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# A Simple Example of an $\omega$-language Topologically Inequivalent to a Regular One

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Landwebers's paper [La69] and the subsequent ones [SW74, TY83] proved a strong relationship between acceptance conditions imposed on finite automata on $\omega$-words and the first classes of the Borel hierarchy in the Cantor space of all $\omega$-words, $(X^\omega, \rho)$, over a finite alphabet $X$. In Theorem 5 of [SW74] it is shown that an $\omega$-language accepted by a finite automaton being simultaneously an $\mathbf{F}_\sigma$- and a $\mathbf{G}_\delta$-set belongs already to the Boolean closure of the class of all open (or, equivalently, closed) subsets of $(X^\omega, \rho)$, $\mathscr{B}(\mathbf{G})$. Thus, an $\omega$-language $F \subseteq X^\omega$ which is simultaneously an $\mathbf{F}_\sigma$- and a $\mathbf{G}_\delta$-set but not a Boolean combination of open sets cannot be accepted by a finite automaton. For a more detailed discussion see e.g. [EH93, St97] or [Th90], for the notation used here see [St97].

The aim of this note is to provide a simple[1] example that a proposition analogous to Theorem 5 of [SW74] is no longer true if we increase the computational power of the accepting device slightly:
We augment the finite control by a so-called blind counter (cf. [EH93, Fi01], these automata are also known as partially blind counter automata [Gr78]), that is, by a counter which has no influence on the computational behavior of the automaton except

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[1]The meaning of the word "simple" here is twofold: on the one hand, as explained above, the topological complexity of our counterexample is the simplest possible one, and, on the other hand, the accepting device has a power only slightly increasing the power of a finite automaton.

that the automaton gets stuck when the counter is decremented below zero. Moreover we require the counter to be one-turn, that is, once we decrease the value of the counter we cannot increase it afterwards.

We are not going to define these one-turn blind one-counter automata in full detail, instead we proceed with the announced example. On reading the first block of $a$'s

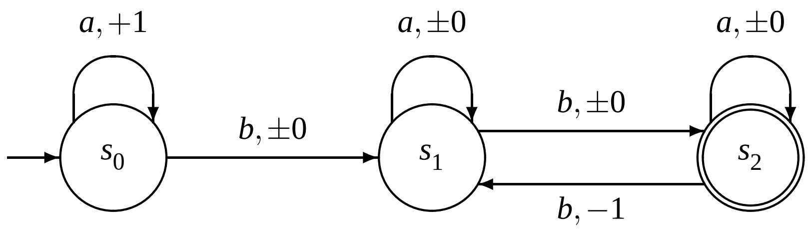


Figure 1: *A Büchi automaton accepting the $\omega$-language of Eq. (1)*

the automaton stores the block length in the counter, and after reading the first $b$ the automaton switches to the cycle of states $s_1, s_2$ where the counter is decremented after every second $b$. Thus the automaton gets stuck when the input $a^i bw$ contains at least $2i+2$ $b$s and, consequently, if the automaton does not get stuck it will finally stay in one of its states. (Looping between $s_1$ and $s_2$ is bounded by the number of initial $a$s.)

Thus depending on the infinite input word $\xi$ our automaton will stay in

$$s = \begin{cases} s_0, & \text{if } \xi = a^\omega \\ s_1, & \text{if } \xi = a^n b \cdot \xi' \text{ and } \xi' \text{ contains an even number of } b\text{'s less than } 2n+1 \\ s_2, & \text{if } \xi = a^n b \cdot \xi' \text{ and } \xi' \text{ contains an odd number of } b\text{'s less than } 2n+2\,. \end{cases}$$

Our acceptance condition is Büchi's condition and the set of final states is $\{s_2\}$, that is, an $\omega$-word $\xi \in \{a,b\}^\omega$ is accepted if and only if the automaton runs infinitely often through state $s_2$ when reading $\xi$. Thus the $\omega$-language accepted by our automaton is

$$F = \bigcup_{n\in\mathbb{N}} a^n b \cdot \bigcup_{i\leq n} (a^* b)^{2i+1} \cdot a^\omega\,. \tag{1}$$

This $\omega$-language is a countable subset of $\{a,b\}^\omega$, thus an $\mathbf{F}_\sigma$-set. Since it is accepted by a deterministic automaton using Büchi acceptance it is also a $\mathbf{G}_\delta$-subset of $\{a,b\}^\omega$ (cf. [EH93, St97]).

We are going to show that our $\omega$-language $F$ cannot be represented as a Boolean combination of open (or closed) $\omega$-languages. Thus, according to Theorem 5 of [SW74] (an even stronger version is Corollary 23 of [St83]), it cannot be accepted by a finite automaton.

Assume the contrary, that is, let $E_i, E'_i$ be open subsets of $\{a,b\}^\omega$ such that

$$F = \bigcup\nolimits_{i=1}^{k} E_i \setminus E'_i\,. \tag{2}$$

Consequently, every subset $F \cap w \cdot \{a,b\}^\omega$ has a similar representation

$$F \cap w \cdot \{a,b\}^\omega = \bigcup_{i=1}^{k} \left(E_i \cap w \cdot \{a,b\}^\omega\right) \setminus \left(E_i' \cap w \cdot \{a,b\}^\omega\right) . \tag{3}$$

as a Boolean combination of open sets $E_i \cap w \cdot \{a,b\}^\omega$ and $E_i' \cap w \cdot \{a,b\}^\omega$.

Consider the $\omega$-languages $F_n := F \cap a^n b \cdot \{a,b\}^\omega = a^n b \cdot \bigcup_{i \le n} (a^* b)^{2i+1} \cdot a^\omega$. Every single $\omega$-language $F_n$ is accepted by a finite automaton. Moreover, $F_n$ is essentially (up to the prefix $a^n b$ and the encoding: $1 \to a$ and $0 \to b$) Wagner's $\omega$-language $c_1^{2n-1} := \bigcup_{i<n} (a^* b)^{2i+1} \cdot a^\omega$ taken over the alphabet $\{a,b\}$.

It is shown in Lemma 11 of [Wa79] that $c_1^{2n+1} \in \widehat{C}_1^{2n+1} \setminus \widehat{C}_1^{2n-1}$ and, consequently, $F_n \in \widehat{C}_1^{2n-1} \setminus \widehat{C}_1^{2n-3}$. For the definition of Wagner classes see [Wa79, p. 139] or Definition 4.1 of [St97].

At the same time Eq. (3) and the fact that the $\omega$-languages $F_n$ are accepted by finite automata imply $F_n \in \widehat{C}_1^{2k+1}$ for all $n \in \mathbb{N}$, a contradiction. Thus Eqs. (3) and (2) cannot hold true, and $F$ is not a Boolean combination of open subsets of Cantor space $(X^\omega, \rho)$.

Finally, we present the Petri net derived from the automaton in Fig. 1 which accepts the same $\omega$-language. For acceptance of $\omega$-languages by Petri nets see [HR86, Va83].

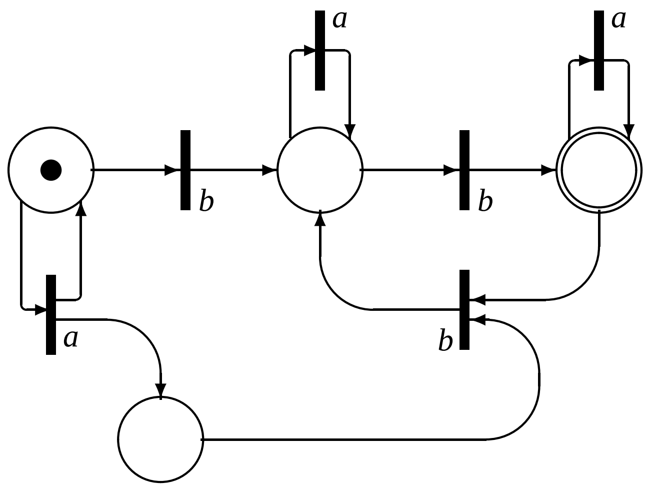


Figure 2: *A Petri net accepting F*

In Fig. 2, the initial marking is represented by a black dot. We also adopt Büchi's acceptance condition with the set of accepting markings having at least one token in the doubly circled place. Likewise we may adopt the co-Büchi acceptance condition where ultimately all accepting markings have a token in the doubly circled place.

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