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Infinite Iterated Function Systems in Cantor Space and the Hausdorff Measure of ω-power Languages

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Infinite Iterated Function Systems in Cantor Space and the Hausdorff Measure of ω-power Languages*

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Abstract

We use means of formal language theory to estimate the Hausdorff measure of sets of a certain shape in Cantor space. These sets are closely related to infinite iterated function systems in fractal geometry.

Our results are used to provide a series of simple examples for the non-coincidence of limit sets and attractors for infinite iterated function systems.

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It is well-known that the estimation of the Hausdorff dimension or measure of even simply definable sets might be rather complicated (cf. [Ed90, Fa90, Fa97]). It was shown in [St93, MS94, FS01] that results from language theory facilitate this task.

In Fractal Geometry the calculation of the Hausdorff dimension of fractals generated by iterated function systems (IFS) is well understood. The papers [MW88, Ba89, ČD93] introduced a combination of IFS controlled by finite automata for the description of a wider class of fractals. A different way of generalising IFS was pursued e.g. in [Fe94b, Ma95, MU96] where the iterated function systems were allowed to contain infinitely many functions. In contrast to usual (finite) IFS an infinite IFS has a fixed point which is not necessarily closed in the topology of the underlying space. Hence the closure of the fixed point (which will be called the attractor of the IFS) might be larger.

The aim of the present paper is twofold. On the one hand, we derive results for the calculation of the Hausdorff measure for a class of fractals generated by infinite IFS in Cantor space. Here we use the setting of the theory of languages and ω-languages.

On the other hand, we use this result to exhibit examples for the possible levels of distinction between the fixed point and the attractor of an infinite IFS. These levels are presented by constructing languages defining infinite IFS for which the fixed point and the attractor have different values in Hausdorff dimension and Hausdorff measure.

Hausdorff dimension and Hausdorff measure in Cantor space are particularly interesting in Algorithmic Information Theory. Here Ryabko's [Ry86] lower bound on Kolmogorov complexity (or, equivalently on constructive dimension [Lu03]) by Hausdorff dimension can be strengthened for subsets of non-null Hausdorff measure (see Lemma 3.1 of [St93] or Corollary 5.5 of[CS06]).

It should be mentioned that these results are not restricted to the Cantor space of infinite words as a direct translation of our results on infinite IFS to the unit interval $[0,1] \subseteq \mathbb{R}$ can be obtained by considering an infinite word $\xi \in \{0,\ldots,r-1\}^\omega$ as the r-ary expansion $0.\xi$ of a real number. As indicated in [MS94], this translation generalises easily to unit cubes in d-dimensional space $\mathbb{R}^d$. Moreover, this translation preserves Hausdorff dimension and, up to a certain linear bound, also Hausdorff measure.

1 Notation and Preliminary Results

Next we introduce the notation used throughout the paper. By $\mathbb{N} = \{0,1,2,\ldots\}$ we denote the set of natural numbers. Let X be an alphabet of cardinality $|X| = r$. By X^* we denote the set (monoid) of words on X, including the *empty word* e, and X^ω is the set of infinite sequences (ω-words) over X. For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a language W let $W^* := \bigcup_{i\in\mathbb{N}} W^i$ be the *submonoid* of X^* generated by W, and by $W^\omega := \{w_1 \cdots w_i \cdots : w_i \in W \setminus$

$\{e\}\}$ we denote the set of infinite strings formed by concatenating words in W. Furthermore $|w|$ is the *length* of the word $w \in X^*$ and $\mathbf{A}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{A}(\eta)$ ($\eta \in X^* \cup X^\omega$) by $w \sqsubseteq \eta$.

As usual, a language $V \subseteq X^*$ is called a *code* provided every word $w \in V^*$ has a unique factorisation into words $v_1, \dots, v_k \in V$. If $e \notin V$ and for arbitrary $w, v \in V$ the relation $w \sqsubseteq v$ implies $w = v$ the language V is called a *prefix code*. Further

we denote by $B/w := \{\eta : w \cdot \eta \in B\}$ the *left derivative* of the set $B \subseteq X^* \cup X^\omega$. As usual a language $W \subseteq X^*$ is *regular* provided its set of left derivatives $\{W/w : w \in X^*\}$ is finite. In the sequel we assume the reader to be familiar with basic facts of language theory (e.g. [BP85, HU79] or Vol. 1 of [RS97])

For a language $W \subseteq X^*$ let $s_W : \mathbb{N} \to \mathbb{N}$ where $s_W(n) := |W \cap X^n|$ be its *structure function*. The *structure generating function* corresponding to s_W is

$$\mathfrak{s}_W(t) := \sum\nolimits_{i \in \mathbb{N}} s_W(i) \cdot t^i. \tag{1}$$

$\mathfrak{s}_W$ is a power series with convergence radius $\operatorname{rad} W := \liminf_{n\to\infty} \frac{1}{\sqrt[n]{s_W(n)}}$. It is convenient to consider $\mathfrak{s}_W$ also as a function mapping $[0, \infty)$ to $[0, \infty) \cup \{\infty\}$.

The convergence radius $\operatorname{rad} W$ is closely related to the entropy of the language (cf. [Ku70, St93]),

$$H_W = \limsup\nolimits_{n\to\infty} \frac{\log_r(1 + s_W(n))}{n} .$$

The parameter $\mathbf{t}_1(W) := \sup\{t : t \geq 0 \wedge \mathfrak{s}_W(t) \leq 1\}$ is important for the calculation of $\operatorname{rad} W^*$. It fulfills the following (see [Ei74, Ku70, St93]).

Lemma 1 *It holds* $\mathfrak{s}_W(\mathbf{t}_1(W)) = 1$ *or* $\mathfrak{s}_W(\operatorname{rad} W) < 1$. *If* $\mathfrak{s}_W(\operatorname{rad} W) \leq 1$*, then* $\mathbf{t}_1(W) = \operatorname{rad} W = \operatorname{rad} W^*$. *If* $\mathfrak{s}_W(\operatorname{rad} W) > 1$ *then* $\operatorname{rad} W^* \leq \mathbf{t}_1(W)$.

If W *is a code then we have always* $\operatorname{rad} W^* = \mathbf{t}_1(W)$.

We consider the set X^ω as a metric space (Cantor space) (X^ω, ρ) of all ω-words over the alphabet X where the metric ρ is defined as follows.

$$\rho(\xi, \eta) := \inf\{r^{-|w|} : w \sqsubset \xi \wedge w \sqsubset \eta\} .$$

This space is a compact, and the mapping $\phi_w(\xi) := w \cdot \xi$ is a contracting similitude if only $w \neq e$. Thus a language $W \subseteq X^* \setminus \{e\}$ defines a possibly infinite IFS (IIFS) in (X^ω, ρ). Moreover, $\mathcal{C}(F) := \{\xi : \mathbf{A}(\xi) \subseteq \mathbf{A}(F)\}$ is the closure of the set F (smallest closed subset containing F) in (X^ω, ρ).

Next we recall the definition of the Hausdorff measure and Hausdorff dimension of a subset of (X^ω, ρ) (see [Ed90, Fa90, Fa97]). In the setting of languages and ω-languages this can be read as follows (see [St93, St98]). For $F \subseteq X^\omega$ and $0 \leq \alpha \leq 1$ the equation

$$\mathbb{L}_\alpha(F) := \lim_{l\to\infty} \inf\Big\{ \sum_{w \in W} r^{-\alpha \cdot |w|} : F \subseteq W \cdot X^\omega \wedge \forall w (w \in W \to |w| \geq l) \Big\} \tag{2}$$

defines the α-dimensional metric outer measure on X^ω. The measure $\mathbb{L}_\alpha$ satisfies the following.

Corollary 2 *If* $\mathbb{L}_\alpha(F) < \infty$ *then* $\mathbb{L}_{\alpha+\varepsilon}(F) = 0$ *for all* $\varepsilon > 0$.

Then the *Hausdorff dimension* of F is defined as

$$\dim F := \sup\{\alpha : \alpha = 0 \vee \mathbb{L}_\alpha(F) = \infty\} = \inf\{\alpha : \mathbb{L}_\alpha(F) = 0\}.$$

It should be mentioned that dim is countably stable and shift invariant, that is,

$$\dim \bigcup_{i \in \mathbb{N}} F_i = \sup\{\dim F_i : i \in \mathbb{N}\} \quad \text{and} \quad \dim w \cdot F = \dim F. \tag{3}$$

We list some relations of the Hausdorff dimension and measure for ω-power languages to the properties of the structure generation functions of the corresponding languages (see [St93, MS94, FS01]).

Proposition 3 $\dim W^\omega = -\log_r \operatorname{rad} W^*$

Proposition 4 *If* $\alpha = \dim W^\omega$ *then* $\mathbb{L}_\alpha(W^\omega) \le 1$.

If W *is a regular language then* $0 < \mathbb{L}_\alpha(W^\omega) \le \mathbb{L}_\alpha(\mathcal{C}(W^\omega)) \le 1$, *and if* W *is regular and a union of codes then* $\mathbb{L}_\alpha(W^\omega) = \mathbb{L}_\alpha(\mathcal{C}(W^\omega))$.

The following direct connections between the structure generation function $\mathfrak{s}_W$ and Hausdorff measure $\mathbb{L}_\alpha(W^\omega)$ or $\dim W^\omega$ are helpful.

Proposition 5 1. *If* $\mathfrak{s}_W(r^{-\alpha}) \le 1$ *then* $\alpha \ge \dim W^\omega$.

2. *If* $\mathfrak{s}_W(r^{-\alpha}) < 1$ *then* $\mathbb{L}_\alpha(W^\omega) = 0$.
3. *If* W *is a code and* $\mathfrak{s}_W(r^{-\alpha}) > 1$ *then* $\alpha < \dim W^\omega$.

2 The Hausdorff Measure of ω-power Languages

As we have seen in Proposition 4 the Hausdorff measure $\mathbb{L}_\alpha(W^\omega)$ may vary only between 0 and 1 when $\alpha = \dim W^\omega$. In this section we give some upper bounds on the measure of $\mathbb{L}_\alpha(V^\omega)$ or $\mathbb{L}_\alpha(V^\omega/w)$. more precise than the ones in Section 1. In particular, we derive a formula for the measure $\mathbb{L}_\alpha(V^\omega)$ when $V$is a prefix code.

We start with the following known properties of the ω-power W^ω.

$$(V \cdot W)^\omega = V \cdot (W \cdot V)^\omega \tag{4}$$
$$(V \cup W)^\omega = (V^* \cdot W)^\omega \cup (V \cup W)^* \cdot V^\omega \tag{5}$$

These properties are called the *rotation* (Eq. (4)) and *union splitting* (Eq. (5)) properties, respectively.

Lemma 6 *Let* $w \in \mathbf{A}(V) \setminus V \cdot X \cdot X^*$, *that is,* $w \sqsubseteq v$ *for some* $v \in V$ *but no* $v' \in V$ *is a proper prefix of* w, *and let* $W := V \cap w \cdot X^*$ *and* $\hat{V} := V \setminus W$. *Then*

$$V^\omega \cap w \cdot X^\omega = W \cdot V^\omega = W \cdot (\hat{V}^* \cdot W)^\omega \cup W \cdot V^* \cdot \hat{V}^\omega \text{ and} \tag{6}$$
$$V^\omega / w = (V/w \cdot \hat{V}^* \cdot w)^\omega \cup (V/w) \cdot V^* \cdot \hat{V}^\omega. \tag{7}$$

Proof. The first identity in Eq. (6) follows from the fact that every $w_1 \cdot w_2 \cdots \in V^\omega$ with $w \sqsubseteq w_1 \cdot w_2 \cdots$ has $w_1 \in W$, and the second one is an application of union splitting of $(W \cup \hat{V})^\omega$ (see Eq. (5)).

The second equation follows from the first one, the rotation property and the observations that $V/w = W/w$ and $w \cdot V/w = W$. ❑

2.1 Upper bounds on the Hausdorff measure

With these prerequisites we derive some general properties of the Hausdorff measure of ω-power languages. First, we get a property of the measure of left derivatives.

Lemma 7 *If* $V \subseteq X^*$ *is a code,* $\alpha \geq \dim V^\omega$*, and* $w \in \mathbf{A}(V) \setminus V \cdot X \cdot X^*$ *then* $I\!\!L_\alpha(V^\omega/w) = I\!\!L_\alpha\Big((V/w \cdot (V \setminus w \cdot X^*)^* \cdot w)^\omega\Big)$. *In particular,* $I\!\!L_\alpha(V^\omega/w) \leq 1$.

Proof. We use $\hat{V} := V \setminus w \cdot X^*$ as in Lemma 6.

Since V is a code, we have $\sum_{v \in V} r^{-\alpha|v|} \leq 1$ for $\alpha \geq \dim V^\omega$. Now $\hat{V} \subset V$ implies $\sum_{v \in \hat{V}} r^{-\alpha|v|} < 1$. Hence $I\!\!L_\alpha(\hat{V}^\omega) = 0$ and $I\!\!L_\alpha((V/w) \cdot V^* \cdot \hat{V}^\omega) = 0$. Thus, the first assertion follows from Eq. (7), and then the second one from Proposition 4. ❑

For prefix codes V we have the property that for every $u \in \mathbf{A}(V)^*$ there is a $w \in \mathbf{A}(V) \setminus \{e\}$ such that $V^\omega/u = V^\omega/w$.

In [MS94, Theorem 11] we proved that for every subset $E \subseteq X^\omega$ having $I\!\!L_\alpha(E) > 0$, $\alpha = \dim E$ it holds $\sup\{I\!\!L_\alpha(E/w) : w \in X^*\} \geq 1$. Using this result and Proposition 4 we obtain as a corollary to Lemma 7 the following.

Corollary 8 *If* V *is a prefix code then* $I\!\!L_\alpha(V^\omega/w) \leq 1$ *for all* $w \in X^*$*, and* $0 < I\!\!L_\alpha(V^\omega)$ *iff* $\sup\{I\!\!L_\alpha(V^\omega/w) : w \in X^*\} = 1$.

We say that a language $V \subseteq X^* \setminus \{e\}$ satisfies the *countable intersection property* provided $|V| = 1$ or V^ω is infinite and the set $w \cdot V^\omega \cap v \cdot V^\omega$ is at most countable for every pair of words $w, v \in V, w \neq v$. It should be noted that every language $V \subseteq X^*$ satisfying the countable intersection property is a code. The converse is not true as Example 2.6 of [DL94] shows.

Theorem 9 *If* $V \subseteq X^*$ *satisfies the countable intersection property,* $\sum\limits_{v \in V} r^{-\alpha|v|} = 1$ *for some* $\alpha, 0 < \alpha \leq 1$*, and* $\sum\limits_{w \sqsubseteq v, v \in V} r^{-\alpha|v|} \geq c \cdot r^{-\alpha|w|}$ *for some word* $w \in \mathbf{A}(V) \setminus V \cdot X \cdot X^*$. *Then* $\alpha = \dim V^\omega$ *and* $I\!\!L_\alpha(V^\omega) \leq c^{-1}$.

Proof. $\alpha = \dim V^\omega$ follows from Lemma 1 and Proposition 3.

Set $W := V \cap w \cdot X^*$ and $\hat{V} := V \setminus w \cdot X^*$ as above and observe that $\sum\limits_{v \in \hat{V}} r^{-\alpha|v|} < 1$. As V satisfies the countable intersection property, we have $I\!\!L_\alpha(w \cdot V^\omega \cap v \cdot V^\omega) = 0$ whenever $w, v \in V, w \neq v$. Consequently, $I\!\!L_\alpha(W \cdot V^\omega) = \sum_{v \in W} I\!\!L_\alpha(v \cdot V^\omega) = \sum_{w \sqsubseteq v, v \in V} r^{-\alpha|v|} \cdot I\!\!L_\alpha(V^\omega)$.

On the other hand, using the identity $w \cdot W/w = w \cdot V/w$ implies $W \cdot V^\omega = w \cdot (V^\omega/w)$. Thus from Lemma 7 the inequality $\mathbb{L}_\alpha(W \cdot V^\omega) = r^{-\alpha\cdot|w|} \cdot \mathbb{L}_\alpha(V^\omega/w) \le r^{-\alpha\cdot|w|}$ follows.

Combining the two estimates for $\mathbb{L}_\alpha(W \cdot V^\omega)$ yields

$$\mathbb{L}_\alpha(V^\omega) \le r^{-\alpha\cdot|w|} \cdot \left(\textstyle\sum_{w\sqsubseteq v, v\in V} r^{-\alpha|v|}\right)^{-1} \le c^{-1}. \qquad ❑$$

Letting the constant c in Theorem 9 tend to infinity (if possible) we obtain the following.

Corollary 10 *Let $V \subseteq X^*$ satisfy the countable intersection property, $\sum_{v\in V} r^{-\alpha|v|} = 1$ for some $\alpha, 0 < \alpha \le 1$, and assume that for all $k \in \mathbb{N}$ there is a word $w \in \mathbf{A}(V) \setminus V \cdot X \cdot X^*$ such that $\sum_{w\sqsubseteq v, v\in V} r^{-\alpha|v|} \ge k \cdot r^{-\alpha|w|}$. Then $\alpha = \dim V^\omega$ and $\mathbb{L}_\alpha(V^\omega) = 0$.*

2.2 A lower bound on the Hausdorff measure

A converse to Theorem 9 can be proved for prefix codes.

Theorem 11 *Let $V \subseteq X^*$ be a prefix code, $\sum_{v\in V} r^{-\alpha|v|} = 1$ for some $\alpha, 0 \le \alpha \le 1$, and assume that there is a constant $c > 0$ such that $\sum_{w\sqsubseteq v, v\in V} r^{-\alpha|v|} \le c \cdot r^{-\alpha|w|}$ for all $w \in \mathbf{A}(V)$. Then $\alpha = \dim V^\omega$ and $L_\alpha(V^\omega) \ge \frac{1}{c}$.*

Before proceeding to the proof we have to state in the setting of formal language theory and Cantor space a major tool for deriving lower bounds on Hausdorff measure, the mass distribution principle [Fa90, Principle 4.2]. To this end we mention that the support $\mathbf{supp}\, \mu$ of a measure μ on X^ω is the smallest closed subset $E \subseteq X^\omega$ having $\mu(E) = \mu(X^\omega)$.

Theorem 12 (Mass distribution principle) *Let μ be a measure on X^ω such that $\mathbf{supp}\, \mu \subseteq F$ and suppose that for some α there are numbers $c_0 > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\forall w (w \in X^* \wedge n_0 \le |w| \to \mu(w \cdot X^\omega) \le c_0 \cdot r^{-\alpha\cdot|w|}) .$$

Then $\mathbb{L}_\alpha(F) \ge \mu(F)/c_0$.

Proof. (of Theorem 11) As in the proofs above $\sum_{v\in V} r^{-\alpha|v|} = 1$ implies $\dim V^\omega = \alpha$ provided V is a code.

Since $\sum_{v\in V} r^{-\alpha|v|} = 1$, in case V is infinite we may choose a sequence of natural numbers $l_n, n \in \mathbb{N}$, such that for $V_n := \{v : v \in V \wedge |v| \le l_n\}$ we have $p_n := \sum_{v\in V_n} r^{-\alpha|v|} \ge 1 - r^{-(n+1)}$. Observe that $1 \ge \prod_{i=0}^\infty p_i > 0$.

If V is finite, we choose $V_n := V$ for all $n \in \mathbb{N}$.

For technical reasons, we introduce the following concepts depending on the sequence $(l_n)_{n\in\mathbb{N}}$:

$$W \;:=\; \bigcup_{i=0}^{\infty}\prod_{n=0}^{i} V_n \text{ , and} \tag{8}$$

$$l(w) \;:=\; \min\{i : \exists w'(w\cdot w' \in \prod_{n=0}^{i} V_n)\} \text{ for } w \in \mathbf{A}(W) \tag{9}$$

In order to apply the mass distribution principle we introduce a set function μ on balls $w\cdot X^\omega$ with $w\in W$ (Observe that $w\in V_0\cdots V_i$ implies $l(w)=i$.):

$$\mu(w\cdot X^\omega) := \prod\nolimits_{n=0}^{l(w)} \frac{1}{p_n}\cdot r^{-\alpha|w|}$$

Due to the choice of the coefficient p_n for $w\in W$ we have the identity

$$\begin{aligned}
\sum_{v\in V_{l(w)+1}} \mu(w\cdot v\cdot X^\omega) &= \sum_{v\in V_{l(w)+1}} \prod_{n=0}^{l(wv)} \frac{1}{p_n}\cdot r^{-\alpha|wv|} \\
&= r^{-\alpha|w|}\cdot \prod_{n=0}^{l(w)} \frac{1}{p_n}\cdot \sum_{v\in V_{l(w)+1}} \frac{1}{p_{l(w)+1}}\cdot r^{-\alpha|v|} \\
&= \left(\prod_{n=0}^{l(w)} \frac{1}{p_n}\right)\cdot r^{-\alpha|w|} = \mu(w\cdot X^\omega)
\end{aligned}$$

Letting $\mu(u\cdot X^\omega) := 0$ for $u\notin \mathbf{A}(W)$ we observe that μ is extendible to a metric outer measure on X^ω with support $\mathbf{supp}\,\mu = V_0\cdot V_1\cdots V_i\cdots \subseteq V^\omega$ and $\mu(\mathbf{supp}\,\mu) = 1$ as follows:

From $\mathbf{supp}\,\mu\cap w\cdot X^\omega \subseteq \bigcup_{\substack{w\sqsubseteq v\\ v\in V_0\cdots V_{l(w)}}} v\cdot X^\omega \subseteq w\cdot X^\omega$ we obtain

$$\mu(w\cdot X^\omega) = \sum_{w\sqsubseteq v, v\in V_0\cdots V_{l(w)}} \mu(v\cdot X^\omega) \qquad \text{for } w\in \mathbf{A}(W) = \mathbf{A}(\mathbf{supp}\,\mu).$$

This yields that $\mu(w\cdot X^\omega) = 0$ or

$$\mu(w\cdot X^\omega) \le \sum_{\substack{w\sqsubseteq v\\ v\in V_0\cdots V_{l(w)}}} \prod_{n=0}^{l(w)} \frac{1}{p_n}\cdot r^{-\alpha|v|} \le \prod_{n=0}^{\infty}\frac{1}{p_n}\cdot \sum_{\substack{w\sqsubseteq v\\ v\in V_0\cdots V_{l(w)}}} r^{-\alpha|v|}$$

Now, $w\in \mathbf{A}(W)$ splits uniquely into the product $w = v'\cdot w'$ where $v'\in \prod_{i=1}^{l(w)-1} V_i$ and $w'\in \mathbf{A}(V_{l(w)})\subseteq \mathbf{A}(V)$. Consequently the inequality assumed in the theorem implies

$$\sum_{\substack{w\sqsubseteq v\\ v\in V_0\cdots V_{l(w)}}} r^{-\alpha|v|} = r^{-\alpha|v'|}\cdot \sum_{\substack{w'\sqsubseteq v\\ v\in V_{l(w)}}} r^{-\alpha|v|} \le c\cdot r^{-\alpha|v'|}\cdot r^{-\alpha|w'|} = c\cdot r^{-\alpha|w|}\,.$$

Thus, $\mu(w \cdot X^\omega) \le c \cdot \prod_{n=0}^{\infty} p_n^{-1} \cdot r^{-\alpha|w|}$ for $w \in X^*$.

Next, we apply the mass distribution principle (Theorem 12) to obtain $\mathbb{L}_\alpha(V^\omega) \ge \frac{\mu(V^\omega)}{c \cdot \prod_{n=0}^{\infty} p_n^{-1}} = \frac{1}{c} \cdot \prod\limits_{n=0}^{\infty} p_n > 0$.

Since the choice of the sequence $(l_n)_{n\in\mathbb{N}}$ is arbitrary, we can make $\prod_{n=0}^{\infty} p_n$ as close to 1 as possible, and we obtain the assertion $\mathbb{L}_\alpha(V^\omega) \ge c^{-1}$. ❑

Combining Proposition 5.2 and Theorems 9 and 11 we obtain the following.

Theorem 13 *Let* $V \subseteq X^*$ *be a prefix code and* $\alpha = \dim V^\omega$.

Then $\mathbb{L}_\alpha(V^\omega) = \begin{cases} 0 & \text{, if } \mathfrak{s}_V(r^{-\alpha}) < 1 \text{, and} \\ \inf\{\mathfrak{s}_{V/w}(r^{-\alpha})^{-1} : w \in \mathbf{A}(V)\}, & \text{if } \mathfrak{s}_V(r^{-\alpha}) = 1 \,. \end{cases}$

Proof. Observe that $\mathfrak{s}_{V/w}(r^{-\alpha}) = r^{\alpha\cdot|w|} \cdot \sum\limits_{w\sqsubseteq v, v\in V} r^{-\alpha|v|}$. ❑

Consider also the following interpretation of the constant $c > 0$ in Theorems 9 and 11. Let $V \cdot F \subseteq F$ and let $w \in \mathbf{A}(V)$. Then $\bigcup\limits_{w\sqsubseteq v, v\in V} v \cdot F \subseteq w \cdot F/w$. Now, if V is a prefix code and $0 < \mathbb{L}_\alpha(F) < \infty$, we have $\sum\limits_{w\sqsubseteq v, v\in V} r^{-\alpha\cdot|v|} \le \frac{\mathbb{L}_\alpha(F/w)}{\mathbb{L}_\alpha(F)} \cdot r^{-\alpha\cdot|w|}$.

If we apply this inequality to a prefix code V and $F = V^\omega$ with $\alpha = \dim V^\omega$ and $0 < \mathbb{L}_\alpha(V^\omega)$ and use Corollary 8 we obtain the upper bound $\mathbb{L}_\alpha(V^\omega) \le \inf\{\mathfrak{s}_{V/w}(r^{-\alpha})^{-1} : w \in \mathbf{A}(V)\}$ of Theorem 13.

2.3 A formula for the measure of a product

We conclude this part providing a formula for the Hausdorff measure of the ω-power of the product of two languages. Since $\mathbb{L}_\alpha$ is a metric outer measure on X^ω, the rotation property Eq. (4) implies the equivalence

$$\mathbb{L}_\alpha((W \cdot V)^\omega) = 0 \quad \text{iff} \quad \mathbb{L}_\alpha((V \cdot W)^\omega) = 0 \,, \tag{10}$$

from which $\dim(W \cdot V)^\omega = \dim(V \cdot W)^\omega$ is immediate.

If, moreover, $0 < \mathfrak{s}_W(r^{-\alpha}), \mathfrak{s}_V(r^{-\alpha}) < \infty$ we have

$$\frac{1}{\mathfrak{s}_W(r^{-\alpha})} \cdot \mathbb{L}_\alpha((W \cdot V)^\omega) \le \mathbb{L}_\alpha((V \cdot W)^\omega) \le \mathfrak{s}_V(r^{-\alpha}) \cdot \mathbb{L}_\alpha((W \cdot V)^\omega) \,.$$

If W, V are prefix codes fulfilling an additional condition we can calculate $\mathbb{L}_\alpha((W \cdot V)^\omega)$ from $\mathbb{L}_\alpha(W^\omega)$ and $\mathbb{L}_\alpha(V^\omega)$. Observe that the product of two prefix codes is again a prefix code (see [BP85]).

Theorem 14 *Let* $V, W \subseteq X^*$ *be prefix codes which satisfy* $\dim(W \cdot V)^\omega \ge \max\{\dim W^\omega, \dim V^\omega\}$.

$$\begin{aligned} \textit{Then} \quad \dim(W \cdot V)^\omega &= \max\{\dim W^\omega, \dim V^\omega\} \quad \textit{and} \\ \mathbb{L}_\alpha((W \cdot V)^\omega) &= \min\{\mathbb{L}_\alpha(W^\omega), \mathbb{L}_\alpha(V^\omega)\} \quad \textit{for } \alpha = \dim(W \cdot V)^\omega \,. \end{aligned}$$

Proof. Since the product of W and V is unambiguous, we have $\mathfrak{s}_{W\cdot V}(t) = \mathfrak{s}_W(t) \cdot \mathfrak{s}_V(t)$. Let $\alpha' \geq \max\{\dim W^\omega, \dim V^\omega\}$. This implies $\mathfrak{s}_W(r^{-\alpha'}) \leq 1$ and $\mathfrak{s}_V(r^{-\alpha'}) \leq 1$ and, consequently, $\mathfrak{s}_{W\cdot V}(r^{-\alpha'}) \leq 1$ whence $\alpha' \geq \dim(W \cdot V)^\omega$. This shows $\dim(W \cdot V)^\omega \leq \max\{\dim W^\omega, \dim V^\omega\}$, hence the first assertion.

To show the second one we distinguish two cases. If $\mathfrak{s}_V(r^{-\alpha}) < 1$ we have $\mathfrak{s}_{W\cdot V}(r^{-\alpha}) = \mathfrak{s}_W(r^{-\alpha}) \cdot \mathfrak{s}_V(r^{-\alpha}) < 1$ and, consequently, $\mathbb{L}_\alpha(V^\omega) = \mathbb{L}_\alpha((W \cdot V)^\omega) = 0$

If $\mathfrak{s}_V(r^{-\alpha}) = 1$ we use the relation

$$\mathfrak{s}_{W\cdot V/u}(t) = \begin{cases} \mathfrak{s}_{V/v}(t) & \text{, if } u = w \cdot v \text{ with } w \in W \text{ and } v \in \mathbf{A}(V)\,, \\ \mathfrak{s}_{W/u}(t) \cdot \mathfrak{s}_V(t), & \text{if } u \in \mathbf{A}(W)\,, \end{cases}$$

for $u \in \mathbf{A}(W \cdot V)$. Then Theorem 13 yields the following estimate.

$$\begin{aligned} \mathbb{L}_\alpha((W \cdot V)^\omega) &= \inf\{\mathfrak{s}_{W\cdot V/u}(r^{-\alpha})^{-1} : u \in \mathbf{A}(W \cdot V)\} \\ &= \min\{\inf\{\tfrac{1}{\mathfrak{s}_{W/w}(r^{-\alpha})} : w \in \mathbf{A}(W)\}, \inf\{\tfrac{1}{\mathfrak{s}_{V/v}(r^{-\alpha})} : v \in \mathbf{A}(V)\}\} \\ &= \min\{\mathbb{L}_\alpha(W^\omega), \mathbb{L}_\alpha(V^\omega)\} \end{aligned}$$

❏

The assumption $\dim(W \cdot V)^\omega \geq \max\{\dim W^\omega, \dim V^\omega\}$ in Theorem 14 is essential as the following simple example shows.

Example 1 Consider $W = \{a\}$, $a \in X$, and $V = X$. Then $0 = \dim W^\omega < 1/2 = \dim(W \cdot V)^\omega < 1 = \dim v^\omega$ and $\mathbb{L}_{1/2}((W \cdot V)^\omega) = (\sqrt{|X|})^{-1} < 1$ whereas $\mathbb{L}_0(W^\omega) = \mathbb{L}_1(V^\omega) = 1$. ❏

3 Construction of prefix codes from languages

In this section we derive our examples which show that limit sets and their closures (attractors) for IIFS in Cantor space do not coincide. We present different levels of non-coincidence using Hausdorff dimension and Hausdorff measure.

We intend to find simple examples for these levels of non-coincidence. Simplicity here means, on the one hand that our examples are prefix codes, which makes the IIFS simple, and on the other hand we try to choose them in low classes of the Chomsky hierarchy, preferably linear context-free languages or only a bit more complex.

3.1 Limit Set and Attractor

The limit set in Cantor space of an IIFS described by a language $L \subseteq X^* \setminus \{e\}$ is L^ω. It is also the largest solution (fixed point) of the equation $F = L \cdot F$ when $F \subseteq X^\omega$ (see [St97b]). The attractor of the IIFS is $\mathcal{C}(L^\omega)$. Using the $\mathbf{ls}$ -limit (or adherence) of [LS77] (see also [St97a]) we can describe the difference $\mathcal{C}(L^\omega) \setminus L^\omega$ more precisely.

Set $\mathbf{ls}\, L := \{\xi : \xi \in X^\omega \wedge \mathbf{A}(\xi) \subseteq \mathbf{A}(L)\}$, for $L \subseteq X^*$. Then (see [LS77, St97a])

$$\mathcal{C}(L^\omega) = \mathbf{ls}\, L^* = L^\omega \cup L^* \cdot \mathbf{ls}\, L \tag{11}$$

Now Eq. (3) implies $\dim \mathcal{C}(L^\omega) = \max\{\dim L^\omega, \dim \mathfrak{ls}\, L\}$. For prefix codes L we have additionally the following identity (see [St98]).

$$\mathbb{L}_\alpha(\mathcal{C}(L^\omega)) = \mathbb{L}_\alpha(L^\omega) + \mathbb{L}_\alpha(\mathfrak{ls}\, L) \cdot \sum\nolimits_{i\in\mathbb{N}} \mathfrak{s}_L(r^{-\alpha})^i \tag{12}$$

From our Eq. (12) we obtain several dependencies between the dimensions $\dim L^\omega$, $\dim \mathcal{C}(L^\omega)$ and the corresponding measures $\mathbb{L}_{\alpha'}(L^\omega)$, $\mathbb{L}_{\alpha'}(\mathcal{C}(L^\omega))$ and $\mathbb{L}_{\alpha'}(\mathfrak{ls}\, L)$.

Proposition 15 *Let $L \subseteq X^*$ be a prefix code. Then the following hold true.*

1. *If $\mathbb{L}_{\alpha'}(L^\omega) > 0$ then*
$$\mathbb{L}_{\alpha'}(\mathcal{C}(L^\omega)) = \begin{cases} \mathbb{L}_{\alpha'}(L^\omega) & \text{, if } \mathbb{L}_{\alpha'}(\mathfrak{ls}\, L) = 0 \text{, and} \\ \infty & \text{, otherwise.} \end{cases}$$
2. *If $0 < \mathbb{L}_{\alpha'}(\mathfrak{ls}\, L) < \infty$ and $\mathfrak{s}_L(r^{-\alpha'}) < 1$ then $\mathbb{L}_{\alpha'}(\mathcal{C}(L^\omega))$ is zero, finite or infinite according to whether $\mathbb{L}_{\alpha'}(\mathfrak{ls}\, L)$ is zero, finite or infinite, respectively.*
3. *If $\dim L^\omega < \alpha'$ then $\mathbb{L}_{\alpha'}(\mathcal{C}(L^\omega)) = \infty$ if and only if $\mathbb{L}_{\alpha'}(\mathfrak{ls}\, L) = \infty$.*
4. *If $\dim L^\omega = \alpha$ then $\mathbb{L}_\alpha(\mathcal{C}(L^\omega)) = \infty$ if and only if $\mathbb{L}_\alpha(\mathfrak{ls}\, L) = \infty$ or $\mathbb{L}_\alpha(\mathfrak{ls}\, L) > 0$ and $\mathfrak{s}_L(r^{-\alpha}) = 1$.*

Proof. All properties are immediate from Eq. (12).
1. follows since $\mathbb{L}_{\alpha'}(L^\omega) > 0$ implies $\mathfrak{s}_L(r^{-\alpha'}) \geq 1$.
2. If $\mathfrak{s}_L(r^{-\alpha'}) < 1$ then $\mathbb{L}_{\alpha'}(L^\omega) = 0$ and $\sum_{i\in\mathbb{N}} \mathfrak{s}_L(r^{-\alpha})^i < \infty$.
3. If $\mathbb{L}_{\alpha'}(L^\omega) = 0$ then $\mathfrak{s}_L(r^{-\alpha'}) < 1$ and 3 follows from 2.
4. This holds, since $\dim L^\omega = \alpha$ implies $\mathfrak{s}_L(r^{-\alpha'}) \leq 1$. ❑

3.2 The Padding Construction

In this section we describe a simple construction of prefix codes L for which the properties guaranteeing $\mathbb{L}_{\alpha'}(L^\omega) > 0$ or $\mathbb{L}_{\alpha'}(L^\omega) = 0$ are easily to decide.

We start with a language $W \subseteq (X \setminus \{d\})^*$ where d is a letter in X, and define for an injective function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n) > n$ when $s_W(n) > 0$

$$L := \{w \cdot d^{f(|w|)-|w|} : w \in W\}\,. \tag{13}$$

Then L is a prefix code and $\mathfrak{s}_L(t) = \sum_{n\geq 0} s_W(n) \cdot t^{f(n)}$. Because f is injective and $s_L(i) > 0$ implies that $i = f(j)$ for some $j \in \mathbb{N}$ we have

$$(\operatorname{rad} W)^{\liminf_{n\to\infty} \frac{n}{f(n)}} \geq \operatorname{rad} L = \liminf_{n\to\infty} \frac{1}{\sqrt[f(n)]{s_W(n)}} \geq (\operatorname{rad} W)^{\limsup_{n\to\infty} \frac{n}{f(n)}}\,. \tag{14}$$

If $\lim_{n\to\infty} \sqrt[n]{s_W(n)}$ exists then we have $\operatorname{rad} L = (\operatorname{rad} W)^{\limsup_{n\to\infty} \frac{n}{f(n)}}$.

Since $\mathcal{C}(L^\omega) = L^\omega$ whenever L is finite, we are interested only in infinite languages $W, L \subseteq X^*$. In this case $0 < \frac{1}{|X|-1} \leq \operatorname{rad} W \leq \operatorname{rad} L \leq 1$.

Lemma 16 *Let* $W \subseteq (X \setminus \{d\})^*$ *be an infinite language and let* L *be constructed according to Eq. (13) and let* $\gamma \geq 1$.

1. *If* $f(n) \geq \gamma \cdot n$ *then* $\mathrm{rad}\, L \geq \sqrt[\gamma]{\mathrm{rad}\, W}$, $\mathfrak{s}_L(t) \leq \mathfrak{s}_W(t^\gamma)$ *for* $0 \leq t \leq \sqrt[\gamma]{\mathrm{rad}\, W}$, *and* $\mathbf{t}_1(L) \geq \sqrt[\gamma]{\mathbf{t}_1(W)}$.
2. *If* $f(n) \leq \gamma \cdot n$ *then* $\mathrm{rad}\, L \leq \sqrt[\gamma]{\mathrm{rad}\, W}$, $\mathfrak{s}_L(t) \geq \mathfrak{s}_W(t^\gamma)$ *for* $0 \leq t \leq \sqrt[\gamma]{\mathrm{rad}\, W}$, *and* $\mathbf{t}_1(L) \leq \sqrt[\gamma]{\mathbf{t}_1(W)}$.
3. *If, moreover,* $\gamma = \lim_{n\to\infty} \frac{f(n)}{n}$ *then* $\mathrm{rad}\, L = \sqrt[\gamma]{\mathrm{rad}\, W}$.

Proof. The first two assertions are immediate consequences of the identity $\mathfrak{s}_L(t) = \sum_{n\geq 0} s_W(n) \cdot t^{f(n)} = \sum_{n\geq 0} s_W(n) \cdot t^{\gamma\cdot n} \cdot t^{f(n)-\gamma\cdot n}$ and the fact that $\mathrm{rad}\, L, \mathrm{rad}\, W \leq 1$, and the last one follows from Eq. (14). ❑

It should be mentioned, however, that Eq. (14) and $\gamma = \lim_{n\to\infty} \frac{f(n)}{n}$ do not imply $\mathbf{t}_1(L) = \sqrt[\gamma]{\mathbf{t}_1(W)}$ (see Example 9).

In order to apply Theorem 13 we are interested in connections between $\mathfrak{s}_{L/w}$ and $\mathfrak{s}_{W/w}$ for $w \in \mathbf{A}(W)$.

Lemma 17 *Let* $W \subseteq (X \setminus \{d\})^*$, $f(n) \geq \gamma \cdot n$ *for* $s_W(n) > 0$. *If* $w \in \mathbf{A}(W)$ *then* $\mathfrak{s}_{L/w}(t) \leq \mathfrak{s}_{W/w}(t^\gamma)$ *for* $0 \leq t \leq \sqrt[\gamma]{\mathrm{rad}\, W}$.

If $w \notin \mathbf{A}(W)$ *then* $\mathfrak{s}_{L/w}(t) \leq 1$ *for* $0 \leq t \leq 1$.

Proof. Let $w \in \mathbf{A}(W)$. We consider the identity $L/w = \{u \cdot d^{f(|wu|)-|wu|} : wu \in W\}$. From this we obtain

$$\mathfrak{s}_{L/w}(t) = \sum_{n\in\mathbb{N}} s_{W/w}(n) \cdot t^{f(|w|+n)-|w|} = \sum_{n\in\mathbb{N}} s_{W/w}(n) \cdot t^{\gamma\cdot n} \cdot t^{f(|w|+n)-|w|-\gamma\cdot n},$$

whence $\mathfrak{s}_{L/w}(t) \leq \mathfrak{s}_{W/w}(t^\gamma)$ if $f(n) \geq \gamma \cdot n$ for $s_W(n) > 0$ and the first assertion is proved.

The second assertion is obvious. ❑

Next we want to bound the values of $\mathfrak{s}_{L/w}(t)$, $w \in \mathbf{A}(W)$, uniformly by $\mathfrak{s}_W(t^\gamma)$.

Lemma 18 *Let* $W \subseteq (X \setminus \{d\})^*$ *be infinite,* $f(n) \geq \gamma \cdot n$ *for* $s_W(n) > 0$, *and suppose there are* $k \in \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$ *and* $c \geq 0$ *such that* $s_{W/w}(n) \leq g(|w|) \cdot (\sum_{j=0}^{k} s_W(n+j) + c)$ *for all* $w \in X^*$ *and* $n \in \mathbb{N}$.

Then $\mathfrak{s}_{L/w}(t) \leq t^{(\gamma-1)|w|} \cdot g(|w|) \cdot \Big((\sum_{j=0}^{k} \frac{1}{t^{\gamma\cdot j}}) \cdot \mathfrak{s}_W(t^\gamma) + \frac{c}{1-t^\gamma} \Big)$ *for* $0 < t < 1$ *and* $w \in \mathbf{A}(W)$.

Proof. We have, for $w \in \mathbf{A}(W)$ and $0 \leq t \leq 1$,

$$\begin{aligned} \mathfrak{s}_{L/w}(t) &= \sum\nolimits_{n\in\mathbb{N}} s_{W/w}(n) \cdot t^{f(|w|+n)-|w|} \\ &= \sum\nolimits_{n\in\mathbb{N}} s_{W/w}(n) \cdot t^{\gamma\cdot n} \cdot t^{f(|w|+n)-\gamma(|w|+n)} \cdot t^{(\gamma-1)\cdot|w|} \end{aligned}$$

$$\leq \quad t^{(\gamma-1)|w|} \cdot g(|w|) \cdot \sum_{n \in \mathbb{N}} \Big(\sum_{j=0}^{k} \frac{1}{t^{\gamma \cdot j}} s_W(n+j) \cdot t^{\gamma \cdot (n+j)} + c \cdot t^{\gamma \cdot n} \Big)$$

$$\leq \quad t^{(\gamma-1)|w|} \cdot g(|w|) \cdot \Big(\big(\sum\nolimits_{j=0}^{k} \frac{1}{t^{\gamma \cdot j}} \big) \cdot \mathfrak{s}_W(t^\gamma) + \frac{c}{1 - t^\gamma} \Big) .$$

❏

Under some special assumptions on the language W we obtain the following estimate for $\mathbb{L}_\alpha(L^\omega)$.

Corollary 19 *Under the hypotheses of Lemma 18 and the additional assumptions that* $\mathfrak{s}_L(\mathbf{t}_1(L)) = 1$, $\mathfrak{s}_W(\mathbf{t}_1(L)^\gamma) < \infty$ *and the function* g *satisfies* $\exists c_1 \forall n (g(n) \cdot \mathbf{t}_1(L)^{(\gamma-1)n} \leq c_1)$, *we have* $\mathbb{L}_\alpha(L^\omega) > 0$ *for* $\alpha = \dim L^\omega = -\log_{|X|} \mathbf{t}_1(L)$.

Proof. First, in view of Lemmata 18 and 17, the conditions $\mathfrak{s}_W(\mathbf{t}_1(L)^\gamma) < \infty$ and $\exists c_1 \forall n (\mathbf{t}_1(L)^{(\gamma-1)n} \cdot g(n) \leq c_1)$ ensure that

$$\mathfrak{s}_{L/w}(\mathbf{t}_1(L)) \leq c_1 \cdot \big(\sum\nolimits_{j=0}^{k} \tfrac{1}{\mathbf{t}_1(L)^{\gamma \cdot j}} \big) \cdot \mathfrak{s}_W(\mathbf{t}_1(L)^\gamma) + \tfrac{c}{1 - \mathbf{t}_1(L)^\gamma} \big) < \infty$$

independently of $w \in X^*$.

Then, $\mathfrak{s}_L(\mathbf{t}_1(L)) = 1$ allows the application of Theorem 13, which yields the assertion. ❏

The next corollary treats the special case of regular languages W.

Corollary 20 *If* $W \subseteq (X \setminus \{d\})^*$ *is an infinite regular language,* $f(n) \geq \gamma \cdot n$ *for* $s_W(n) > 0$, $\mathfrak{s}_W(\mathbf{t}_1(L)^\gamma) < \infty$ *and* $\mathfrak{s}_L(\mathbf{t}_1(L)) = 1$ *then* $\dim L^\omega = -\log \mathbf{t}_1(L)$ *and* $\mathbb{L}_\alpha(L^\omega) > 0$ *for* $\alpha = \dim L^\omega$.

Proof. With $\mathfrak{s}_L(\mathbf{t}_1(L)) = 1$ the first hypothesis of Theorem 13 is fulfilled, and $\dim L^\omega = -\log \mathbf{t}_1(L)$.

Observe that $t^{|w|} \cdot \mathfrak{s}_{W/w}(t) \leq \mathfrak{s}_W(t)$ whenever $0 \leq t$. If W is regular, there is a constant $k \in \mathbb{N}$ such that for every $w \in X^*$ there is a $\hat{w}, |\hat{w}| \leq k$ with $W/w = W/\hat{w}$.

According to Lemma 17 we have $\mathfrak{s}_{L/w}(\mathbf{t}_1(L)) \leq \max\{1, \frac{\mathfrak{s}_W(\mathbf{t}_1(L)^\gamma)}{\mathbf{t}_1(L)^k}\}$ for arbitrary $w \in X^*$, and Theorem 13 shows $\mathbb{L}_\alpha(L^\omega) > 0$ for $\alpha = \dim L^\omega$. ❏

If we change the order in the construction of Eq. (13) we obtain for $\widetilde{d} \in X$ and $\widetilde{W} \subseteq (X \setminus \{\widetilde{d}\})^*$

$$\widetilde{L} := \{\widetilde{d}^{f(|w|)-|w|} \cdot w : w \in \widetilde{W}\} , \tag{15}$$

and the results on the structure generating function Eq. (14) and Lemma 16 remain valid. In particular, $\widetilde{L}$ is also a prefix code[1]. Moreover we have a lower bound for $\mathfrak{s}_{\widetilde{L}/w}$.

Proposition 21 *If* $w = \widetilde{d}^{f(n)}$ *then* $\mathfrak{s}_{\widetilde{L}/w}(t) \geq s_{\widetilde{W}}(n) \cdot t^n$.

[1]This is, however, not true in general. But if $e \notin \widetilde{W}$ and $f(n) - f(m) \neq n - m$ for $n \neq m$ then the assertion is true. These hypotheses are, in particular, satisfied for the constructions in Section 3.3.

Observe that then $\mathbf{t}_1(\widetilde{L}) > \operatorname{rad}\widetilde{W}$ implies $\limsup_{n\to\infty} s_{\widetilde{W}}(n)\cdot \mathbf{t}_1(\widetilde{L})^n = \infty$. This enables us to apply Corollary 10 and we obtain the following.

Corollary 22 *Let* $\widetilde{W} \subseteq (X \setminus \{\widetilde{d}\})^*$ *be infinite,* $f : \mathbb{N} \to \mathbb{N}$ *injective and* $f(n) > n$ *for* $s_{\widetilde{W}}(n) > 0$. *If* $\widetilde{L} = \{\widetilde{d}^{f(|w|)-|w|}\cdot w : w \in \widetilde{W}\}$ *and* $\mathbf{t}_1(\widetilde{L}) > \operatorname{rad}\widetilde{W}$ *then* $I\!\!L_\alpha(\widetilde{L}^\omega) = 0$ *for* $\alpha = \dim \widetilde{L}^\omega$.

It should be mentioned that for linear functions $f : \mathbb{N} \to \mathbb{N}$, $f(n) = \gamma\cdot n + \delta$ with rational coefficients, and regular languages $W, \widetilde{W}$ the resulting languages L and $\widetilde{L}$ are one-turn deterministic one-counter languages, simple cases of unambiguous linear context-free languages [AB97]. Thus they have rational structure generating functions $\mathfrak{s}_L$ and $\mathfrak{s}_{\widetilde{L}}$, respectively (see [Ku70]).

Their (unambiguous) product, $L\cdot\widetilde{L}$, where we may start with different regular languages $W, \widetilde{W}$ is a two-turn deterministic one-counter language, and has also a rational structure generating function $\mathfrak{s}_{L\cdot\widetilde{L}} = \mathfrak{s}_L\cdot\mathfrak{s}_{\widetilde{L}}$.

For rational structure generating functions $\mathfrak{s}_V$ we have the restriction that $\mathfrak{s}_V(\operatorname{rad} V) = \infty$ whence $\mathfrak{s}_V(\mathbf{t}_1(V)) = 1$.

3.3 Examples

In this section we give our announced examples. Here we consider the following cases which might appear for $\alpha = \dim L^\omega$ and $\hat{\alpha} = \dim \mathcal{C}(L^\omega)$, $I\!\!L_\alpha(L^\omega)$ and $I\!\!L_{\hat{\alpha}}(\mathcal{C}(L^\omega))$. The principal possibilities are shown in the figure below. The case $I\!\!L_\alpha(L^\omega) = \infty$ is excluded by Proposition 4.

We try to derive our examples as simple as possible. Therefore, on the one hand, we consider only prefix codes L. In this case Eq. 12 and Proposition 15 give some principal limitations.

On the other hand, in the light of the discussion concluding Section 3.2 we want our examples to be languages to be simple with respect to their accepting devices (cf. [AB97]).

In Figures 1 and 2 we list the twelve possible cases for relations between $\dim L^\omega$, $\dim \mathcal{C}(L^\omega)$, $I\!\!L_{\dim L^\omega}(L^\omega)$ and $I\!\!L_{\dim \mathcal{C}(L^\omega)}(\mathcal{C}(L^\omega))$.

What concerns $I\!\!L_{\dim L^\omega}(L^\omega)$ and $I\!\!L_{\dim \mathcal{C}(L^\omega)}(\mathcal{C}(L^\omega))$ we distinguish only the cases of null-measure, finite non-null measure and infinite measure. According to Proposition 4 the case $I\!\!L_{\dim L^\omega}(L^\omega) = \infty$ is impossible.

In virtue of Proposition 4 we cannot choose regular languages as examples (except for Case 2). Moreover, Proposition 15.1 shows that, for $\dim L^\omega = \dim \mathcal{C}(L^\omega)$ and prefix codes L, the Case 4 is impossible.

Observe that in Figure 1 we have $\dim L^\omega \geq \dim \mathbf{ls}\, L$, and in Cases 3, 5 and 6 necessarily $\alpha = \dim L^\omega = \dim \mathbf{ls}\, L$ and $I\!\!L_\alpha(\mathbf{ls}\, L) > 0$.

In Figure 2 we have always $\dim L^\omega = \alpha < \hat{\alpha} = \dim \mathbf{ls}\, L$.

The construction of our ten examples follows a general line. We let X consist of the four letters a, b, d and $\widetilde{d}$, and we arrange our examples according to increasing complexity. All examples, except for Example 9, have $f(n) = \gamma\cdot n$ with $\gamma \in \{2, 3, 4\}$.

	fixed point L^ω	**attractor** $\mathcal{C}(L^\omega)$	**Example**
1.	$\mathbb{L}_\alpha(L^\omega) = 0$	$\mathbb{L}_\alpha(\mathcal{C}(L^\omega)) = 0$	Example 2
2.	$\mathbb{L}_\alpha(L^\omega) > 0$	$\mathbb{L}_\alpha(\mathcal{C}(L^\omega)) = \mathbb{L}_\alpha(L^\omega)$	Proposition 4
3.	$\mathbb{L}_\alpha(L^\omega) = 0$	$0 < \mathbb{L}_\alpha(\mathcal{C}(L^\omega)) < \infty$	Example 9
4.	$\mathbb{L}_\alpha(L^\omega) > 0$	$\mathbb{L}_\alpha(L^\omega) < \mathbb{L}_\alpha(\mathcal{C}(L^\omega)) < \infty$	impossible
5.	$\mathbb{L}_\alpha(L^\omega) = 0$	$\mathbb{L}_\alpha(\mathcal{C}(L^\omega)) = \infty$	Example 6
6.	$\mathbb{L}_\alpha(L^\omega) > 0$	$\mathbb{L}_\alpha(\mathcal{C}(L^\omega)) = \infty$	Example 3

Figure 1: Measures of fixed point and attractor when $\alpha = \dim L^\omega = \dim \mathcal{C}(L^\omega)$

	fixed point L^ω	**attractor** $\mathcal{C}(L^\omega)$	**Example**
7.	$\mathbb{L}_\alpha(L^\omega) = 0$	$\mathbb{L}_{\widehat{\alpha}}(\mathcal{C}(L^\omega)) = 0$	Example 11
8.	$\mathbb{L}_\alpha(L^\omega) > 0$	$\mathbb{L}_{\widehat{\alpha}}(\mathcal{C}(L^\omega)) = 0$	Example 10
9.	$\mathbb{L}_\alpha(L^\omega) = 0$	$0 < \mathbb{L}_{\widehat{\alpha}}(\mathcal{C}(L^\omega)) < \infty$	Example 7[2]
10.	$\mathbb{L}_\alpha(L^\omega) > 0$	$0 < \mathbb{L}_{\widehat{\alpha}}(\mathcal{C}(L^\omega)) < \infty$	Example 4
11.	$\mathbb{L}_\alpha(L^\omega) = 0$	$\mathbb{L}_{\widehat{\alpha}}(\mathcal{C}(L^\omega)) = \infty$	Example 8
12.	$\mathbb{L}_\alpha(L^\omega) > 0$	$\mathbb{L}_{\widehat{\alpha}}(\mathcal{C}(L^\omega)) = \infty$	Example 5

Figure 2: Measures of fixed point and attractor when $\dim L^\omega = \alpha < \widehat{\alpha} = \dim \mathcal{C}(L^\omega)$

In the first seven examples we use the languages $W^{(1)} := \{a,b\}^* \setminus \{e\}$, $W^{(2)} := (\{a,b\}\cdot a)^* \setminus \{e\}$ and $W^{(3)} := \{a,b\}^* \cdot \widetilde{d} \cdot \{a,b\}^*$ with $\mathbf{ls}\, W^{(1)} = \{a,b\}^\omega$, $\mathbf{ls}\, W^{(2)} = (\{a,b\}\cdot a)^\omega$, $\mathbf{ls}\, W^{(3)} = \{a,b\}^\omega \cup \{a,b\}^* \cdot \widetilde{d} \cdot \{a,b\}^\omega$ and the parameters:

$$\begin{array}{llll}
\mathfrak{s}_{W^{(1)}}(t) = \frac{2t}{1-2t} & , \mathbf{t}_1(W^{(1)}) = \frac{1}{4} & \text{and} & \mathbb{L}_{\frac{1}{2}}(\mathbf{ls}\, W^{(1)}) = 1 \\
\mathfrak{s}_{W^{(2)}}(t) = \frac{2t^2}{1-2t^2} & , \mathbf{t}_1(W^{(2)}) = \frac{1}{2} & \text{and} & \mathbb{L}_{\frac{1}{4}}(\mathbf{ls}\, W^{(2)}) = 1 \\
\mathfrak{s}_{W^{(3)}}(t) = \frac{t}{(1-2t)^2} & , \mathbf{t}_1(W^{(3)}) = \frac{1}{4} & \text{and} & \mathbb{L}_{\frac{1}{2}}(\mathbf{ls}\, W^{(3)}) = \infty
\end{array}$$

(see [MS94, Example B])

The first four examples are one-turn deterministic one-counter languages.

Example 2 Set $W_2 := W^{(1)}$, $\gamma_2 := 4$ and use the construction of Eq. (15). Lemma 16 shows $\mathbf{t}_1(L_2) = \frac{1}{\sqrt{2}} > \operatorname{rad} W_2 = \frac{1}{2}$. Since $\mathbf{ls}\, L_2 = \{\widetilde{d}\}^\omega$, we have $\mathbb{L}_\alpha(\mathcal{C}(L_2^\omega)) = \mathbb{L}_\alpha(L_2^\omega)$, for $\alpha = -\log_4 \mathbf{t}_1(L_2) = \frac{1}{4}$, and Corollary 22 yields $\mathbb{L}_\alpha(L_2^\omega) = 0$. ❑

In Examples 3, 4 and 5 we use the construction of Eq. (13) and Corollary 20 to show that $\mathbb{L}_\alpha(L^\omega) > 0$. Observe that the construction of Eq. (13) yields $\mathbf{ls}\, L = \mathbf{ls}\, W$.

[2]For Case 9, an example of a language generated by a simple context-free grammar was given in Example 6.3 of [St93].

Example 3 We set $W_3 := W^{(1)}$ and $\gamma_3 := 2$. Then $\alpha = \dim L_3^\omega = -\log_4 \mathbf{t}_1(L_3) = \frac{1}{2}$ and $\mathfrak{s}_{L_3}(\mathbf{t}_1(L_3)) = 1$. Now, Proposition 15 implies $I\!\!L_\alpha(\mathcal{C}(L_3^\omega)) = \infty$. ❏

Example 4 We use $W_4 := W^{(1)}$ and $\gamma_4 := 4$. Then $\alpha = \dim L_4^\omega = \frac{1}{4}$, $\hat{\alpha} = \dim \mathbf{ls}\, L_4 = \frac{1}{2}$, $\mathfrak{s}_{L_4}(4^{-\hat{\alpha}}) = \mathfrak{s}_{W_4}(\frac{1}{16}) = \frac{1}{7}$ and, finally, $I\!\!L_\alpha(\mathcal{C}(L_4^\omega)) = \frac{7}{6}$. ❏

Example 5 Set $W_5 := W^{(3)}$ and $\gamma_5 := 4$. This yields $\alpha = \dim L_5^\omega = -\log_4 \mathbf{t}_1(L_5) = \frac{1}{4}$ and $\hat{\alpha} = \dim \mathbf{ls}\, L_5 = \frac{1}{2}$ and $I\!\!L_{\hat{\alpha}}(\mathbf{ls}\, L_5) = \infty$ (cf. Example B of [MS94]). ❏

The next three examples and Example 11 are products of languages L_i' and $\widetilde{L}_i$ constructed according to Eqs. (13) and (15), respectively. Then we can use Theorem 14 to show that $I\!\!L_\alpha((L_i' \cdot \widetilde{L}_i)^\omega) = 0$. Since $\mathbf{ls}\, \widetilde{L}_i = \{\widetilde{d}\}^\omega$, we have $I\!\!L_{\alpha'}(\mathbf{ls}\,(L_i' \cdot \widetilde{L}_i)) = I\!\!L_{\alpha'}(\mathbf{ls}\, L_i')$ for $\alpha' > 0$.

Example 6 Define L_6' using Eq. (13) and the parameters $W_6' := W^{(2)}$ and $\gamma' := 2$. This yields $\mathbf{t}_1(L_6') = \frac{1}{\sqrt{2}}$ and $\alpha = \dim L_6'^\omega = \frac{1}{4}$. Now $\widetilde{L}_6 := L_2$ has also $\dim \widetilde{L}_6^\omega = \frac{1}{4}$ and, consequently, $I\!\!L_{\frac{1}{4}}((L_6' \cdot \widetilde{L}_6)^\omega) = 0$.

Finally, $I\!\!L_\alpha(\mathbf{ls}\,(L_6' \cdot \widetilde{L}_6)) = I\!\!L_\alpha(\mathbf{ls}\, L_6') = 1$ and $\mathfrak{s}_{L_6'}(4^{-\alpha}) = \mathfrak{s}_{\widetilde{L}_6}(4^{-\alpha}) = 1$ yield $I\!\!L_\alpha(\mathcal{C}((L_6' \cdot \widetilde{L}_6)^\omega)) = \infty$. ❏

Example 7 Here we use $L_7' := L_4$ and $\widetilde{L}_7' := L_2$ and argue in the same way as in the preceding example.

Example 8 This example uses the language $L_8' := L_5$ and concatenates it with $\widetilde{L}_8 := L_2$. ❏

Because of $I\!\!L_\alpha(L^\omega) = 0$ and $\infty > I\!\!L_\alpha(\mathbf{ls}\, L) > 0$ Item 3 requires $\mathfrak{s}_L(\mathsf{rad}\, L) < 1$. This is not possible with languages having a rational structure generating function.

Example 9 Set $W := \{a, b, \widetilde{d}\}^* \setminus \{e\}$ and $f(n) := n + 2\lceil\sqrt{n}\rceil$. Then $\mathfrak{s}_L(t) = \sum_{i=1}^\infty 3^n t^{n+2\lceil\sqrt{n}\rceil}$.

Since $\lim_{n\to\infty} \frac{f(n)}{n} = 1$, in virtue of Lemma 16, we have $\mathsf{rad}\, L = \mathsf{rad}\, W = 1/3$. Thus we obtain $\mathfrak{s}_L(\mathsf{rad}\, L) = \sum_{i=1}^\infty (\frac{1}{3})^{2\lceil\sqrt{n}\rceil} = \frac{5}{32} < 1$[3], and consequently $0 = I\!\!L_\alpha(L^\omega) < I\!\!L_\alpha(\mathcal{C}(L^\omega)) = \frac{32}{27} < \infty$ for $\alpha := \dim L^\omega = \dim \mathbf{ls}\, L = \log_4 3$. ❏

In view of $\alpha < \hat{\alpha}$ and $I\!\!L_{\hat{\alpha}}(\mathcal{C}(L_i^\omega)) = 0$ the final two examples require $I\!\!L_{\hat{\alpha}}(\mathbf{ls}\, L) = 0$. Following Lemma 4.3 of [St93] W cannot be a regular language.

Example 10 Let $F := \{a, b\} \cdot \prod_{i=0}^\infty (\{a, b\}^{2^i - 1} \cdot a)$ and set $W_{10} := \mathbf{A}(F) \setminus \{e\}$. Then $s_W(n) = 2^{n - \lfloor \log_2 n \rfloor}$ for $n > 0$.

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[3]This follows from the identity $\sum_{n=1}^\infty t^{\lceil\sqrt{n}\rceil} = \sum_{i=1}^\infty (2i-1) \cdot t^i = \frac{t+t^2}{(1-t)^2}$.

Since F is closed in (X^ω, ρ) and $s_{\mathbf{A}(F/w)}(n) = s_{\mathbf{A}(F/v)}(n)$ whenever $w, v \in \mathbf{A}(F)$ and $|w| = |v|$, Theorem 4 of [St89] shows that $\dim F = \liminf_{n\to\infty} \frac{\log_4 s_{\mathbf{A}(F)}(n)}{n} = \frac{1}{2}$. Moreover, it is easy to calculate that $I\!\!L_{1/2}(F) = 0$.

Choose $\gamma_{10} = 3$ and use the construction of Eq. (13). Then $\mathbf{ls}\, L_{10} = F$,

$$\begin{aligned} -\ln(1-2t^3) \; = \; \sum_{i=1}^{\infty} \tfrac{(2t^3)^n}{n} \; < \; \mathfrak{s}_{L_{10}}(t) \; &< \; 2 \cdot \sum_{i=1}^{\infty} \tfrac{(2t^3)^n}{n} - 2t^3 \\ &= \; -2(\ln(1-2t^3) + t^3) \end{aligned}$$

for $0 < t \le \frac{1}{\sqrt[3]{2}}$, and we obtain $\mathfrak{s}_{L_{10}}(\frac{1}{\sqrt[3]{4}}) < 1 < \mathfrak{s}_{L_{10}}(\frac{1}{\sqrt[3]{3}}) < \infty$. Therefore, $\mathbf{t}_1(L_{10}) < \frac{1}{\sqrt[3]{3}}$, $\mathfrak{s}_{L_{10}}(\mathbf{t}_1(L_{10})) = 1$ and $\alpha = \dim L_{10}^\omega = -\log \mathbf{t}_1(L_{10}) < \frac{1}{2}$. Although we do not know the exact value of $\alpha = \dim L_{10}^\omega$, this allows us to show $I\!\!L_\alpha(L^\omega) > 0$ using Corollary 19 in the following way:

From the preceding considerations we know that the hypotheses $\mathfrak{s}_{L_{10}}(\mathbf{t}_1(L_{10})) = 1$ and $\mathfrak{s}_{W_{10}}(\mathbf{t}_1(L_{10})^3) < \mathfrak{s}_{W_{10}}(\frac{1}{3}) < \sum_{i=1}^{\infty} (\frac{2}{3})^i < \infty$ of Corollary 19 are satisfied.

Now, the funktion $g : \mathbb{N} \to \mathbb{N}$ with $g(n) = n$ satisfies the remaining assumption $\mathbf{t}_1(L_{10})^{2n} \cdot g(n) < (\frac{1}{3})^{2n} \cdot n \le 1$, for all $n \in \mathbb{N}$. Hence $I\!\!L_\alpha(L^\omega) > 0$. ❑

Example 11 Let $L'_{11} := L_{10}$ and let $\widetilde{L}_{11}$ be constructed according to Eq. (15) with $\widetilde{W}_{11} := W_{10}$ and $\widetilde{\gamma}_{11} := \gamma_{10} = 3$.

Arguing in the same way as in Examples 2 and 6 we calculate that Corollary 22 is applicable and obtain $\alpha = \dim(L'_{11} \cdot \widetilde{L}_{11})^\omega < \hat{\alpha} = \frac{1}{2}$, and $I\!\!L_\alpha((L'_{11} \cdot \widetilde{L}_{11})^\omega) = 0$ as well as $I\!\!L_{\hat{\alpha}}(\mathcal{C}((L'_{11} \cdot \widetilde{L}_{11})^\omega)) = I\!\!L_{\hat{\alpha}}(\mathcal{C}(L'^{\omega}_{11})) = 0$. ❑

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