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Abstract

We provide for the first time a complete list of forbidden minors (obstructions) for the family of graphs with vertex cover 6. This paper shows how to limit both the search space of graphs and improve the efficiency of an obstruction checking algorithm when restricted to k–VERTEX COVER graph families. In particular, our upper bounds $2k+1$ ($2k+2$) on the maximum number of vertices for connected (disconnected) obstructions are shown to be sharp for all $k > 0$.

1 Introduction

The main contribution of this paper is the characterization of graphs with vertex cover at most 6 by its obstruction set (forbidden minors). The general problem of vertex cover (which is $\mathcal{N}P$-complete; see [GJ79]) asks whether a graph has a set of vertices of size at most k that covers all edges (a more formal definition is given below). Earlier Cattell and Dinneen in [CD94] classified the families of graphs with vertex cover at most 5 by using the computational machinery now described in [CDD+97]. Our current results are based on a more family-specific approach where we limit the search space of graphs. In this paper, as our primary limiting factor, we prove an exact upper bound on the number of vertices for an obstruction to any k–VERTEX COVER family.

The numbers of obstructions for 1–VERTEX COVER to 5–VERTEX COVER, along with our new result for 6–VERTEX COVER, are listed below in Table 1.

Table 1: Numbers of obstructions for k–VERTEX COVER, $1 \leq k \leq 6$.

k	Connected obstructions	Disconnected obstructions	Total obstructions
1	1	1	2
2	2	2	4
3	3	5	8
4	8	10	18
5	31	25	56
6	**188**	**72**	**260**

We had known that the set of obstructions for 6–VERTEX COVER is finite by the now-famous Graph Minor Theorem (GMT) of Robertson and Seymour [RS85]. They proved Wagner's conjecture which states that there are a finite number of obstructions for any graph family closed under the minor order. Unfortunately the proof of the GMT does not indicate how to find these obstructions. The set of planar graphs is the best known example of a family with "forbidden graphs", where Kuratowski's characterization provides us with K_5 and $K_{3,3}$ as the only obstructions to planarity. Another example is the set of 103 irreducible graphs (35 minor-order obstructions) for the projective plane [GHW79, Arc80]. A lot of work has recently been done concerning the development of general methods for computing minor-order obstructions, such as mentioned in the papers [FL89, APS91, LA91, Pro93].

1.1 Preliminary definitions

In this paper we use standard graph theory definitions (e.g. see [CL86]). A graph is a pair (V, E), where V is a finite set of vertices and E is a set of undirected edges connecting two vertices of V. An edge between vertices x and y of V will be denoted by xy.

A *partial order* is a reflexive, transitive and antisymmetry binary relation. A graph H is a *minor* of a graph G, denoted $H \leq_m G$, if a graph isomorphic to H can be obtained from G by a sequence of a operations chosen from:

i. delete a vertex,
ii. delete an edge, or
iii. contract an edge (removing any multiple edges or loops that form).

The *minor order* is the set of finite graphs ordered by $\leq_m$ and is easily seen to be a partial order.

A family $\mathcal{F}$ of graphs is a *lower ideal*, under a partial order $\leq_p$, if whenever a graph $G \in \mathcal{F}$ implies that $H \in \mathcal{F}$ for any $H \leq_p G$. For this paper we will always take $\leq_p$ to be the minor order $\leq_m$. An *obstruction* (often called *forbidden minor*) O for a lower ideal $\mathcal{F}$ is a minor-order minimal graph not in $\mathcal{F}$. Thus, for example, K_5 and $K_{3,3}$ are the 'smallest' non-planar graphs (under the minor order). Recall that by the GMT, a complete set of obstructions provides a *finite characterization* for any (minor-order) lower ideal $\mathcal{F}$.

The graph families of interest in this paper are based on the following problem.

Problem 1. Vertex Cover
Input: Graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Is there a subset $V' \subseteq V$ with $|V'| \leq k$ such that V' contains at least one vertex from every edge in E?

A set V' in the above problem is called a *vertex cover* for the graph G. The family of graphs that have a vertex cover of size at most k will be denoted by k–Vertex Cover. For a given graph G, let $VC(G)$ denote the least k such that G has vertex cover of cardinality k. Figure 1 shows an example of a graph G with $VC(G) = 4$.

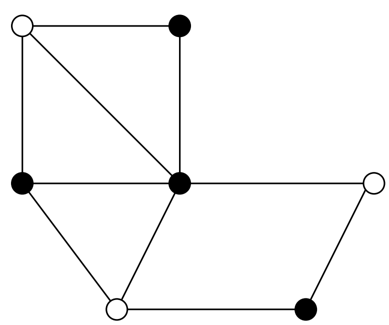

Figure 1: An example of a graph of vertex cover 4. The black vertices denote one possible vertex cover.

For completeness, we repeat from [CD94] the simple proof that the k–Vertex Cover graph families are closed under the minor-order operators.

Lemma 2. *The graph family k–*Vertex Cover *is a lower ideal in the minor order.*

Proof. Assume a graph $G = (V, E)$ has a minimal vertex cover $V' \subseteq V$ of size k. If $H = G \backslash uv$ for some $uv \in E$ (edge deletion), then V' is also a vertex cover for H. Likewise, if $u \in V$ is an isolated vertex of G, V' also covers $H = G \backslash u$ (vertex deletion). For any edge $uv \in E$, observer that $|\{u, v\} \cap V'| \geq 1$. Let w be the new vertex created from u and v in $H = G/uv$ (edge contraction). Clearly, $V'' = (V' \cup w) \backslash \{u, v\}$ is a vertex cover of H with size at most k. Since any minor of G can be created by repeating the above operations, k–Vertex Cover is a lower ideal. □

Finally, as another related reference, we define the following graph problem (see [GJ79, DCF96]).

Problem 3. Feedback Vertex Set
Input: A graph $G = (V, E)$ and a non-negative integer $k \leq |V|$.
Question: Is there a subset $V' \in V$ with $|V'| \leq k$ such that V' contains at least one vertex form every cycle in G?

A set V' in this problem is called a *feedback vertex set* for the graph G. The family of graphs that have a feedback vertex set of size at most k is denoted by k–FEEDBACK VERTEX SET. Similar to Lemma 2 it is easy to see that k–FEEDBACK VERTEX SET is a lower ideal in the minor order.

1.2 Outline of the paper

We now describe the organization of our paper. In the next section we explain our computational model and prove some general results relating to how to compute obstructions for any k–VERTEX COVER lower ideal. This is followed by Section 3 where we prove some specific results regarding 6–VERTEX COVER, namely edge bounds. After a short conclusion and references, we list all of the connected obstructions for 6–VERTEX COVER in Figures 11–16.

2 Computing Minor-Order Obstructions

We now begin to describe how we compute the obstructions for k–VERTEX COVER. Our basic method is simply to generate and check all graphs that are potential obstructions. In practice, this search method can be used for an arbitrary lower ideal if one can (1) bound the search space of graphs to a reasonable size and (2) easily decide whether an arbitrary graph is an obstruction. With respect to k–VERTEX COVER, we show how to do both of these tasks efficiently in this and the next sections.

In earlier work, Cattell and Dinneen in [CD94] bounded the search space to graphs of pathwidth at most $k+1$ when they computed the k–VERTEX COVER obstructions for $1 \leq k \leq 5$. Their generation process was self-terminating but, unfortunately, many isomorphic graphs were created during the generation of the search space. These superfluous graphs had to be either caught by an isomorphism checking program or eliminated by other means. For $k = 6$ using this approach did not seem feasible. Armed with McKay's graph generation program `geng` (part of his `Gtools/Nauty` package [McK90]) and some tight new upper bounds on the structure of vertex cover obstructions, we have succeeded in the characterization of 6–VERTEX COVER.

Although we are primarily interested in computing the obstructions for the specific case of 6–VERTEX COVER, the ideas presented here may be extended to other

k–Vertex Cover families, or even to other parameterized families of minor-order lower ideals. One example is the observation that the search space can be restricted to connected graphs; the disconnected obstructions for k–Vertex Cover are easily obtained from the connected obstructions for smaller values of k (see below and [CD94, Din97]).

2.1 Directly checking non-isomorphic graphs

We now discuss a natural method for finding a set of obstructions for a lower ideal from an available set of non-isomorphic graphs. We need to generate (e.g., using `nauty`) a complete set of non-isomorphic connected graphs, which is large enough to contain the set of obstructions sought. Our initial programming model is simply described as follows.

Program FindAllObstructions(GraphFamily $\mathcal{F}$)
    repeat
        Get a graph G from the input stream
        if G is an obstruction of $\mathcal{F}$ **then** Save G
    until no more graphs
end

By using a graph membership algorithm for a given lower ideal $\mathcal{F}$, an algorithm to decide if a graph is an obstruction is almost trivial. If a given graph G is an obstruction for $\mathcal{F}$, then G is a minimal graph such that $G \notin \mathcal{F}$. For a lower ideal $\mathcal{F}$ we only need to check that each 'one-step' minor of G is in $\mathcal{F}$. A general algorithm to decide if G is an obstruction for $\mathcal{F}$ is presented below.

Procedure IsObstruction(GraphMembershipAlgorithm GA, Graph G)
    if $GA(G) =$ `true` **then return** `false`
    for each edge e **in** G **do**
        $G' =$ the resulting graph after deleting e in G
        if $GA(G') =$ `false` **then return** `false`
$+$ $G'' =$ the resulting graph after contracting e in G
$+$ **if** $GA(G'') =$ `false` **then return** `false`
    endfor
    return `true`
end

To decide if a graph is an obstruction it helps to have an efficient membership algorithm GA for the targeted lower ideal $\mathcal{F}$. For example, our decision algorithm for the 6–Vertex Cover family of graphs is an implementation of the Balasubramanian *et. al.* linear-time vertex cover algorithm [BFR98].

The above algorithm `IsObstruction` will work for any minor-order lower ideal. For a given input finite graph G, if the graph family membership algorithm GA operates in polynomial time, then this algorithm `IsObstruction` has polynomial-time complexity. However, for some particular lower ideals, the algorithm can be simplified (e.g., remove lines marked with +'s). For the k–VERTEX COVER families, the following theorem is very useful.

Theorem 4. *A graph $G = (V, E)$ is an obstruction for k–*VERTEX COVER *if and only if $VC(G) = k + 1$ and $VC(G\backslash uv) = k$ for all $uv \in E$.*

Proof. Let the graph G be an obstruction for k–VERTEX COVER. This implies that if any edge in G is deleted then the vertex cover decreases, by the definition of an obstruction. The size of the vertex cover decreases by one to exactly k, otherwise $VC(G) < k + 1$.

Now we prove the other direction. Let $G = (V, E)$ be a graph, and $\mathcal{F}$ denote a fixed k–VERTEX COVER lower ideal such that $G \notin \mathcal{F}$. Suppose if any edge in G is deleted, then the resulting graph $G' \in \mathcal{F}$. Thus for each edge uv in G, a set of vertices V' of cardinality k can be found which covers all edges in G except edge uv.

Let u be the reserved vertex and v be the deleted vertex after uv is contracted. Since V' covers all edges in G except uv, V' covers each edge wv where $w \neq u$. We have $w \in V'$. After contraction of uv, for each edge wv, a new edge wu is made in G'. Since $w \in V'$, V' covers wu. Thus all new edges are covered by V'. Hence after doing any edge contraction for any edge uv in G, a vertex cover V' where $|V'| = k$ covers all edges in the resulting graph G'. That is $G' \in \mathcal{F}$. Therefore if each edge deletion causes $G' \in \mathcal{F}$ and $G \notin \mathcal{F}$, then G is an obstruction. □

Thus according to the above theorem, for k–VERTEX COVER, our obstruction deciding algorithm is simplified. This means that it does not have to check edge-contraction minors (i.e., create the graphs G'' in the `IsObstruction` procedure). This greatly reduces the overall computation time for deciding if a graph is an obstruction since doing any one edge contraction, with most graph data structures, is not relatively efficient (as compared with the actual time needed to delete any one edge).

Unfortunately the previous theorem does not hold for most other lower ideals, such as k–FEEDBACK VERTEX SET. For example, in the Figure 2, the graph G has a feedback vertex set with size 2. If we delete any edge of G then the resulting graph contains a feedback vertex set with size 1. But if the edge uv is contracted, the resulting graph still requires a feedback vertex set of size 2.

2.2 Properties of connected k–VERTEX COVER obstructions

The number of connected non-isomorphic graphs of order n increases exponentially. For $n = 11$, the number of connected graphs is 1006700565. If we could process one

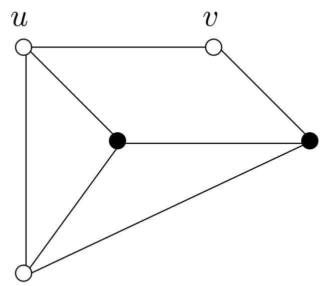

Figure 2: A graph G illustrating that checking edge contractions for k–FEEDBACK VERTEX SET is important.

graph per microsecond we would still need over 11 days of running time. Since there exist obstructions larger than this for 6–VERTEX COVER we clearly need a more restricted search space. Thus we will filter the input by exploiting other properties of the k–VERTEX COVER obstructions.

A few important properties for k–VERTEX COVER obstructions are now systematically presented. As mentioned above, we need these results to tightly constrain the search space to a manageable number of graphs. We first mention some basic properties concerning k–VERTEX COVER here. (The interested reader may consult [DCF96] for similar properties for the k–FEEDBACK VERTEX SET lower ideals.) In the next section we will give additional results that specialize for the 6–VERTEX COVER lower ideal.

Lemma 5. *Any connected obstruction for the k–*VERTEX COVER *lower ideal is a biconnected graph.*

Proof. Suppose v is a cut-vertex in a connected obstruction O for k–VERTEX COVER. Let $C_1, C_2, \ldots, C_{m\geq 2}$ be the connected components of $O\backslash\{v\}$ and $C'_i = O[V(C_i)\cup\{v\}]$, where $O[X]$ denotes the subgraph induced by vertices X. Each C'_i denotes the part of the graph containing the component C_i, the vertex v, and the edges between v and C_i.

Since O is an obstruction to k–VERTEX COVER. We have

$$\sum_{i=1}^{m} VC(C_i) = k \quad .$$

Any vertex cover for $\bigcup_{i=1}^{m} C'_i$ is also a vertex cover for O, where vertex v may be repeated in several C'_i. Thus,

$$\sum_{i=1}^{m} VC(C'_i) \geq VC(O) = k+1 \quad .$$

This implies that there exists an i such that $VC(C_i') = VC(C_i) + 1$. Now $O' = (\bigcup_{j \neq i} C_j) \cup C_i'$ is a proper subgraph of O. But $VC(O') = k + 1$ contradicts O being an obstruction. So O does not have any cut-vertices. Therefore, O is a biconnected graph. □

Lemma 5 implies that any vertex in a connected obstruction O for k–VERTEX COVER has at least n edges, where n is the order of O.

Lemma 6 (Cattell–Dinneen). *In an obstruction for k–*VERTEX COVER*, any two vertices with degree 2 do not have the same neighborhood.*

Proof. Suppose u and v are two vertices with degree 2 in an obstruction O for k–VERTEX COVER and have the same neighborhood $\{x, y\}$.

To cover the four edges: ux, vx, uy, vy in O, either $\{u, v\}$ or $\{x, y\}$ is sufficient. If any edge of above four edges is deleted. To cover all edges in this structure, two vertices are still necessary. For example, if ux is deleted, v and y have to be used to cover the remaining 3 edges. Thus the graph O' obtained by deleting any edge incident to u or v satisfies $VC(O') = VC(O)$. Thus O is not an obstruction for k–VERTEX COVER.

Therefore in an obstruction of k–VERTEX COVER, any two vertices with degree 2 do not have the same neighbors. □

A nice filter for our graph generator is the following result.

Lemma 7. *A vertex in an obstruction for k–*VERTEX COVER *has degree at most $k + 1$.*

Proof. Suppose u is a vertex with degree at least $k+2$ in an obstruction $O = (V, E)$ for k–VERTEX COVER. Let O' be the resulting graph by deleting any edge uv of O. Since O is an obstruction for k–VERTEX COVER, we have $VC(O') = k$. Hence, in G there is a set of vertices $V' \subseteq V$ which covers all edges in G except uv and $|V'| = k$. Since V' does not cover edge uv, $u \notin V'$ and $v \notin V'$.

Thus V' must contain all the neighbors of u except v. Hence V' contains at least $k + 2 - 1$ vertices in the neighborhood of u. Thus we have $|V'| \geq k + 1 > k$, contradicting V' being a witness vertex cover to O'.

Hence the degree of u is at most $k + 1$. Therefore a vertex in an obstruction for k–VERTEX COVER has maximum degree $k + 1$. □

Lemma 8. *There is only one connected obstruction for k–*VERTEX COVER *with $k+2$ vertices, which is the complete graph K_{k+2}. Furthermore, no other obstruction for k–*VERTEX COVER *has fewer vertices.*

Proof. Let K_{k+2} be the complete graph with $k+2$ vertices. Choose any vertex set $V' \subseteq V$ such that $|V'| = k$. Then,

$$K_{k+2} \backslash V' \simeq K_2 \quad .$$

Thus V' in K_{k+2} covers all edges in K_{k+2} except one edge.

Hence, if any edge in K_{k+2} is deleted, the resulting graph K' has $VC(K') = k$. Thus K_{k+2} is an obstruction for k–Vertex Cover (using Theorem 4).

Note for any connected graph G with order $n \leq k+2$ we have $G \leq_m K_{k+2}$. Since K_{k+2} is an obstruction for k–Vertex Cover, G is not an obstruction. □

The previous lemma shows that K_{k+2} is the smallest connected obstruction for k–Vertex Cover. The next lemma shows this is the unique obstruction with maximum degree.

Lemma 9. *If a vertex in a connected obstruction for k–*Vertex Cover *has degree $k+1$, then this obstruction is the complete graph with $k+2$ vertices (K_{k+2}).*

Proof. Suppose $O = (V, E)$ is a connected obstruction for k–Vertex Cover and u is a vertex in O with degree $k+1$ and let $N(u)$ be the neighborhood of u. Thus $|N(u)| = k+1$.

Let uv be any edge incident to u. Let $O' = O \backslash uv$; we know $VC(O') = k$. This is illustrated in Figure 3. Let V' be a minimum vertex cover of O'. We have $u \notin V'$ and $v \notin V'$ (otherwise O' is not an obstruction).

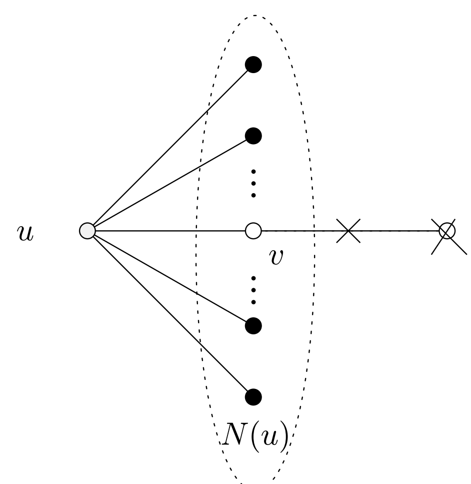

Figure 3: A vertex u in O with degree $k+1$. The black vertices must be in the vertex cover whenever uv is deleted.

Since V' covers all edges incident to u except uv, we have $V' = N(u) \backslash \{v\}$. Since $N(u) \backslash \{v\}$ is the vertex cover for O', v is not adjacent to any other vertex w where $w \notin N(u)$, otherwise, w must also be in V'. Thus $N(v) \subseteq N(u) \cup \{u\}$.

Since v is any vertex in $N(u)$, any vertex in $N(u)$ is connected to u or vertices in $N(u)$. Thus $\{u\} \cup N(u) = V$. Hence O has $k+2$ vertices.

By Lemma 8, a connected obstruction with $k+2$ vertices for k–Vertex Cover is the complete graph K_{k+2}, thus $O = K_{k+2}$ in this case. □

This lemma shows that for any obstruction $O = (V, E)$ for k–Vertex Cover, if Δ is the maximum degree of O and $|V| > k+2$ then $\Delta < k+1$.

2.3 Vertex bounds for k–Vertex Cover

We now present two important results that yield sharp upper bounds on the number of vertices for connected and disconnected k–Vertex Cover obstructions. In the next section, when focusing on 6–Vertex Cover, we will give some edge bounds that may be generalized (with some effort) to k–Vertex Cover.

Theorem 10. *A connected obstruction for k–Vertex Cover has at most $2k+1$ vertices.*

Proof. Assume graph $O = (V, E)$ is a connected obstruction for k–Vertex Cover, where $|V| = 2k+2$. We prove O is not a connected obstruction for k–Vertex Cover by contradictions. The same argument also holds for graphs with more vertices.

If O is a connected obstruction of k–Vertex Cover, then $VC(O) = k+1$. Hence V can be split into two subsets V_1 and V_2, as indicated in Figure 4, such that V_1 is a $k+1$ vertex cover and $V_2 = V \backslash V_1$. Thus $|V_2| = 2k+2-(k+1) = k+1$. Obviously,

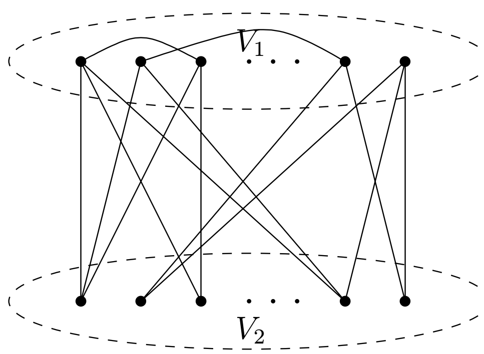

Figure 4: Splitting the vertex set of O into two subsets.

no edge exists between any pair of vertices in V_2, otherwise V_1 is not a vertex cover. Each vertex in V_1 has at least one vertex in V_2 as a neighbor, otherwise it can be moved from V_1 to V_2. (i.e., the vertex is not needed in this minimal vertex cover).

Thus the neighborhood of V_2, $N(V_2)$ is V_1. We now prove that no subset S of V_2 has $|N(S)| < |S|$. [This result, in fact, will immediately exclude the case $|V| > 2k+2$.]

By way of contradiction, assume $V_3 = S$ is a minimal subset in V_2 such that $|N(V_3)| < |V_3|$. We say V_3 is minimal whenever if T is any subset in V_3 then $|N(T)| \geq |T|$. Note V_3 will contain at least 3 vertices since every vertex has degree at least 2 in biconnected graphs (see Lemma 5).

Let $V_4 = N(V_3)$. Thus all edges adjacent to V_3 are covered by V_4. Thus $V_4 \subseteq V_1$ and $|V_4| < |V_3|$. Let $V_5 = V_2 \backslash V_3$ and $V_6 = V_1 \backslash V_4$. O can be further split as indicated in Figure 5.

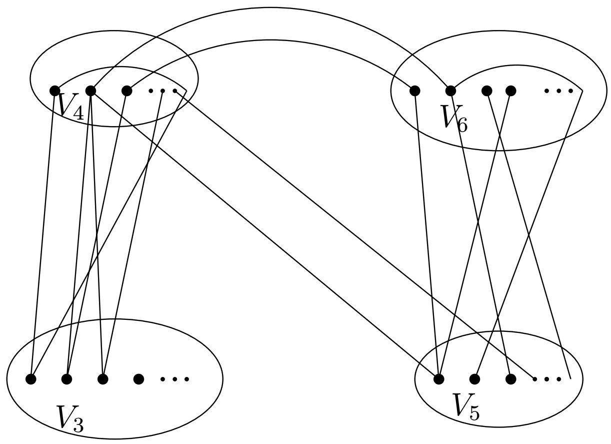

Figure 5: Splitting the vertex set of O into four subsets.

We summarize some facts about this partition of $2k + 2$ vertices of O.

1. $V_4 \cup V_6$ is the vertex cover of size $k + 1$.
2. $V_3 \cup V_5 = V_2$, where no edge connects V_3 and V_5 since no edges are in V_2.
3. $N(V_3) = V_4$ and $|V_4| < |V_3|$. And thus $|V_5| < |V_6|$.
4. Some edges must exist between V_4 and V_6 or between V_4 and V_5, otherwise O is not connected.
5. Some edges might exist within V_4 or within V_6.

We now prove the disconnected case of part (4) must hold (which will contradict the existence of V_3).

Since V_3 is assumed minimal, no subset T in V_3 has $|N(T)| < |T|$. Thus if we delete any vertex in V_3, leaving V_3', then any subset T in V_3' has $|N(T)| \geq |T|$.

Recall that a matching in a bipartite graph is a set of independent edges (with no common end points). Recall Hall's Marriage Theorem [Hal35] (e.g., see [CL86]):

Hall's Marriage Theorem:
A bipartite graph $B = (X_1, X_2, E)$ has a matching of cardinality $|X_1|$ if and only if for each subset $A \subseteq X_1$, $|N(A)| \geq |A|$.

Thus there is a matching of cardinality $|V_3'|$ in the bipartite induced subgraph $O' = (V_3', N(V_3))$ in O. Thus there are $|V_3'| = |V_3| - 1$ independent edges between the set V_3 and the set V_4.

Let us now delete all edges between V_4 and V_5 and all edges between V_4 and V_6. Let $C_1 = O[V_3 \cup V_4]$, $C_2 = O[V_5 \cup V_6]$ be these disconnected components in the resulting graph. (The induced graph C_1 is connected, while the induced graph C_2 may or may not be.)

As discussed above, there exists $|V_3| - 1$ independent edges in C_1. Thus to cover these edges, $VC(C_1) \geq |V_3| - 1$. We know $|V_4| < |V_3|$ and all edges incident to V_3 are covered by V_4, thus $VC(C_1) = |V_4|$.

Now consider the graph C_2. Suppose, $VC(C_2) < |V_6|$. Since all deleted edges are also covered by V_4, $V_4 \cup VC(C_2)$ must cover all edges in O. Thus $VC(O) = |V_4| + VC(C_2) < |V_4| + |V_6| = k + 1$. This contradicts that O is an obstruction for k–Vertex Cover. Thus the assumption that $VC(C_2) < |V_6|$ is not correct. Hence $VC(C_2) = |V_6|$ even though edges between C_1 and C_2 were deleted.

Thus $VC(C_1 \cup C_2) = |V_4| + |V_6| = k + 1$. Therefore O can not be a connected obstruction for k–Vertex Cover since the resulting graph still requires a $k + 1$ vertex cover whenever all edges between V_4 and V_5 and all edges between V_4 and V_6 are deleted.

Therefore the assumption that there exists a minimal subset V_3 in V_2 is not correct. Hence any subset S in V_2 has $|N(S)| \geq |S|$.

Once again, by applying Hall's Marriage Theorem, there is a matching of cardinality $k+1$ in the induced bipartite subgraph $O' = (V_2, V_1)$ of O. To cover these $k+1$ independent edges, a vertex cover of size $k + 1$ is necessary. We know that if O is a connected graph, there must exist other edges in O except these $k + 1$ independent edges. If those edges are deleted, the resulting graph still has vertex cover $k + 1$. Thus O can not be a connected obstruction for k–Vertex Cover if it has more than $2k + 1$ vertices. □

By extending the above result, we have the following corollary.

Corollary 11. *Any obstruction for k–*Vertex Cover *has order at most $2k + 2$.*

Proof. Let $\mathcal{O}(i$–Vertex Cover$)$ represent the set of obstructions for the lower ideal i–Vertex Cover. First we use induction to prove any disconnected obstruction for k–Vertex Cover has order at most $2k + 2$.

Basis step. $k = 0$: we know that when $k = 0$, the only obstruction K_2 has order at most 2. In this case, the theorem holds.

Induction step. Induction hypothesis: assume for $i = 0, 1, 2, \ldots, k-1$, the theorem holds.

Let $O_1 \in \mathcal{O}((k-1-i)$–Vertex Cover) and $O_2 \in \mathcal{O}(i$–Vertex Cover) where $0 \le i \le k-1$. Then a disconnected obstruction $O_3 \in \mathcal{O}(k$–Vertex Cover) can be obtained as $O_3 = O_1 \cup O_2$ (e.g., see [Din97]).

We know a connected obstruction for i–Vertex Cover has order at most $2i+1$ by Theorem 10 and, by our hypothesis, a disconnected obstruction for i–Vertex Cover has order at most $2i+2$ whenever $i = 0, 1, 2, \ldots, k-1$. Thus any obstruction $O_1 \in \mathcal{O}((k-1-i)$–Vertex Cover) has order at most $2(k-1-i)+2 = 2k-2i$ and obstruction $O_2 \in \mathcal{O}(i$–Vertex Cover) where $i = 0, 1, 2, \ldots, k-i$ has order at most $2i+2$. Thus a disconnected obstruction $O_3 \in \mathcal{O}(k$–Vertex Cover) has order at most $(2k-2i)+(2i+2) = 2k+2$.

Therefore any obstruction for k–Vertex Cover has order at most $2k+2$. □

We conclude this section, which has been discussing properties of k–Vertex Cover obstructions, with some observations. The known obstructions for the small cases (see [CD94] for $k \le 5$, in addition to this paper's $k = 6$) indicate a very interesting feature: **the more vertices, the fewer edges**. As proven above, a complete graph K_{k+2} is the smallest obstruction (and only one), but it seems to have the largest number of edges. The cycle obstruction C_{2k+1} appears to be the only connected obstruction with the largest number of vertices (and appears to have the smallest number of edges). We strongly believe:

Conjecture 12. *The cycle C_{2k+1} is the only (and largest) connected obstruction for k–*Vertex Cover *with $2k+1$ vertices.*

It would be nice to have some general results for k–Vertex Cover relating the order and the size of the obstructions. In the next section we concentrate bounding the number of edges of the 6–Vertex Cover family which further enables us to characterize this family by forbidden minors.

3 Computing all obstructions for 6–Vertex Cover

Now we are ready to present specific edge bounds for the 6–Vertex Cover obstructions. In the previous section, we have proved some useful vertex properties for k–Vertex Cover. In conjunction with these properties we can further reduce the search space for finding all of the obstructions for 6–Vertex Cover.

Let us briefly recap what we know about any connected obstruction $O = (V, E)$ for 6–Vertex Cover:

1. It is biconnected.

2. $8 \leq |V| \leq 13$.

3. The maximum degree for any vertex is 7.

4. If one vertex has a maximum degree 7 then $O = K_8$.
   Thus if $|V| > 8$ then the maximum degree is at most 6. Hence $|E| \leq |V| \cdot 3$.

5. Two vertices of degree 2 can not have the same neighborhood.

The search space can be reduced by (1) to (4), and (5) can be taken as a pretest condition so that the time for finding all obstructions for 6–VERTEX COVER is improved.

However, these constraints are not sufficient for finding all 6–VERTEX COVER obstructions in a reasonable amount of time. Considering there are about 4.0×10^{10} biconnected graphs with 12 vertices, maximum degree 7, and at most 42 edges, we needed to develop other bounds to reduce the search space. Furthermore, although we believe Conjecture 12 is true, it is still open. So we had to check other potential graphs with 13 vertices by the way of brute force. Thus we divide our process for finding all the connected obstructions for 6–VERTEX COVER into two steps.

(1) We find all connected obstructions with order at most 11 for 6–VERTEX COVER. This step is very straight forward. Our algorithm `IsObstruction` is applied to all non-isomorphic biconnected graphs with a number of vertices between 9 and 11, of maximum degree 6, and of maximum number of edges 33.

(2) To find all connected obstructions of order 12 and 13 for 6–VERTEX COVER, new degree and edge bounds were found and used, as indicated below in the next section.

3.1 6–VERTEX COVER obstructions with 12 and 13 vertices

By the Lemma 9, we know that if a connected obstruction O has a vertex of degree 7 for 6–VERTEX COVER, then this obstruction must be K_8. Thus we only have to consider graphs with maximum degree 6. Furthermore, if the degree for O is 6, we can prove the following statement.

Statement 1: If a connected obstruction $O = (V, E)$ for 6–VERTEX COVER has a vertex of degree 6, then $|V| \leq 10$. Furthermore, if $|V| = 10$ then $|E| \leq 24$, and if $|V| = 9$ then $|E| \leq 25$.

Proof. Consider a connected obstruction $O = (V, E)$ for 6–VERTEX COVER with degree 6. Let u be a vertex in O which has degree 6. Let $N(u)$ be the neighborhood of u.

Let v be any vertex in $N(u)$. If the edge uv is deleted, with O' being the resulting graph, then $VC(O') = 6$. Let V' be a vertex cover of O', then $u \notin V'$ and $v \notin V'$. Thus $(N(u)\backslash\{v\}) \subset V'$. We know $|N(u)\backslash\{v\}| = 5$, thus only one other vertex w can exist in V' such that $w \notin N(u)$. This means v has at most one neighbor vertex w where $w \notin N(u) \cup \{u\}$. Since v can be taken to be any vertex in $N(u)$, if any edge incident to u is deleted, then a vertex cover V' must contain 5 vertices in $N(u)$ plus a possible outside vertex w.

Let W be the set $V \backslash (\{u\} \cup N(u))$. To achieve the maximum number of vertices and the maximum number of edges of O, consider the following two cases.

case 1: No edges inside W.

In this case, no edge connects two vertices in W. Since any vertex in $N(u)$ can have only one edge incident to a vertex in W, there are at most 6 edges between $N(u)$ and W. Since no edge connects any pair of vertices in W and O is a biconnected graph (any vertex in O must have degree at least 2), any vertex in W must have at least two edges connected to vertices in $N(u)$. Thus in this case, there are at most 3 vertices in W as showed in Figure 6. Hence in this case, $V = \{u\} \cup N(u) \cup W$, and *the maximum number of vertices is* $1 + 6 + 3 = 10$.

Since any vertex v in $N(u)$ has one edge adjacent to u and possibly at most one edge adjacent to a vertex in W, and with degree at most 6, v can have at most 4 edges connected to vertices in $N(u)$. Thus there are at most 12 edges between the set of vertices in $N(u)$. Hence in this case, *the maximum number of edges is* $6 + 12 + 6 = 24$.

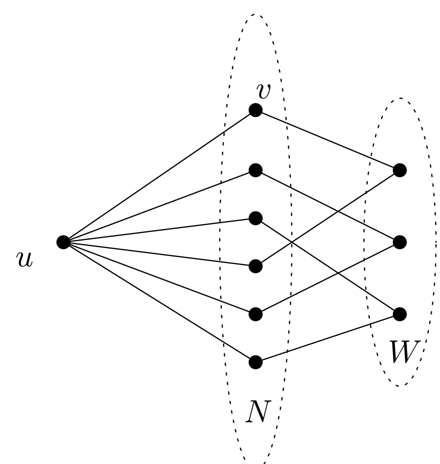

Figure 6: Case 1: no edge between vertices of W.

case 2: At least one edge inside W.

Let H be the induced subgraph $O[W]$. and let H' be the resulting graph after deleting all isolated vertices in H. We assume H has at least one edge. As in case 1, if any edge incident to u is deleted, then only one vertex in W can be introduced into the vertex cover V' of O'. Thus $VC(H') = 1$, otherwise all edges in H' can not be covered by V'. Only stars have vertex cover 1.

Suppose H is a star which has at least two edges wx and wy. Because O is biconnected, x and y must have more than one neighbor. Thus x must be connected to a vertex $s \in N(u)$ and y must be connected to a vertex $t \in N(u)$. Since any vertex in $N(u)$ can have only one edge out from $N(u)$, we have $s \neq t$. If edge us is deleted, then only x can be in the vertex cover V' of O' and both w and y can not be in V'. Thus edge wy can not be covered by V'. Similarly, if edge ut is deleted, then edge wx can not be covered. Thus H' can have only one edge.

Let wx be the edge in H', and suppose H has at least one isolated vertex y. Thus y must have at least two edges connect to two vertices in $N(u)$. Let yv be an edge where $v \in N(u)$. If edge uv is deleted, then y must be in the vertex cover V' of O' and both w and x can not be in V'. Thus the edge wx can not be covered. Thus V' can not cover all edges in the resulting graph O'. This contradicts O being an obstruction of 6–Vertex Cover. Thus H can not have any isolated vertices.

Thus in this case, $H = H'$ and only one edge in H'. Thus $|W| = 2$. Hence in this case, the maximum number of vertices of O is $1 + 6 + 2 = 9$, as showed in Figure 7.

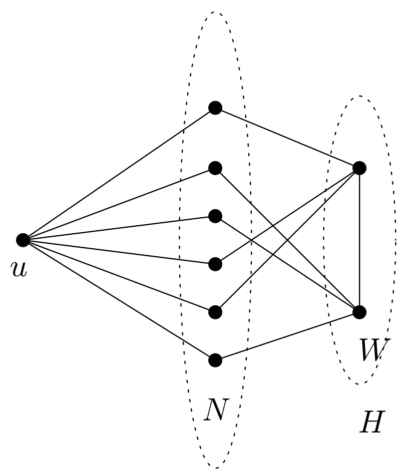

Figure 7: Case 2: at least one edges connects two vertices in W.

We now calculate the maximum number of edges of O in this case. There are at most 6 edges between $N(u)$ and W and there are at most 12 edges inside $N(u)$. Thus the maximal number of edge of O in this case is $6+12+6+1 = 25$. Hence in this case, *the maximum number of vertices is 9 and the maximum number of edges is 25.*

By case 1 and case 2, the statement is true. □

We note that the arguments of the above proof may be applied for k–Vertex Cover families for larger k.

A consequence of this proof is the following: if a connected obstruction O for 6–Vertex Cover has 11 or more vertices then the degree of every vertex is less than or equal to 5. Thus any obstruction O for 6–Vertex Cover has at most 30 or 32 edges for orders 12 and 13, respectively. We improve these edge bounds below.

Statement 2: If a connected obstruction $O = (V, E)$ for 6–Vertex Cover has 12 vertices then $|E| \leq 24$. Further, if O has 13 vertices then $|E| \leq 26$.

Proof. Consider a connected obstruction $O = (V, E)$ for 6–Vertex Cover of degree 5. Let u be a vertex in O with degree 5. Let v be any vertex in $N(u)$. If edge uv is deleted, with O' being the resulting graph, then $VC(O') = 6$. Let V' be the vertex cover of O', then $|V'| = 6$ and $(N(u)\backslash\{v\}) \subset V'$. Thus v can have at most two edges connected to two vertices that are not in $N(u) \cup \{u\}$. Since v is any neighbor of u, if any edge incident to u is deleted then only two vertices can be in the vertex cover V' which are not in $N(u)$.

Let W be the set of all vertices which are not in $\{u\} \cup N(u)$. Let H be the induced subgraph $O[W]$. Let H' be H minus isolated vertices. Since in O, if any edge incident to u is deleted then all edges of H' must be covered by at most two vertices, we have $VC(H') \leq 2$.

1. If H' is an empty graph.

   In this case, all vertices in H are isolated. Thus all vertices in W are only connected to $N(u)$. Since any vertex in $N(u)$ can only be connected to at most two vertices in W, the total number of edges between W and $N(u)$ is 10. Since O is biconnected, any vertex in W must have at least 2 edges connected to $N(u)$. Thus $|W| \leq 5$. Thus in this case the maximum number of vertices of O is $|\{u\} \cup N(u) \cup W| = 1 + 5 + 5 = 11$.

2. If H' is a disconnected graph.

   Since $VC(H') \leq 2$, H can have at most two components C_1, C_2 and $VC(C_1) = 1$, $VC(C_2) = 1$. Thus C_1 and C_2 must be two stars.

   Now we prove each component can have only one edge. Suppose without losing generality, C_1 has only one edge st, and C_2 has two edges xy and xz. Since O is a biconnected graph, y and z must have degree at least 2. Thus y must connected to a vertex $v \in N(u)$. If uv is deleted, then in order to cover the edge yv, y must be in the vertex cover V' of the resulting graph O'. Thus only one vertex can be in V' from s, t, x, z. But to cover the edge st and the edge xz, two vertices are necessary. Thus at least one edge can not be covered. Thus C_2 can not have more than one edge. Hence if H' is disconnected then H' can have at most two components and each of them has only one edge.

Suppose H has at least one isolated vertex w, then w must have at least two edges connected to two vertices in $N(u)$. Let wv be an edge in O and $v \in N(u)$. If edge uv is deleted, w must be in the vertex cover set V' of O'. Only other one vertex can be in V' from H'. But in order to cover those two edges (stars) in H', 2 additional vertices from H' must be in V'. Thus V' would have more than 6 vertices. Thus this case can not happen. Therefore, if H' has two disconnected edges, then $H = H'$. Thus H' contains all vertices in W. Hence in this case, the maximum number of vertices of O is $1 + 5 + 4 = 10$.

3. If H' is a connected graph (with at least one edge).

   For this case, we discuss the structure of H' as follows.

   (a) H' has cycles.

   Since $VC(H') \leq 2$ and C_5 is an obstruction for 2–Vertex Cover, H' can only have one cycle, either C_3 (a cycle with 3 vertices) or C_4 (a cycle with 4 vertices).

   To cover all edges in C_3 or C_4, two vertices on the cycle are necessary in the vertex cover V' of the graph O' (created by deleting an edge incident to u in O). If a vertex w in H is not on the cycle, then w must have at least one edge incident to a vertex v in $N(u)$. If uv is deleted, w must be in the vertex cover V'. Thus the remaining cycle can not be covered by V'. Hence w does not exist in H. Thus in this case $H = H'$ and H' can either be
   (1) C_3,
   (2) C_4 or
   (3) C_4 with a chord.
   Thus in this case the maximum number of vertices of O is $|\{u\} \cup N(U) \cup W| = 10$.

   (b) H' is a tree.

   If H' is a tree, since $VC(H') \leq 2$, H' can only have a path with length at most 3.

   - H' has a path with length 3.
     Let $wxyz$ be the path with length 3 in H. Thus at least two vertices in the path are necessary to cover all edges in this path. Since H' is a tree of diameter 3, if H' has any extra edges which are not on the path, these edges must be incident to x, or y. Suppose x has an edge connected to an extra vertex s. Since O is biconnected, s must have degree at least 2. Thus s must have an edge connected to a vertex v in $N(u)$. If uv is deleted in O then $s \in V'$ where $|V'| = 6$ and V' is the vertex cover for the resulting graph. Thus only one other vertex in $\{w, x, y, z\}$ can be in V'. But in order to cover all edges on the path, two vertices are necessary. Thus s can not exist in H'. Thus $H' = P_3$, where P_3 is a path with 3 edges. Similarly by earlier arguments, H

can not have isolated vertices. Therefore in this case, $H = H' = P_3$, the maximum order is $1 + 5 + 4 = 10$.

- H' has diameter at most 2.
  Here H' must be a star S_i. As we know, each vertex in O has a degree at most 5, thus $H' = S_i$ where $1 \leq i \leq 5$.
  - $H' = S_1$.
    In this case. H' is an edge between 2 vertices. Thus at least two edges must exist between vertices in $N(u)$ and vertices in H'. To cover this edge in H', any vertex in H' is necessary.
    Recall that any vertex in $N(u)$ can have at most two edges connected to vertices in H'. To achieve maximum number of isolated vertices in H, suppose two edges come from H' are all incident to a vertex in $N(u)$, then there are at most 4 vertices in $N(u)$ that have edges connected to isolated vertices in H.
    We say any vertex v in $N(u)$ can have at most one edge connected to an isolated vertix in H. Otherwise if v has at least two edges connected to isolated vertices i_1 and i_2 in H, then when uv is deleted, both i_1 and i_2 have to be in the vertex cover V'. Thus the edge of H' can not be covered. Since each isolated vertex in H must have at least two edges connected to vertices in $N(u)$, the maximum number of isolated vertices in H is 2. Thus in this case, the maximum number of vertices of O is $1 + 5 + 2 + 2 = 10$.
  - $H' = S_2$.
    In this case, H' has 3 vertices and 2 edges. In order to achieve the maximum number of vertices of O, H must have isolated vertices. Let v be any vertex in $N(u)$, then v can only have one edge connected to an isolated vertex in H, otherwise the edges in H' can not be covered once uv being deleted.
    Because O is biconnected, the two leaves of H' must have at least two edges connected to $N(u)$. If they all connect to one vertex in $N(u)$, then at most 4 vertices in $N(u)$ can have edges connect to isolated vertices in H. Because O is biconnected, any isolated vertex in H must have at least two edges connected to vertices in $N(u)$. Thus the maximum number of isolated vertices in H is 2. Thus in this case, the maximum number of vertices of O is $1 + 5 + 3 + 2 = 11$, as showed in Figure 8.
  - $H' = S_3$.
    In this case H' has 4 vertices and 3 edges. Similar to the previous case, the three leaves of H' have at least three edges connected to $N(u)$. If any vertex v in $N(u)$ has one edge connected to a leaf of H', then this vertex can not have any edge connected to an isolated vertex in H and also it can not have any edge connected to any other leaf of H'. Thus there are at least 3 vertices in $N(u)$ that can

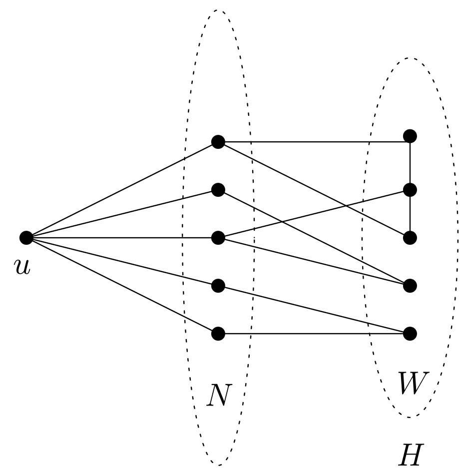

Figure 8: The case $H' = S_2$.

not have any edge connected to an isolated vertex in H. Thus at most 2 edges can between vertices in $N(u)$ and isolated vertices in H. Hence H can have at most 1 isolated vertex. Therefore in this case, the maximum number of vertices of O is $1 + 5 + 4 + 1 = 11$ as showed in Figure 9.

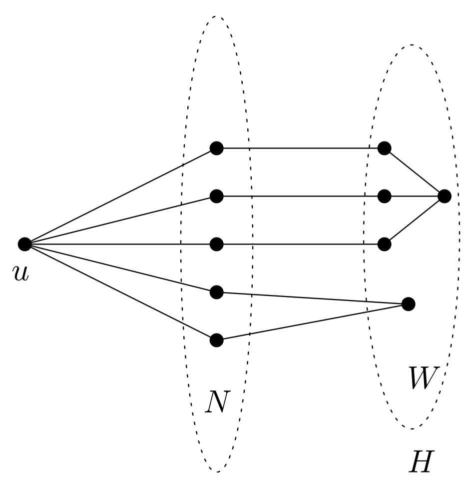

Figure 9: The case $H' = S_3$.

— $H' = S_4$.
In this case, H' has 5 vertices and 4 edges. Similar to the previous cases, the four leaves of H' have at least 4 edges connected to $N(u)$. Each vertex in $N(u)$ can have at most one edge connected to a leaf in H'. Thus there are at least 4 vertices in $N(u)$ that can not have any edges connected to any isolated vertices in H. Thus there is at most 1 edge between $N(u)$ and the isolated vertices in H. Thus the number of isolated vertices in this case is 0. Since

each isolated vertex has at least two edges connected to $N(u)$. Therefore in this case, the maximum number of vertices of O is $1 + 5 + 5 + 0 = 11$.

— $H' = S_5$.

In this case H' has 6 vertices and 5 edges. In this case, H can not have any isolated vertices because the maximum degree in O is 5. Thus $H = H'$. Thus in this case the maximal number of vertices of O is $1 + 5 + 6 = 12$. In this case O has the maximum number of vertices compared with all cases above.

Now we compute the maximum number of edges in this case.

Let r be the root vertex of H'. Thus r has degree 5. Thus r can not have any edges connected to $N(u)$. Let v be any vertex in $N(u)$. Thus v can have only one edge connected to a leaf of H'. If v has two edges connected to two leaves of H', then if uv is deleted in O, the two leaves of H' have to be in the vertex cover V' of the resulting graph O'. Thus the root of H' has no chance to be in V', thus all edges in H' can not be covered by V'. Thus in this case each vertex in $N(u)$ can have at most one edge connected to H'.

Since O is biconnected, each leaf of H' must have at least one edge connected to $N(u)$, otherwise these leaves would have degree 1. Hence there must be exactly 5 edges between vertices in $N(u)$ and leaves in H'.

Each vertex v in $N(u)$ have one edge connected to u and one edge connected to a leaf of H'. Thus each $v \in N(u)$ can have 3 edges connected to the vertices in $N(u)$. Thus there are at most 7 edges within the set $N(u)$.

Therefore in this case, the maximum number of edges of O is $5 + 7 + 5 + 5 = 22$ as showed in Figure 10.

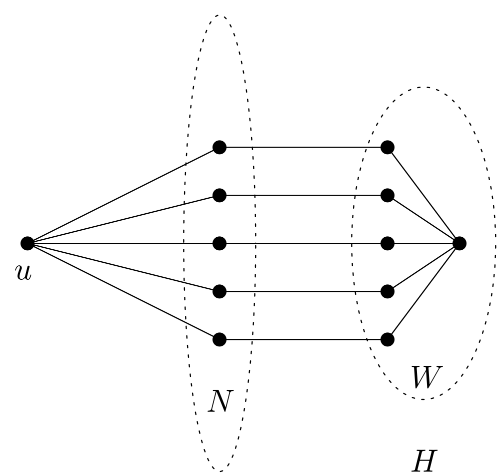

Figure 10: The case $H' = S_5$.

By cases 1–3, it is shown that *if any vertex in $O = (V, E)$ has a degree* 5*, then* $|V| \leq 12$ (In case 3b when H' is a star with 5 edges). Here when $|V| = 12$, $|E| \leq 22$.

Earlier in statement 1, we have proved that if any vertex in a connected obstruction $O = (V, E)$ for 6–Vertex Cover has maximum degree 6, then $|V| \leq 10$. And since $|E| \leq 24$ when $|V| = 12$ and O has maximum degree 4, the two results show that if $|V| = 12$ and maximum degree of O is at most 5, then $|E| \leq 24$. By same reasoning, if $|V| = 13$, where the maximum degree is at most 4, then $|E| \leq 26$. □

According to statement 2, we have two new edge bounds for an obstruction with 12 vertices or 13 vertices, and one new degree bound for an obstruction with 13 vertices. Now the search space for finding all obstructions with orders 12 and 13 for 6–Vertex Cover has been extremely reduced.

In summary, to find all obstructions with 12 vertices for 6–Vertex Cover, we only need to test all non-isomorphic (biconnected) graphs with maximum degree 5 and at most 24 edges. For finding all obstructions with 13 vertices for 6–Vertex Cover, we only need to check all non-isomorphic graphs with maximum degree 4 and at most 26 edges. This search space is very manageable; it requires about two months of computation time! In Figures 11–16, we display all 188 connected obstructions for 6–Vertex Cover.

4 Conclusion

Our main result was the successful characterization of the 6–Vertex Cover graph family by computing its 260 forbidden minors. From the current counts of all the known vertex cover obstructions sets (see Table 1), we notice that the number of connected obstructions grows quite rapidly. We suspect that there are between 1500 and 2000 obstructions for 7–Vertex Cover. The present search space for tackling the case $k = 7$ is probably too large for the filtered method used here. Thus a more direct approach will be needed. Can the bounded pathwidth scheme of [CD94] be combined with this approach? It would help if we had available an efficient bounded pathwidth (non-isomorphic) graph generator.

We finish by mentioning a couple of other areas left open by our research. It would be nice to have a proof that C_{2k+1} is the only connected obstruction for k–Vertex Cover, since we now known that $2k + 1$ is an upper bound on the number of vertices. There are some interesting open questions regarding edge bounds for k–Vertex Cover. As we pointed out earlier, the obstructions start having fewer edges as the number of vertices increases. More theoretical results that generalizes our specific bounds for $k = 6$ seem possible.

References

[APS91] Stefan Arnborg, Andrzej Proskurowski, and D. Seese. Monadic second order logic, tree automata and forbidden minors. In *Proceedings of the 4th Workshop on Computer Science Logic, CSL'90*, volume 533 of *Lecture Notes on Computer Science*, pages 1–16. Springer-Verlag, 1991.

[Arc80] Dan S. Archdeacon. *A Kuratowski Theorem for the projective plane.* Ph.D. dissertation, The Ohio State University, 1980. Also see *Journal of Graph Theory* 5:243–246 (1981).

[BFR98] R. Balasubramanian, Michael R. Fellows, and Venkatesh Raman. An improved fixed-parameter algorithm for vertex-cover. *Information Processing Letters*, 65(3):163–168, 1998.

[CD94] Kevin Cattell and Michael J. Dinneen. A characterization of graphs with vertex cover up to five. In Vincent Bouchitte and Michel Morvan, editors, *Orders, Algorithms and Applications, ORDAL'94*, volume 831 of *Lecture Notes on Computer Science*, pages 86–99. Springer-Verlag, July 1994.

[CDD+97] Kevin Cattell, Michael J. Dinneen, Rodney G. Downey, Michael R. Fellows, and Michael A. Langston. On computing graph minor obstruction sets, August 1997. To appear *Theoretical Computer Science A.*

[CL86] Gary Chartrand and Linda Lesniak. *Graphs and Digraphs.* Wadsworth Inc., 2nd edition, 1986.

[DCF96] Michael J. Dinneen, Kevin Cattell, and Michael R. Fellows. Forbidden minors to graphs with small feedback sets. Report CDMTCS-019, Centre for Discrete Mathematics and Theoretical Computer Science, University of Auckland, New Zealand, October 1996. To appear *Discrete Mathematics.*

[Din97] Michael J. Dinneen. Too many minor order obstructions (for parameterized lower ideals). *Journal of Universal Computer Science*, 3(11):1199–1206, December 1997.

[FL89] Michael R. Fellows and Michael A. Langston. An analogue of the Myhill-Nerode Theorem and its use in computing finite-basis characterizations. In *IEEE Symposium on Foundations of Computer Science Proceedings*, volume 30, pages 520–525, 1989.

[GHW79] Henry H. Glover, J. Philip Huneke, and Chin San Wang. 103 graphs which are irreducible for the projective plane. *Journal of Combinatorial Theory, Series B*, 27:332–370, 1979.

[GJ79] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness.* W. H. Freeman and Company, 1979.

[Hal35] P. Hall. On representation of subsets. *J. London Math. Soc.*, 10:26–30, 1935.

[LA91] Jens Lagergren and Stefan Arnborg. Finding minimal forbidden minors using a finite congruence. In *Proceedings of the International Colloquium on Automata, Languages and Programming*, volume 510 of *Lecture Notes on Computer Science*, pages 533–543. Springer-Verlag, 1991. 18th ICALP.

[McK90] B. D. McKay. `nauty` User's Guide (version 2.0). Technical Report TR-CS-90-02, Australian National University, Department of Computer Science, 1990.

[Pro93] Andrzej Proskurowski. Graph reductions, and techniques for finding minimal forbidden minors. In Neil Robertson and Paul D. Seymour, editors, *Graph Structure Theory*, volume 147 of *Contemporary Mathematics*, pages 591–600. American Mathematical Society, 1993.

[RS85] Neil Robertson and Paul D. Seymour. Graph Minors – A survey. In *Surveys in Combinatorics*, volume 103, pages 153–171. Cambridge University Press, 1985.

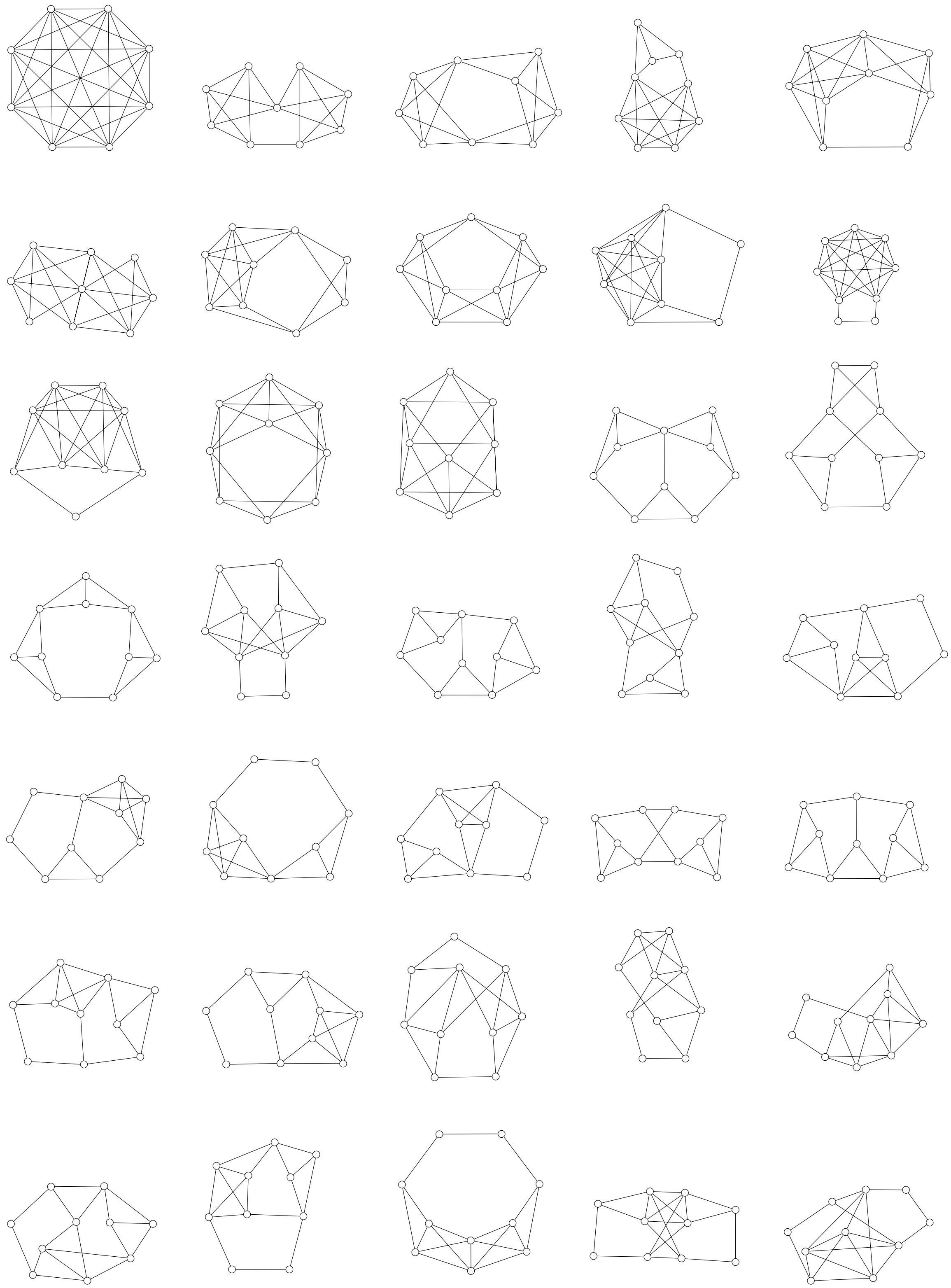

Figure 11: All connected obstructions for 6–Vertex Cover.

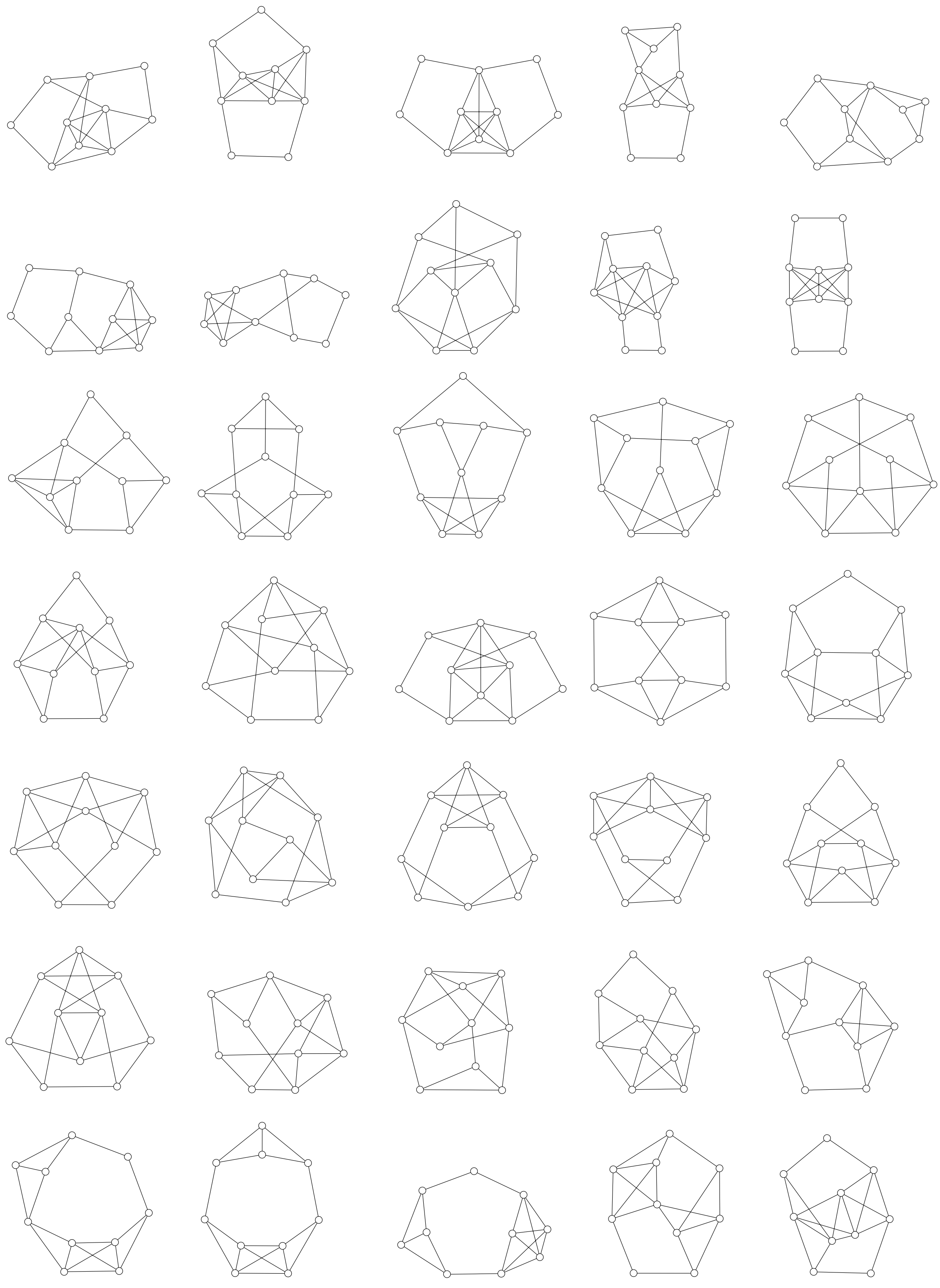

Figure 12: All connected obstructions for 6–Vertex Cover (continued).

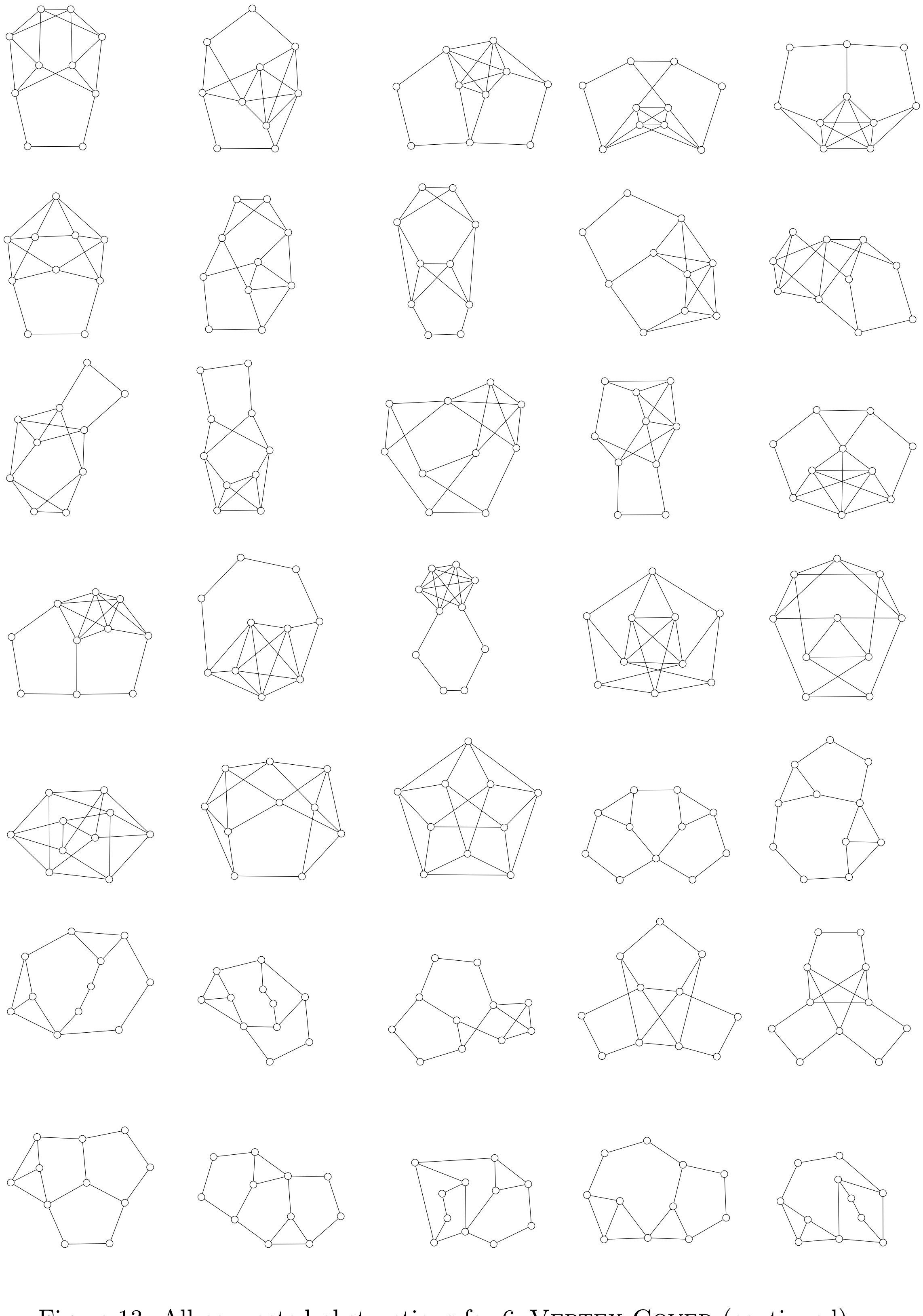

Figure 13: All connected obstructions for 6–Vertex Cover (continued).

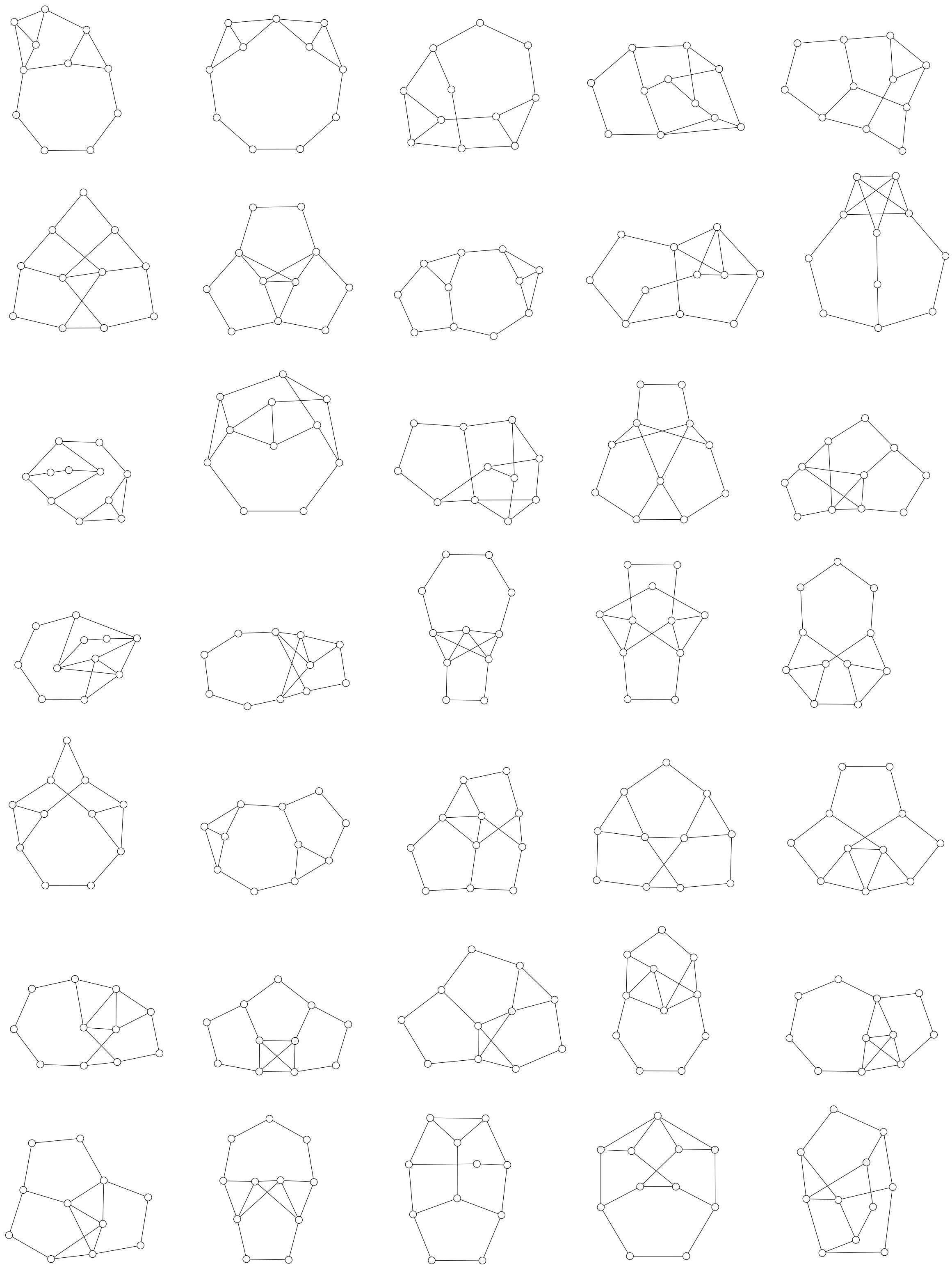

Figure 14: All connected obstructions for 6–Vertex Cover (continued).

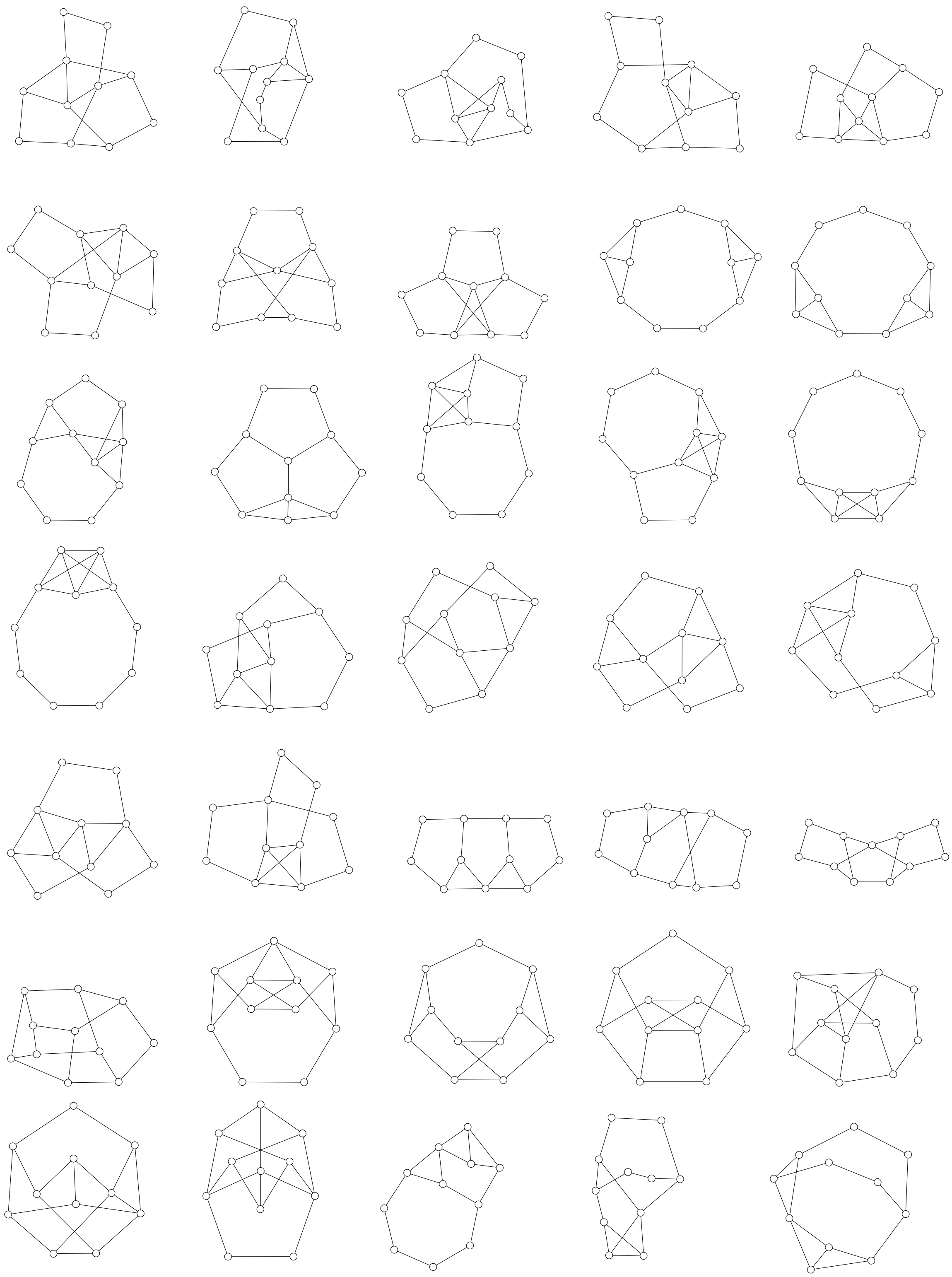

Figure 15: All connected obstructions for 6–Vertex Cover (continued).

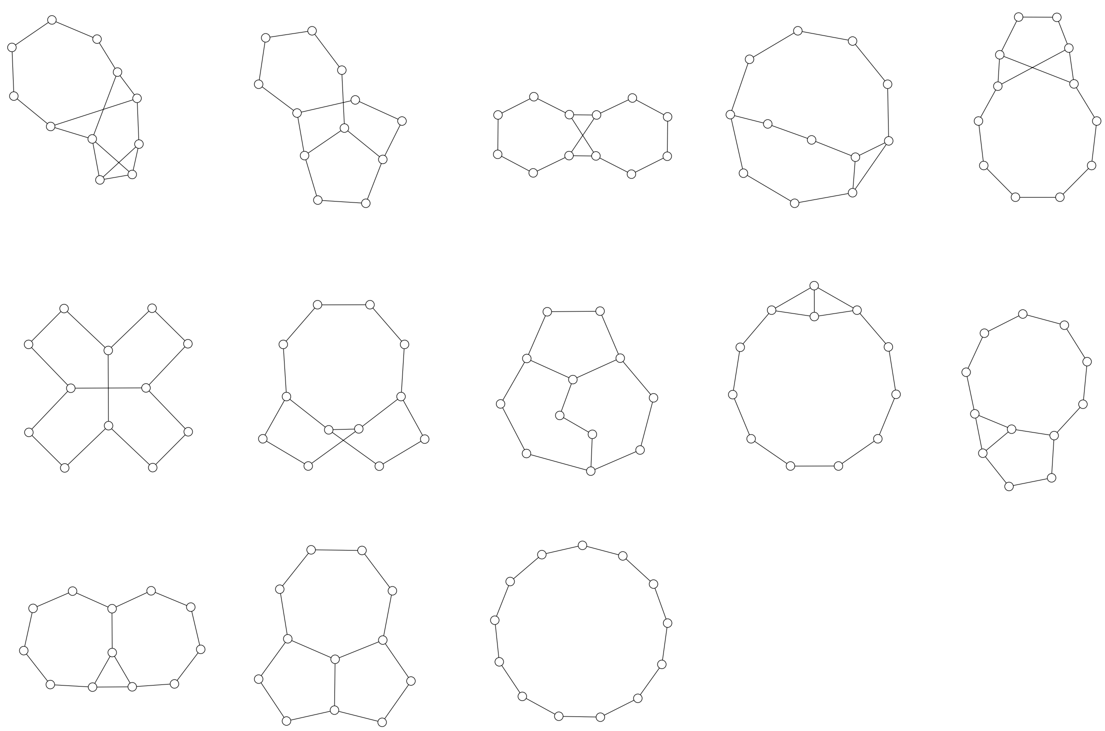

Figure 16: All connected obstructions for 6–Vertex Cover (continued).