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IS MOVEMENT AN ILLUSION ?

Zeno's Paradox From A Modern Viewpoint

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A B S T R A C T

The Greek philosopher Zeno presented for the first time in history the problems derived from assuming (or rejecting) the infinite divisibility of space and time. He showed that knowledge of the physical world is dependent on what axioms concerning reality are admitted: either space ant time are atomic or dividable ad infinitum*. Aristotle, differential calculus, Einstein's relativity, nonstandard mathematics, and modern philosophers such as Heidegger, all tried to cope with this problem. However, their "solutions" always imply adding controversial new axioms. Thus, a fundamental aspect of how humans understand Nature or, equivalently, the problem of determining which of the possible but indispensable axioms should be given pre-eminence, is reflected in the study of this famous paradox. Finally, a recently characterised subset of the real numbers called "Lexicons" adds a surprising twist to this notorious paradox.*

Zeno, the Greek philosopher from Elea in southern Italy, who lived from *circa* 495 to 445 BC, wanting to prove his teacher's Parmenides thesis of the impossibility of all motion, conceived the notorious "paradox" of Achilles and the tortoise, the solution of which has challenged mathematicians and philosophers throughout the centuries. He claimed: Not even Achilles, the fastest Greek hero of the *Illiad*, can ever catch a slow tortoise if that animal is given a head start of, for instance, s_1 meters. Obviously, to catch the tortoise, Achilles must first run that distance, say in t_1 seconds. But once he gets there the tortoise has already moved further off a distance of s_2 meters, a distance Achilles will cover in t_2 seconds. And so on, the procedure must be repeated, *ad infinitum*. However close Achilles gets to the tortoise, there will always remain some infinitesimally small distance yet to go. But we do see Achilles in real life easily catch the ponderous animal. Conclusion: Real life motion is an illusion!

It is clear that the paradox assumes as true by axiom one of the most consequential ideas of the Greek philosophers: space and time are a continuum that can be divided indefinitely, and there is neither an *atom* of space nor an *instant* of time. And if this is so, Zeno claims, there will always remain some physical distance, and some duration of time, before Achilles catches the tortoise, no matter how small. In other words, what Zeno's paradox asked philosophers to explain, if motion is assumed to be real, is *how an infinity of acts can be serially completed in finite time*, even if each act is infinitesimally small.

As stated, the problem raised by Zeno has been the subject of two and a half millennia of analysis. Plato treated the question *in extenso* in his dialogue *Parmenides* showing, in truly "Platonic" form, that whichever approach is favoured: divisible *ad infinitum* or atomic, there will always inescapably unacceptable paradoxes. After Plato, Aristotle's eight books of *Physics* address the problem. He wrote: "Since Nature is the principle of movement and change, and it is Nature that we are studying, we must understand what movement is" (*Physics* III 200b 12-13)[i]. He further claimed that "infinity cannot exist as an actualised entity, [for then it] must be either altogether indivisible or divisible into infinities. But for one and the same thing to be many infinities is impossible" (*Physics* III 204a 21-28). Since "we are engaged in the study of things cognizable by the senses" (*Physics* III 204b 2), and motion is a fact of the senses, the question is: How is motion possible?

For Plato and Aristotle, "motion" (*kinêsis*) means any kind of *change*, not just how something, Achilles for instance, can pass from being at rest to being in motion, but also the contrary, how can motion stop, or how something, *viz*. Achilles, comes into being, or ceases to be, that is, dies. For if time and space are infinitely divisible Zeno´s paradox applies, and to be born or to die are then both equally pure illusion! How does Aristotle "solve" the problem? Here is what he says: "If we are asked whether it is possible to go through an unlimited number of points, whether in a period of time or in a length, we must answer that in one sense it is possible but in another not. If the points are actual, it is impossible, but if they are potential it is possible. For one who moves continuously traverses an illimitable number of points [of time and space] only in an accidental, not unqualified, sense; it is an accidental characteristic of the line that it is an illimitable number of half-lengths; its essential nature is something different" (*Physics* VIII 263 b 4-8). Otherwise stated, Aristotle is distinguishing between different kinds of infinities: the actual and the potential. Traversing a region of space (or of time) does not involve moving across an *actual* infinity, which would be impossible. However, it is consistent with crossing a *potentially* infinite number of sub-regions of space (or time intervals), in the sense that there can be no end to the process of dividing space (or time). Thus Zeno´s paradox is pertinent, but only potentially, whereas our senses prove that actually Achilles does catch the tortoise. In summary, the question raised by Zeno is: To which of the two possible means of acquisition of knowledge about the physical world are we to give priority: to pure (mathematical) reason or to our (common) senses? Aristotle's answer was: Pure (i.e., independent of all experience) reason shows what is potentially possible, whereas the senses (i.e., results of performed measurements) teach us the actual world. As a side note, remark that in our days we see this Aristotelian idea in quantum mechanics: the quantum wave function Ψ is said to represent the linear superposition of all the different "potentially" possible states of some physical system, whereas nothing but the "actual" measurement provides the senses with the one and only objective (i.e., "eigen-") value. The rest of the maybe infinite possibilities are then said to have "collapsed" or vanished somehow.

Western philosophy spent almost two-thousand years trying to shake loose from Aristotelian physics and recover the Platonic approach—put forward in the *Timaeus* (reference as in note iii below)—, claiming to give absolute pre-eminence to the mathematical (pure reason) *model* over the (common) sense apperception of Nature. Consistent with this paradigm shift, after Newton and Leibniz sanctioned differential calculus as the pre-eminent tool to "explain" the world, a new solution to Zeno's paradox was proposed. Clearly, before catching the tortoise Achilles must traverse ever-smaller segments of space in ever-smaller intervals of time. Does the infinite (as admitted by the pure-reason hypothesis) sum of these ever smaller terms converge to a finite value, or is it itself infinite? What is the *limit* of such a summation when its infinite terms tend to zero? At the limit, the ratio of the space segments divided by the time intervals in which Achilles traverses them approaches the ratio of 0/0, which is indeterminate. But this ratio of the space segments over the time intervals is equal to Achilles' instantaneous speed: how can that be indeterminate precisely at the point where he finally catches the tortoise? Thus, Achilles can only catch the tortoise if the infinite terms of the above mentioned sum converge to some finite value. It follows that in the real world Achilles' continuous velocity function is at least one-time differentiable everywhere.

But, initially, Achilles is at rest. How can he pass from rest to motion? According Zeno this is impossible since, if time is illimitable divisible, at the immediate next "instant" in time after rest he must already possess some finite velocity; otherwise, he remains at rest, and so never could start moving. Whatever that initial (finite) velocity, when divided by the very first interval of time, and if time is infinitely divisible then that "first" interval may be taken as close to zero as you wish, it results that the quotient of that initial (finite) speed divided by the very first interval of time, that is, Achilles *acceleration*, now tends to infinity. And since the reverse argument also applies, motion can neither start nor stop! Unless, of course, the summation of the ratio of the ever smaller velocities over the ever smaller time intervals converges. That is, throughout the interval, the acceleration of our hero must furthermore conform to a *function everywhere two-times differentiable*[ii]. But the subset of everywhere two-times differentiable functions is a vanishing small fraction of the set of all possible continuous functions; and so, in spite of all the effort, we are back where Aristotle left us! Since we do observe Achilles catching the tortoise, we must conclude that the "potential" set of all possible continuous functions that are *not* everywhere two-times differentiable will simply not obtain in the "actual" case. For, unless everything our senses transmit is pure illusion, Achilles can be born, start running, catch the tortoise, stop, and someday die, only if he always manages to traverse the space-time continuum in this particular everywhere two-times differentiable manner. Potentially anything can happen, or everything can be an illusion; actually Achilles has a speed and an acceleration conforming to differential calculus. The physical world may be "explained" by our (pure-reason) mathematical model, but only as far as it conforms to observed phenomena; all potentially possible but unobserved outcomes "collapse" and vanish.

More recent approaches to Zeno's paradox, while still adhering to the Platonic point of view, come however to the contrary solution: all movement *is* illusion. This can be seen in what may be called "Einstein's solution". In Einstein's Theory of Relativity the continuous (i.e., non-atomic) space and the continuous (i.e., non-atomic) time are fused into one continuous (i.e., non-atomic) entity: space-time. This four-dimensional *topological space*[iii] possesses this characteristic: it is entirely frozen, in it there is simply no change, no movement, no *kinêsis*. This follows from the condition that time is already incorporated as the fourth dimension of space-time; so, how could anything *change?* Consequently, in four-dimensional space-time all movement, all change is an illusion, as in a reel of film, where "time" appears as a number, a dimension: the number of each frame. The illusion of *kinêsis* (from which "cinema") is achieved in the usual manner, and the illusion of future and past are the result of running the film in one or the other direction. Zeno—and Parmenides—would have enthusiastically endorsed this way of understanding reality.

A different approach originated in the 1960s with the work of A. Robinson, followed by that of E. Nelson[iv]: nonstandard mathematics. Once again, the question is: Is a line segment divisible without limit? Take a segment of finite length, call the first "point" on it *zero* and the last "point" *one*. The *distance* from the origin (zero) to any point on the segment is given by a (standard) real number of the form: zero-point (0.), followed by an *infinite* expansion (sequence) of digits (for instance, 0.2854618326580009276492651648206517848...). This is the mathematical version of Zeno's axiom, inaccessible to the Greek who did not know the zero. Accepting the "existence" of the real numbers is a strong hypothesis and, consequently, their study has become a key element in the search for a set of axioms or fundamental assumptions on which elementary number theory, and by extension, the whole of mathematics, might be firmly based. It is generally accepted that the set of 10 or so statements supporting most mathematical systems is the Zermelo-Fraenkel set theory. To these statements the nonstandard approach adds three additional axioms. They are based on the definition of a new nonstandard number: the infinitesimal. An infinitesimal nonstandard number is a new type of number: by definition, it is greater than zero but always less than any standard real number, however small.

These infinitesimals possess a thoroughly elusive character because they can never be captured through any possible measurement. The reason: measurements have always as result a standard real number. Furthermore, the difference between two standard real numbers can never be a nonstandard number, which is by definition always less than any standard number. Thus the interval between two nonstandard points on the line, or two nonstandard intervals of time, can never be measured, and so these intervals are forever beyond the range of observation. They exist only by axiomatic (Platonic) definition but can never become actual in Aristotle's sense.

The nonstandard theory adds two more nonstandard numbers as axioms. The nonstandard unlimited number, which is the inverse of an infinitesimal number, is greater than any standard number but nevertheless smaller than infinity. The nonstandard unlimited numbers are thus very large, larger than any standard number, but always finite, that is, always less than the truly infinite numbers. The nonstandard mixed numbers are so to speak in between: around each standard number, on both sides of it on the line segment, a particular set of nonstandard numbers is, on the left, greater than any other standard number but less than this particular standard number; on the right side, it is smaller than any standard number greater than this particular number, but it is still greater than this number. In summary, between zero and infinity, a new infinity of nonstandard numbers has been added by axiom.

How does this nonstandard mathematics "solve" Zeno's paradox? Achilles, as he gets closer and closer to the tortoise, will be traversing an infinite series of ever smaller space segments until eventually he will be at nonstandard infinitesimal distance from the tortoise. From this point on, his progress until he catches his prey escapes all possibility of measurement: all the final segments being nonstandard infinitesimal distances. In other words, what "really" happens when Achilles catches the tortoise can never be known, by definition, and so the case rests.

In the early twentieth century some philosophers have been particularly intrigued by the time continuum: if the instant is zero, when do we *exist*? The past is already gone, the future not yet here, and the present instant zero: when can we claim that we *are?* Martin Heidegger's attempt to crack this problem in *Being and Time*, first published in 1927, may in a certain way be considered to be one more "solution" to Zeno´s paradox. This philosopher suggests that humans are never authentically "being"; instead, from the very moment one is born, one is already dying, i.e., not-being. "The moment you are born you are old enough to die". He furthermore claims that the only "time" that has a sense is the unknown period a human still has before he dies, that is, only the yet non-existent future is real. If one is asked: Will you die? The answer is: Yes, of course, *but not yet*. Like Achilles: will he ever catch the tortoise? Yes, of course, *but not yet*. So, Heidegger states, there are two manner of being: the inauthentic and the authentic. In the former, which is where most of us choose to *be*, our allotted time-span is this *not yet*, this unknown future which allows us to escape from, to conceal, the unbearably displeasing fact: we are mortal. In this manner of being, we never *actually* die, we are *always* alive; death is "only" a *potentiality*. And we can say such strange things as: I *have* no time, don't *waste* your time, etc. Whereas *being* authentically is equivalent, in a sense, to *dying!* That is, to fully accept human mortality. Put differently, inauthentic *being* is equivalent to live, to *be*, in the standard part of the time scale: there we can "measure" time with watches in standard numbers. Whereas authentic being is traversing our life-span in the nonstandard numeration, which is beyond measure, and where, in a sense, as soon as we are born we are already dying[v].

Recently[vi] a new, fascinating twist to this old conundrum appeared: Cris Calude's *Lexicon*. Here is succinctly how it applies. Zeno's paradox, and its possible "solutions", must be somehow explicitly stated in a communicable language. This implies some linear sequence of symbols; the set of allowed symbols constituting the pertinent alphabet. Any *finite* sequence can be unambiguously coded in binary (or decimal) and thus corresponds exactly to some rational number. This paper, for example, corresponds to the rational number "**w**". On the other hand, real numbers are infinite sequences of digits (in whatever chosen code or *base*). Question: Is there a real number that with certainty contains the *word* **w** (i.e.: *exactly* this paper)? Answer, Yes, and furthermore, there exists a real number that contains *every possible "word"*. That is, that contains *everything that can be explicitly stated, coded, communicated.* Here is how that number is constructed, in binary: simply add one after the other every possible binary sequence of 1,2,3,4,… bits:

0,1,00,01,10,11,000,001,011,111,110,100, 010,101, 0000,0001,…

all the way to infinity. By construction, absolutely everything that can be explicitly stated is represented, at least once, in this sequence.

Now it can be shown that this special real number not only contains, by construction, every possible finite linear sequence, say William Shakespeare's complete works, but also that it contains every possible linear sequence *infinitely many times!* This is easily proved. Again, call some sequence, say this paper, **w**. Now construct these sequences:

w0
w00
w000
w0000
....

all the way to infinity. Since by construction, each of them is already on our specially constructed binary real number, all of these "words" or binary sequences must also appear, at least once. But in each of them **w** appears, hence **w** appears infinitely many times. And this is the case for every possible **w**, QED.

It has been shown in 1998 by Calude and Zamfirescu[vi] that there exist real numbers that present this remarkable property *independent of the employed code or alphabet* (binary, decimal, or, for instance, all the symbols on a computer keyboard). These are the Lexicons. Thus a Lexicon contains infinitely many times anything imaginable and not imaginable, everything ever written, or that will ever be written, any description of anything, of any phenomenon, real or imaginary, etc., etc. But where are these monsters to be found? The amazing result[vi] is: almost every real number is, both geometrically and measure-theoretically, a Lexicon ! In particular, if you put all the reals in an urn, and blindly pick out one, with almost certainty it will be a Lexicon.

So what is the relation of this surprising result with Zeno's paradox? We humans are limited and mortal. We can only name, we can only put our fingers on, we can only *actually* exhibit, the rational (finite) numbers. But underlying all our mathematics, and Zeno's paradox, are the real numbers. These, at least the immense majority of them, contain *potentially* everything, and infinitely many times. What we can *actually* exhibit is only a vanishing small subset of the underlying *potentially* possible number set. Now, we may call some "fact"—corresponding to some rational number—a *novelty*. But this is just a Zeno-type illusion: everything is already contained (or expressed) infinitely many times in the infinite set of the reals almost all being Lexicons.

But what is a fact? Something that may have happened and that is amenable to a communicable description (measurement). For instance: the result of the sense-perception by which we observe Achilles catching the tortoise. Thus, something that corresponds to some finite sequence, to some rational number. But this “fact”, this sequence, this change, movement, *kinêsis*, is infinitely many times recorded in infinitely many real numbers ... if they exist, of course. Then it follows from the admittance of the existence of the real numbers, of Zeno’s infinite divisibility axiom of space and time, that there can never be any novelty. Everything *is*, always, and nothing ever *becomes*: Parmenides was right after all!

“The thing that has been, it is that which shall be; and that which is done is that which shall be done : and there is no new thing under the sun” ; so says the Ecclesiastes 1.9 (composed after 250 BC). Everything, the theory tells us, is right there under our nose, so to speak, potentially, but inaccessible. Whereas anything that actually happens, happened, or will happen, is an illusion, since it has been “there”, already, always. A final question remains, however: has anybody anytime laid his fingers on such a Lexicon? Surprisingly, yes, Greg Chaitin’s marvellous and mysterious real number Ω is a Lexicon. But that is another story.

In conclusion, after twenty-five centuries Zeno’s paradox is still with us: if we admit the existence of the real numbers we run into trouble; we deny it, and we find a different set of equally intractable problems since now mathematics, and thus physics describing “change”, become problematic. We are left with admiration for those early Greek philosophers, who unveiled the fundamental *limits* of human reason (and of mathematics[vii]).

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NOTES and REFERENCES

[i]All quotations from : *Aristotle, Physics*, Books I-VIII, translated by P.H.Wicksteads and F.M.Cornford (1929), Loeb Classical Library, Cambridge, Mass. and London (1980).

[ii] The function ought to be continuous, otherwise Achilles disappears at some point to reappear later on !

[iii] More precisely, a *four-dimensional Hausdorff* C^{∞} *manifold, connected, without boundaries and paracompact.* For details, see : Luc Brisson and F. Walter Meyerstein, *Inventing the Universe*, State University of New York (SUNY) Press, New York, 1995.

[iv] Edward Nelson, *Internal Set Theory : A New Approach to Nonstandard Analysis*, Amer. Math. Soc., **83** (1977), 1165-1198. See also the excellent article by William I. McLaughlin, *Resolving Zeno´s Paradoxes*, Scientific American, November 1994, 66-71.

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[v] One way to look at this strange assertion is to remember that, of someone whom the doctor has diagnosed a terminal cancer, one says : “He is dying of cancer”. Here one says : “He is dying from having been born”.

[vi] See Cristian Calude and Tudor Zamfirescu, *The Typical Number is a Lexicon*, in New Zealand Journal of Mathematics, Volume 27 (1998), 7 -13.

[vii] See Gregory J, Chaitin, *The Limits of Mathematics*, Springer-Verlag Singapore, 1998.