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Constructive Dimension equals Kolmogorov Complexity

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Abstract

We derive the coincidence of Lutz's constructive dimension and Kolmogorov complexity for sets of infinite strings from Levin's early result on the existence of an optimal left computable cylindrical semi-measure **M** via simple calculations.

1 Introduction

Constructive dimension in relation to algorithmic randomness can be seen in the same way as Hausdorff dimension theory relative to Lebesgue measure theory. This is supported by Lutz's characterisation of Hausdorff dimension in terms of *s*-(super-)gales (see [L1], [L2] and [L3]). His investigations led to a theory of constructive dimension in relation to Algorithmic randomness similar to that of Hausdorff dimension to Lebesgue measure theory.

As it was observed in Section 6 of [L3] parts of this theory were discovered earlier by different people and from a different point of view: When one observes that random sequences are those of maximal Kolmogorov complexity (see [LV, Sections 3.6 and 4.5.7]), it is only natural to consider levels of Kolmogorov complexity of individual sequences (cf. Eq. (4) and (12) below) as a natural counterpart to their constructive dimension.

In [Ma] Mayordomo proved that for individual infinite strings Lutz's [L3] constructive dimension coincides with their lower Kolmogorov complexity, $\underline{\kappa}(\xi)$. The present paper shows that the generalisation of Mayordomo's result to sets of infinite sequences is already an immediate consequence of early results in this area, for instance, of the existence of the optimal left computable cylindrical semi-measure **M** proved already in Zvonkin's and Levin's seminal paper [ZL].

Moreover, the derivation presented here makes also obvious that the constructive strong dimension defined in [AH] and upper Kolmogorov complexity $\kappa(\xi)$ coincide.

This elucidates the relations between the papers [CH] [R2], [R3], [S1] [S2] and also [Ta] and the subject matter of [AH], [L1], [L2] and [L3] in a more precise manner than the mere remark in [L2] that "Moreover, Ryabko, Staiger, and Cai and Hartmanis have all proven results establishing quantitative relationships between Hausdorff dimension and Kolmogorov complexity." As a consequence, several results derived in the mentioned papers are applicable to constructive dimension theory and vice versa. Unfortunately, the corresponding definitions and results are spread into a bunch of different papers, mixed with other (interesting) things and obscured with a lot of technical details.

The aim of this note is to set up a starting point that might lead to the understanding of the whole picture of constructive dimension and Kolmogorov complexity as one theory by giving a derivation of the coincidence of Lutz's constructive dimension and Kolmogorov complexity for sets of infinite strings from Levin's [ZL] early result via simple calculations.

2 Preliminaries

We consider the Cantor space of infinite strings (ω-words), X^ω over a finite alphabet X of cardinality $|X| \geq 2$. By X^* we denote the set of finite strings (words) over X.

The Kolmogorov complexity $K : X^* \to \mathbb{N}$ of strings assigns to every word $w \in X^*$ a natural number. The definition and basic properties of Kolmogorov complexity, K, can be found in the book by Li and Vitányi [LV] or in the papers [US] or [ZL]. Observe that there are several variants of Kolmogorov complexity of finite words in use (see [LV, Section 5.5.4] or [US]).

For reasons which will become clear later we prefer the variant of Kolmogorov complexity based on Levin's universal lower semi-computable semi-measure. To this end, we recall that a function $f : X^* \to \mathbb{R}$ is referred to as *left computable (or lower semi-computable)* provided the set $\{(w,q) \mid q \in \mathbb{Q} \wedge q < f(w)\}$ is computably enumerable.

A *cylindrical semi-measure* on X^* is a function $\mu : X^* \to \mathbb{R}_+$ which satisfies the inequality

$$\forall w(w \in X^* \to \mu(w) \geq \textstyle\sum_{a \in X} \mu(wa)). \qquad (1)$$

A cylindrical semi-measure **M** is called multiplicatively *optimal* for a class $\mathscr{M}$ of cylindrical semi-measures provided

$$\forall \mu(\mu \in \mathscr{M} \to \exists c(c > 0 \wedge \forall w(w \in X^* \to \mathbf{M}(w) \geq c \cdot \mu(w)))). \tag{2}$$

In [ZL] it was shown the following.

Lemma 1 *There is a left computable cylindrical semi-measure* $\mathbf{M}$ *optimal for the class of all left computable cylindrical semi-measures.*

Moreover, any of the variants of Kolmogorov complexity considered in [LV, Section 5.5.4] or [US] satisfies the inequality.

$$|K(w) - (-\log \mathbf{M}(w))| \leq O(\log |w|) \tag{3}$$

Then

$$\underline{\kappa}(\xi) := \liminf_{w \to \xi} \frac{K(w)}{|w|} = \liminf_{w \to \xi} \frac{-\log \mathbf{M}(w)}{|w|} \tag{4}$$

is referred to as the *lower Kolmogorov complexity* of an ω-word $\xi \in X^\omega$. Here $\liminf_{w \to \xi}$ is a shorthand for the limit inferior taken over all (finite) prefixes w of ξ when the length $|w|$ tends to infinity.

3 Constructive Dimension

Next we are going to introduce the constructive dimension of a set $F \subseteq X^\omega$. An *s-supergale* is a function $d : X^\omega \to \mathbb{R}_+$ which satisfies the condition

$$\forall w(w \in X^* \to d(w) \geq |X|^{-s} \sum_{a \in X} d(wa)). \tag{5}$$

An *s-gale* is an s-supergale that satisfies Eq. (5) with equality.

Observe that a cylindrical semi-measure is a 0-supergale.

As in [L1], [L2] or [L3] we say that an s-supergale d *succeeds* on an ω-word $\xi \in X^\omega$ if

$$\limsup_{w \to \xi} d(w) = \infty. \tag{6}$$

The *success set* of an s-supergale d is $S^\infty[d] := \{\xi \mid \xi \in X^\omega \wedge d \text{ succeeds on } \xi\}$. Left computable supergales were called *constructive* (cf. [L1], [L2], [L3] or [Ma]). The *constructive dimension*, $\mathrm{cdim}(F)$, of a set $F \subseteq X^\omega$ is

$$\mathrm{cdim}(F) := \inf\{s \mid s \in \mathbb{R}_+ \wedge \exists d(d \text{ is a constructive } s\text{-supergale } \wedge F \subseteq S^\infty[d])\}.$$

Quite recently Hitchcock [Hi] proved that one can replace the term “supergale” by “gale” in the above definition. But for gales there is no analogous version of

Lutz's [L3] Theorem 3.6 (see Corollary 3 below) yielding a universal family of optimal supergales. One easily infers that

$$d(w) := |X|^{s\cdot|w|} \cdot \mu(w) \tag{7}$$

is an s-supergale if μ is a cylindrical semi-measure, and vice versa (cf. [L2, Observation 3.1]). Moreover, we have the following.

Lemma 2 *Let s be a computable real number, $\mu : X^* \to \mathbb{R}_+$ and let d be defined by Eq. (7). Then μ is a left computable (semi-)measure iff d is a constructive s-(super)gale.*

Analogously to measures, we call a constructive s-supergale $\mathbf{d}$ optimal provided for every constructive s-supergale d there is a constant c_d such that $\forall w(w \in X^* \to \mathbf{d}(w) \geq c_d \cdot d(w))$. Then the following is obvious.

Corollary 3 ([L3, Theorem 3.6]) *For every computable real number $s \in [0,1]$ the mapping $\mathbf{d}^{(s)} : X^* \to \mathbb{R}_+$ defined by*

$$\mathbf{d}^{(s)}(w) := |X|^{s\cdot|w|} \cdot \mathbf{M}(w) \tag{8}$$

is an optimal constructive s-supergale.

Now, using the family of optimal constructive s-supergales described in Corollary 3, we can prove our assertion. First we mention a first consequence of our corollary.

$$\begin{aligned} \mathrm{cdim}(F) &= \inf\{s \mid \forall\xi(\xi \in F \to \limsup_{w\to\xi} \mathbf{d}^{(s)}(w) = \infty)\} \\ &= \inf\{s \mid \forall\xi(\xi \in F \to \limsup_{w\to\xi} \mathbf{d}^{(s)}(w) > 0)\}. \end{aligned} \tag{9}$$

The latter identity holds, since $\limsup\limits_{w\to\xi} \mathbf{d}^{(s)}(w) > 0$ implies $\limsup\limits_{w\to\xi} \mathbf{d}^{(s')}(w) = \infty$ for $s' > s$.

3.1 Constructive Dimension equals Kolmogorov Complexity

Next we derive a relation between the condition $\limsup_{w\to\xi} \mathbf{d}^{(s)}(w) > 0$ and the lower Kolmogorov complexity of ξ.

Lemma 4 *Let $\xi \in X^\omega$. Then*

1. $\limsup_{w\to\xi} \mathbf{d}^{(s)}(w) > 0$ *implies* $\underline{\kappa}(\xi) \leq s$, *and*
2. $\underline{\kappa}(\xi) < s$ *implies* $\limsup_{w\to\xi} \mathbf{d}^{(s)}(w) = \infty$.

Proof. Let $\limsup_{w\to\xi} \mathbf{d}^{(s)}(w) > 0$. Then there is a $c > 0$ such that $|X|^{s\cdot|w|}\cdot\mathbf{M}(w) \geq c$ for infinitely many prefixes w of ξ . Consequently, $\frac{-\log \mathbf{M}(w)}{|w|} \leq s - \frac{\log_{|X|} c}{|w|}$ infinitely often for prefixes w of ξ. Taking $\liminf_{w\to\xi}$ on both sides yields $\underline{\kappa}(\xi) \leq s$.

Let now $\underline{\kappa}(\xi) < s$ and set $\varepsilon := \frac{s-\underline{\kappa}(\xi)}{3}$. Then there are infinitely many prefixes w of ξ such that $\frac{-\log_{|X|}\mathbf{M}(w)}{|w|} < s-\varepsilon$. All these prefixes w satisfy $|X|^{\varepsilon\cdot|w|} < |X|^{s\cdot|w|}\mathbf{M}(w)$, whence $\limsup_{w\to\xi} \mathbf{d}^{(s)}(w) = \infty$. ❑

Now, Lemma 4 together with Eq. (9) yields our main theorem.

Theorem 5

$$\forall F (F \subseteq X^\omega \to \mathrm{cdim}(F) = \underline{\kappa}(F))$$

4 Strong dimension

Recently, in [AH] effective strong dimension cDim was introduced using limit inferior instead of the limit superior in Eq. (6). It was said that an s-supergale d *strongly succeeds* on an ω-word $\xi \in X^\omega$ if

$$\liminf_{w\to\xi} d(w) = \infty, \tag{10}$$

and the *strong success set* of an s-supergale d was defined as $S^\infty_{str}[d] := \{\xi \mid \xi \in X^\omega \wedge d \text{ strongly succeeds on } \xi\}$. Then *strong constructive dimension,* $\mathrm{cDim}(F)$, of a set $F \subseteq X^\omega$ is

$$\mathrm{cDim}(F) := \inf\{s \mid s \in \mathbb{R}_+ \wedge \exists d (d \text{ is a constructive } s\text{-supergale} \wedge F \subseteq S^\infty_{str}[d])\},$$

and, utilising Corollary 3, we obtain the respective counterparts to Eq. (9) and Lemma 4.

$$\begin{aligned} \mathrm{cDim}(F) &= \inf\{s \mid \forall\xi(\xi \in F \to \liminf_{w\to\xi} \mathbf{d}^{(s)}(w) = \infty)\} \\ &= \inf\{s \mid \forall\xi(\xi \in F \to \liminf_{w\to\xi} \mathbf{d}^{(s)}(w) > 0)\}. \end{aligned} \tag{11}$$

Substituting, in the proof of Lemma 4, lim sup by lim inf and simultaneously infinitely often by almost all yields a proof of its counterpart.

Lemma 6 *Let* $\xi \in X^\omega$. *Then*

1. $\liminf_{w\to\xi} \mathbf{d}^{(s)}(w) > 0$ *implies* $\kappa(\xi) \leq s$, *and*

2. $\kappa(\xi) < s$ *implies* $\liminf_{w\to\xi} \mathbf{d}^{(s)}(w) = \infty$,

Here, using the notation of [S1], we refer to

$$\kappa(\xi) := \limsup_{w\to\xi} \frac{\kappa(w)}{|w|} = \limsup_{w\to\xi} \frac{-\log \mathbf{M}(w)}{|w|} \tag{12}$$

as the *upper Kolmogorov complexity*, of an ω-word $\xi \in X^\omega$. If we set $\kappa(F) := \sup\{\kappa(\xi) : \xi \in F\}$ we obtain likewise the analogue of Theorem 5 for strong constructive dimension and upper Kolmogorov complexity.

Theorem 7

$$\forall F (F \subseteq X^\omega \to \mathrm{cDim}(F) = \kappa(F))$$

5 Conclusion

A mayor achievement was the characterisation of Hausdorff dimension $\dim_H$ and packing dimension $\dim_P$ via s-gales in [L1] and [AH]. (For a good introduction to fractal dimensions see [Fa].)

$$\dim_H(F) = \inf\{s \mid s \in \mathbb{R}_+ \wedge \exists d (d \text{ is an } s\text{-supergale } \wedge F \subseteq S^\infty[d])\} \tag{13}$$
$$\dim_P(F) = \inf\{s \mid s \in \mathbb{R}_+ \wedge \exists d (d \text{ is an } s\text{-supergale } \wedge F \subseteq S^\infty_{str}[d])\} \tag{14}$$

Theorems 5 and 7 together with the obvious inequalities $\dim_H(F) \le \mathrm{cdim}(F)$ and $\dim_P(F) \le \mathrm{cDim}(F)$ thus yield, on the one hand, a new proof of Ryabko's inequality (see [R2, Theorem 2] or [S1, Corollary 3.14])

$$\dim_H(F) \le \underline{\kappa}(F)\ , \tag{15}$$

and, on the other hand,

$$\dim_P(F) \le \kappa(F)\ . \tag{16}$$

Both inequalities show that large (in dimension) sets must contain complex infinite strings. Conditions under which equality holds in Eq. (15) are discussed in [S1] and [S2]. Moreover, several propositions in the theory of constructive dimension can be derived using previous results relating Hausdorff dimension and Kolmogorov complexity.

As an example we consider the sets $\underline{\mathrm{DIM}}^{<\alpha} := \{\xi \mid \xi \in X^\omega \wedge \mathrm{cdim}(\{\xi\}) < \alpha\}$ and $\underline{\mathrm{DIM}}^{\alpha} := \{\xi \mid \xi \in X^\omega \wedge \mathrm{cdim}(\{\xi\}) = \alpha\}$. Then Theorem 5 together with results of [R1] and [CH, Theorem 3.8], proves the following assertion.

$$\forall \alpha (0 \le \alpha \le 1 \to \dim_H(\underline{\mathrm{DIM}}^{<\alpha}) = \dim_H(\underline{\mathrm{DIM}}^{\alpha}) = \alpha).$$

It is interesting to note that for $\mathrm{DIM}^{<\alpha} := \{\xi \mid \xi \in X^\omega \wedge \mathrm{cDim}(\{\xi\}) < \alpha\}$ the same identity holds even for Hausdorff dimension (see [S1, Eq. (5.6)]).

$$\forall \alpha (0 \le \alpha \le 1 \to \dim_H(\mathrm{DIM}^{<\alpha}) = \alpha).$$

In contrast to this, we have

$$\dim_P(\underline{\mathrm{DIM}}^0) = 1 \text{ and } \forall\alpha(0 < \alpha \leq 1 \rightarrow \dim_P(\underline{\mathrm{DIM}}^{<\alpha}) = 1). \tag{17}$$

Proof. It is known from [S3] that $\underline{\mathrm{DIM}}^0$ contains the set of expansions of Liouville numbers L. The set of numbers L is a set of second Baire category in X^ω [Ox]. By [Ed, Exercise 1.8.4] it follows $\dim_P(L) = 1$. ❏

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References

[AH] Athreya, K.B., Hitchcock, J.M., Lutz, J.H., and Mayordomo E., Effective strong dimension, algorithmic information, and computational complexity, in: *Proc. 21st Symposium on Theoretical Aspects of Computer Science*, Springer-Verlag, Berlin, 2004, 632 – 643.

[CH] Cai, J., and Hartmanis J., On Hausdorff and topological dimensions of the Kolmogorov complexity of the real line. Journal of Computer and Systems Sciences, 49 (1994), 605 – 619.

[Ed] Edgar, G.A., *Integral, Probability, and Fractal Measures*, Springer-Verlag, New York, 1998.

[Fa] Falconer, K., *Fractal Geometry*, Wiley, New York, 1990.

[Hi] Hitchcock, J.M., Gales suffice for constructive dimension. Inform. Process. Letters 86 (2003), 9 – 12.

[LV] Li, M., and Vitányi, P.M.B., *An Introduction to Kolmogorov Complexity and its Applications*, 2nd edition, Springer-Verlag, Berlin, 1997.

[L1] Lutz, J.H., Dimension in complexity classes, SIAM Journal on Computing, 32 (2003), 1236 – 1250.

[L2] Lutz, J.H., Gales and the constructive dimension of individual sequences, in: *Proc. 27th International Colloquium on Automata, Languages, and Programming*, Springer-Verlag, Heidelberg, 2000, 902 – 913.

[L3] Lutz, J.H., The dimensions of individual strings and sequences, Inform. and Comput. 187 (2003), 49 – 79.

[Ma] Mayordomo, E., A Kolmogorov complexity characterization of constructive Hausdorff dimension, Inform. Process. Letters 84 (2002), 1 – 3.

[Ox] Oxtoby, J.C., *Measure and Category*, Springer-Verlag, Berlin, 1971.

[R1] Ryabko, B.Ya., Coding of combinatorial sources and Hausdorff dimension, Soviet Math. Dokl. 30 (1984) 1, 219 – 222.

[R2] Ryabko, B.Ya., Noiseless coding of combinatorial sources. Problems of Information Transmission 22 (1986), 170 – 179.

[R3] Ryabko, B.Ya., The complexity and effectiveness of prediction problems, J. Complexity 10 (1994), 281 – 295.

[S1] Staiger, L., Kolmogorov complexity and Hausdorff dimenson, Inform. and Comput. 103 (1993), 159 – 194.

[S2] Staiger, L., A tight upper bound on Kolmogorov complexity and uniformly optimal prediction, Theory Comput. Systems 31 (1998), 215 – 229.

[S3] Staiger, L., The Kolmogorov complexity of real numbers, Theoret. Comput. Sci. 284 (2002), 455 – 466.

[Ta] K. Tadaki. A generalization of Chaitin's halting probability Ω and halting self-similar sets, Hokkaido Math. J. 31 (2002), 219 – 253.

[US] Uspensky, V.A., and Shen, A., Relations between varieties of Kolmogorov complexities. Math. Systems Theory 29 (1996), 271 – 292.

[ZL] Zvonkin, A.K., and Levin, L.A., Complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms, Russian Math. Surveys 25 (1970), 83 – 124.