llbracket \varphi_\beta \rrbracket_{(\underline{w},\sigma)} = \beta(q_{|w|}) \tag{20}$$

Next we determine the semantics of φ_δ under an assignment σ that defines a partition $\{\sigma(X_q) \mid q \in Q\}$ of $\mathsf{pos}(w)$. Since

$$\llbracket X_p(x) \wedge X_q(y) \wedge P_a(y) \rrbracket_{(\underline{w},\sigma')} = \begin{cases} 1_W & \text{if } p = q_i,\ q = q_j, \text{ and } w_j = a \\ 0_W & \text{otherwise} \end{cases}$$

and for $\sigma' = \sigma[x \mapsto i, y \mapsto j]$, we obtain

$$\begin{aligned}
&\left\llbracket \neg S(x,y) \vee \left(S(x,y) \wedge \bigvee_{\substack{a \in A \\ p,q \in Q}} (X_p(x) \wedge X_q(y) \wedge P_a(y) \wedge \delta(a)(p,q)) \right) \right\rrbracket_{(\underline{w},\sigma')} \\
&= \begin{cases} \delta(w_j)(q_i, q_j) & \text{for } i = j-1 \\ 1_W & \text{otherwise} \end{cases}
\end{aligned}$$

Since 1_W is neutral for $\cdot$, we have

$$\llbracket \varphi_\delta \rrbracket_{(\underline{w},\sigma)} = \prod_{j \in \mathsf{pos}(w) \setminus \{0\}} \delta(w_j)(q_{j-1}, q_j) \tag{21}$$

Finally we combine our results from the equations 17, 19, 20 and 21 to

$$\llbracket \varphi_{\mathcal{A}} \rrbracket_{\underline{w}} = \sum_{\substack{\sigma : Q \longrightarrow 2^{\mathsf{pos}(w)} \\ \text{defines partition}}} \left(\alpha(q_0) \cdot \prod_{i \in \mathsf{pos}(w) \setminus \{0\}} \delta(w_i)(q_{i-1}, q_i) \ \cdot \beta(q_{|w|}) \right) \tag{22}$$

Equation 18 associates a unique sequence $(q_0, \ldots, q_{|w|})$ of states to every assignment $\sigma : \{X_q \mid q \in Q\} \longrightarrow \mathsf{pos}(w)$ that defines a partition of $\mathsf{pos}(w)$.

$$\llbracket \varphi_{\mathcal{A}} \rrbracket_{\underline{w}} = \sum_{\substack{(q_0, \ldots, q_{|w|}) \\ \in Q^{|w|+1}}} \left(\alpha(q_0) \cdot \prod_{i \in \mathsf{pos}(w) \setminus \{0\}} \delta(w_i)(q_{i-1}, q_i) \ \cdot \beta(q_{|w|}) \right)$$

Hence for every word $w \in A^*$,

$$S_{\varphi_{\mathcal{A}}}(w) = \|\mathcal{A}\|(w)$$

i.e. the behavior of $\mathcal{A}$ and the semantics of $\varphi_{\mathcal{A}}$ coincide. $\square$

It is easy to check that in an idempotent semiring $\mathbb{W}$ where 1_W is maximal w.r.t. the natural ordering of $\mathbb{W}$, an even simpler formula suffices.

$$
\begin{aligned}
\varphi'_{\mathcal{A}} &= \exists X_1 \ldots \exists X_n \left(\varphi_p \wedge \varphi'_\alpha \wedge \varphi'_\delta \wedge \varphi'_\beta\right) \quad \text{where} \\
\varphi'_\alpha &= \forall x \left(\neg \mathsf{first}(x) \vee \bigvee_{q \in Q} (X_q(x) \wedge \alpha(q))\right) \\
\varphi'_\delta &= \forall x \forall y \left(\neg S(x,y) \vee \bigvee_{\substack{a \in A \\ p,q \in Q}} (X_p(x) \wedge X_q(y) \wedge P_a(y) \wedge \delta(a)(p,q))\right) \\
\varphi'_\beta &= \forall x \left(\neg \mathsf{last}(x) \vee \bigvee_{q \in Q} (X_q(x) \wedge \beta(q))\right)
\end{aligned}
\tag{23}
$$

The sentence $\varphi'_{\mathcal{A}}$ is a straightforward extension by truth values of the classical sentence in the proof of Büchi's and Elgot's theorem [17, 15].

Since every MV-semiring $\mathbb{W}$ is idempotent and 1_W is maximal w.r.t. the natural ordering of $\mathbb{W}$, the following theorem is immediate.

Theorem 9.3. *Let $\mathcal{A} = (Q, \alpha, \delta, \beta)$ be an weighted automaton over an MV-semiring, $\varphi'_{\mathcal{A}}$ the sentence defined in Equation 23 , and t the translation in Definition 8.1. Then*

$$\|\mathcal{A}\| = S_{t(\varphi'_{\mathcal{A}})}$$

Since semirings derived from the standard MV-algebra and its sub-MV-algebras are locally finite, the following result from [7] is even more interesting in our context.

Theorem 9.4 ([7]). *Let $\mathbb{W}$ be a locally finite commutative semiring and A an alphabet. Then a series $S : A^* \longrightarrow W$ is recognizable iff S is $\mathsf{MSO}(W, A)$-definable.*

Now, the following corollary is immediate.

Corollary 9.1. *A series $S : A^* \longrightarrow [0,1]$ is Ł-recognizable iff S is definable in $\mathsf{MSO}_\vee^{(Ł,[0,1])}(A, \mathbb{X})$.*

Remark 9.1. Corollary 9.1 is also true for all semirings derived from MV-algebras that are (isomorphic to) sub-MV-algebras of $[0,1]^{Ł}$.

10 Conclusion

We detected a connection between the new concept of weighted logics from [7] and Łukasiewicz logic, a well-established many-valued logic.

Normally, formulas in Łukasiewicz logic are interpreted in the standard MV-algebra $[0,1]^{Ł}$. We used a slight generalization of this logic to arbitrary MV-algebras. Due to [8], MV-semirings can be derived from MV-algebras. For every MV-semiring $\mathbb{W}$, we defined fragments $\mathsf{MSO}_\vee^{(Ł,W)}(A, \mathbb{X})$ and $\mathsf{MSO}_\wedge^{(Ł,W)}(A, \mathbb{X})$ of Łukasiewicz logic appropriate for the definition of formal series. We presented a straightforward translation between $\mathsf{MSO}_\vee^{(Ł,W)}(A, \mathbb{X})$ and the weighted logics $\mathsf{MSO}(\mathbb{W}, A)$ over the MV-semiring $\mathbb{W}$. The semirings derived from the standard MV-algebra are locally finite. Hence we could carry over general results about recognizability and decidability of formal series from [7].

Since the strong connectives in Łukasiewicz logic satisfy the normal condition of many-valued logics, the formula proving that every MV-recognizable series is definable in $\mathsf{MSO}_\vee^{(Ł,W)}(A, \mathbb{X})$ could be simplified for this special case.

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