CDMTCS
Research Report Series

Degree- and Time-Constrained Broadcast Networks

Michael J. Dinneen
Department of Computer Science
University of Auckland
Auckland, New Zealand

Geoffrey Pritchard
Department of Statistics
University of Auckland
Auckland, New Zealand

Mark C. Wilson
Department of Mathematics
University of Auckland
Auckland, New Zealand

CDMTCS-080
April 1998

Centre for Discrete Mathematics and
Theoretical Computer Science

DEGREE- AND TIME- CONSTRAINED BROADCAST NETWORKS

MICHAEL J. DINNEEN*, GEOFFREY PRITCHARD, AND MARK C. WILSON

Abstract. We consider the problem of constructing networks with as many nodes as possible, subject to upper bounds on the degree and broadcast time. This paper includes the results of an extensive empirical study of broadcasting in small regular graphs using a stochastic search algorithm to approximate the broadcast time. Significant improvements on known results are obtained for cubic broadcast networks.

1. Introduction

Broadcasting is the process of sending a message originating at one node of a network to all other nodes, with the restriction that each node can only forward the message to one of its neighbors at a time. In other words, this is the familiar telephone (or point-to-point) communication model. For a comprehensive survey of this and other communication models see [11].

The classic broadcast design problem was introduced by Farley and others (see [8]). This is the problem of finding graphs of a given order with the least number of edges such that one can broadcast in minimum time from each vertex. It is easy to observe that the minimum time to broadcast in a network of order n is $\lceil \log_2 n \rceil$, since at each time step the number of vertices that have received the message can at most double. The current state of this broadcast problem is presented in [7].

Formally, a *broadcast protocol* for a vertex v (called the *originator*) of a graph $G = (V, E)$ may be defined as follows. It is a sequence $V_0 = \{v\}, E_1, V_1, E_2, \ldots, E_t, V_t = V$ such that each $V_i \subseteq V$, each $E_i \subseteq E$, and for $1 \leq i \leq t$:

1. each edge in E_i has exactly one endpoint in V_{i-1},
2. no two edges in E_i share a common endpoint, and
3. $V_i = V_{i-1} \cup \{w \mid \{u, w\} \in E_i\}$.

Date: March 30, 1998.

1991 *Mathematics Subject Classification.* Primary 90B12. Secondary 05C90, 68R10, 94A05.

Key words and phrases. broadcast networks, bounded degree, diameter, girth, Cayley graphs.

* Corresponding author.

The third author was supported by a NZST Postdoctoral Fellowship.

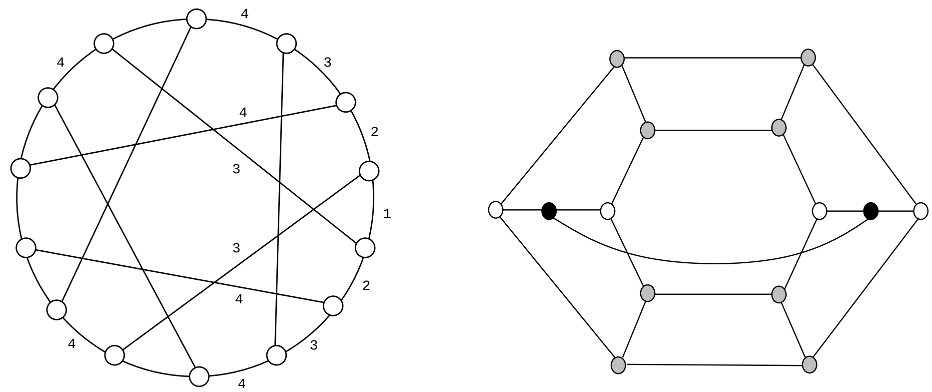

FIGURE 1. Two different $(3, 4)$ broadcast graphs.

Here V_i is the set of vertices which have been informed after i steps. During time step i, each vertex in $V_i \setminus V_{i-1}$ receives the message from a unique vertex in V_{i-1}, and each informed vertex in V_{i-1} sends to at most one uninformed neighbor.

The *broadcast time for a vertex* v of G, denoted $b(G, v)$, is the minimal length t of a broadcast protocol for v. The *broadcast time of the graph* G is $b(G) = \max(b(G, v) \mid v \in G)$.

This paper focuses on the broadcasting problem from a slightly different perspective. Instead of fixing the order and minimizing the number of edges, we constrain both the degree Δ and the broadcast time T while maximizing the order. A (Δ, T) *broadcast graph* is a graph G such that (1) the degree of every vertex $v \in V(G)$ is at most Δ and (2) the broadcast time of G is at most T. We define $B(\Delta, T)$ (respectively $B^{\text{tr}}(\Delta, T)$) to be the maximum number of vertices possible for a graph (respectively, a transitive graph) with maximum degree Δ and broadcast time T.

Two examples of $(3, 4)$ broadcast graphs of order 14 are given in Figure 1. One broadcast protocol is indicated for the symmetric Heawood graph by labeling the edges with the transmission times. The non-symmetric graph on the right requires three different broadcast protocols (one for each white/black/gray node).

The *degree- and time- constrained broadcast problem*, also called the (Δ, T) problem, was introduced in [4] as an engineering alternative to the previously-mentioned broadcast problem of Farley. The (Δ, T) problem is related to the degree/diameter network problem. In that situation, the communication model allows a node to simultaneously send a message to all its neighbors in one time step (the *multi-cast model*). Network designers want the largest possible architecture that satisfies the physical constraints on the number of connections per processor (degree) and limitations on overall communication time (diameter). In the classic broadcast problem, it is observed that these sparse graphs often have a few vertices of very high degree, making networks modeled on these graphs impractical.

We say that a (Δ, T) broadcast graph G is *optimal* if $|V(G)| = B(\Delta, T)$. It is trivial to see that the cycle C_{2T} is optimal for $\Delta = 2$ and $T \geq 2$. However for $\Delta \geq 3$ the (Δ, T) problem is decidedly nontrivial. The main reason is that, in general, the problem of computing the broadcast time of a given graph is very difficult.

It is straightforward to reduce the instances of the three dimensional matching problem to a corresponding minimum broadcast time problem and thereby prove NP-completeness for the general case of unbounded-degree networks [9]. A simple proof showing that the problem of finding the broadcast time of networks of maximum degree 3 is NP-complete is presented in [6]. A more involved proof by Middendorf [13] shows that, in the context of broadcasting with multiple originators in cubic planar graphs, even the problem of determining whether the broadcast time is at most 2 is NP-complete. Since exact algorithms are impractical for large networks, several heuristics have been proposed (for example, see [20, 12, 16]). Also, because of the general hardness of this problem, some research has been restricted to specific families of "nice" graphs. For example, a near-optimal broadcasting algorithm for the pancake graphs, a family of Cayley graphs, is given in [10].

The difficulty of exhibiting broadcast protocols (evident even in the second graph in Figure 1) is one reason for concentrating on *transitive* (also called vertex-transitive or vertex-symmetric) graphs. By definition, each vertex of such a graph G may be mapped to any other vertex by a suitable automorphism of the graph (in other words, the automorphism group $\mathrm{Aut}(G)$ acts transitively on $V(G)$). Hence it suffices to find a protocol for a single originator instead of one for each possible originator.

An important subclass of transitive graphs is the class of *Cayley graphs*. Recall that given a group $(G, \cdot)$ and a set S of generators of G which is closed under inverses, the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is defined by $V(\Gamma) = G$, and $E(\Gamma) = \{\{x, x \cdot s\} \mid x \in G, s \in S\}$. Cayley graphs, because of their accessibility and their transitivity properties, have been systematically and successfully used to model many of the largest known degree/diameter networks (e.g., see [5, 18]). Thus an investigation of these graphs is a natural starting point in an attempt to establish lower bounds for the (Δ, T) problem.

An outline of the paper follows. In Section 2, we present the best lower bounds known for $B(\Delta, T)$ and $B^{\mathrm{tr}}(\Delta, T)$, for small values of Δ and T. We discuss explicit examples of graphs which achieve these bounds. Section 3 contains various necessary conditions on (Δ, T) networks, including some upper bounds on $B(\Delta, T)$ and $B^{\mathrm{tr}}(\Delta, T)$. Some further graph constructions, establishing lower bounds on $B(\Delta, T)$ and $B^{\mathrm{tr}}(\Delta, T)$, appear in Section 4, including a result on the asymptotic behaviour of $B(\Delta, T)$ as $T \to \infty$. Section 5 contains comments on our methodology and Section 6 finishes with a list of selected open problems.

We end this introduction by mentioning two more broadcasting problems related to the (Δ, T) problem, neither of which we study in this paper. Both of these place limits on the broadcast time but have no degree constraints (as they were originally proposed).

One variation of time-constrained broadcasting problem is the *bounded depth broadcasting* problem of Peters and Peters [14], where there is a limit on the number of times information can be retransmitted before it becomes unusable. They define a (t, d)-broadcast graph to be a graph in which broadcasting can be completed from any originator in time t and depth d.

Another time-restricted broadcasting problem was introduced by Shastri [19] where the goal is to find the sparsest networks of order n with broadcast time slightly more than the minimum time. Here a *t-relaxed minimal broadcast network G* is a network in which broadcasting can be accomplished in $\lceil \log_2 v(G) \rceil + t$ time units from any node. It turns out that for relatively small t the sparsest networks are trees, so this parametrized problem is probably not of much interest to engineers.

2. Numerical results

In this section we present the best known lower bounds on $B(\Delta, T)$ for small values of Δ and T, and explicitly present graphs attaining these bounds.

For $\Delta > 2$, there are only two infinite families of graphs which are known to be optimal for the (Δ, T) problem. For $T = \Delta$, the hypercube Q_Δ (the Cayley graph of $(\mathbb{Z}_2)^\Delta$ with respect to the standard generating involutions) is optimal. For $T = \Delta + 1$, the Cayley graph of the dihedral group $D_{2^{\Delta-1}-1} = \langle a, b \mid a^2 = b^{2^\Delta - 1} = (ab)^2 = 1 \rangle$, with respect to generators $\left\{ ab^{2^i - 1} \mid 0 \le i \le \Delta - 1 \right\}$, is optimal (see [4]). In each of these cases a protocol exists which is as simple as possible. Specifically, there is an ordering $s_0 < s_1 < \cdots < s_{\Delta-1}$ of the set of generators such that at time step i, vertex x sends to vertex xs_j, where $0 \le j \le \Delta - 1$ and $j \equiv i \mod \Delta$. In other words, at a given time step all transmissions are in a fixed "dimension", and these dimensions cycle through the elements of S. We shall call such a protocol a *simple protocol*. We believe that simple protocols are rather rare amongst graphs that are close to optimal for this problem.

We now present our numerical results. Details of our methodology are delayed until Section 5.

Table 1 presents the best known lower bounds on $B(\Delta, T)$ for small values of Δ and T, $T \ge \Delta \ge 3$. In Table 1, bold entries are known to be optimal. All of these in fact attain the upper bound on $B(\Delta, T)$ given in Table 2. Italicized entries are new results. All entries in Table 1 are obtained explicitly from Cayley graphs unless indicated by a superscript. An

TABLE 1. Orders of the largest known broadcast networks with degree $\leq \Delta$ and broadcast time $\leq T$.

Δ \ T	3	4	5	6	7	8	9	10	11	12
3	**8**	**14**	**24**	**40***	*64*	*96*	*144*	*216*	*324*	*506*
4		**16**	**30**	**56**	90	*156*	*260*	*444*	*710*	*1220*
5			**32**	**62**	108	186	336	*612*	*1088*	*1958*
6				**64**	**126**	220	390	750	*1320*	*2430*
7					**128**	**254**	440†	816	*1500*†	*2712*
8						**256**	**510**	880	*1632*†	*3000*†
9							**512**	**1022**	*1760*†	*3264*†
10								**1024**	**2046**	*3520*†

TABLE 2. Upper bounds on $B(\Delta, T)$.

Δ \ T	3	4	5	6	7	8	9	10	11	12
3	8	14	24	40	66	108	176	286	464	752
4		16	30	56	104	192	354	652	1200	2208
5			32	62	120	232	448	864	1666	3212
6				64	126	248	488	960	1888	3712
7					128	254	504	1000	1984	3936
8						256	510	1016	2024	4032
9							512	1022	2040	4072
10								1024	2046	4088

asterisk (*) indicates that the entry is transitive but not a Cayley graph, while a dagger (†) means that the entry is obtained from a compound construction as explained in Section 4. We note that, while it is possible that $B(3,7) = 66$, so that the $(3,7)$ entry need not be optimal, it can be shown that $B^{\mathrm{tr}}(3,7) = 64$ (since all cubic transitive graphs of order 66 are known, and can be eliminated by the methods of Section 3). For comparison we have included, in Table 2, some upper bounds on $B(\Delta, T)$. For the origin of these bounds, see Section 3.

Table 3 shows the properties of the largest cubic broadcast graphs for $T \leq 12$. All of these graphs are transitive. In brackets we list how many non-isomorphic graphs that we have found of the given order. For comparison we list, in the third column, the order of the largest known cubic (transitive) graph with diameter T. We believe that our bounds for diameters

TABLE 3. Vital statistics of the largest-known cubic broadcast networks.

	General transitive lower bounds		Properties of best broadcast graph			
T	Max $\#V$ ($b(G)=T$) broadcast graph	Max $\#V$ ($\mathrm{diam}(G)=T$) multi-cast graph	Girth	Diameter	#Aut	Symmetric?
3	8 [2]	14	4	3	48	yes
4	14 [1]	26	6	3	336	yes
5	24 [4]	60	6	4	144	yes
6	40 [1]	82	8	6	480	yes
7	64 [6]	168	8	7	384	yes
8	96 [1]	300	10	7	96	no
9	144 [3]	506	10	8	288	no
10	216 [3]	882	12	8	1296	yes
11	324 [2]	1220	12	9	324	no
12	506 [1]	1830	14	9	506	no

$T = 4$, $T = 6$ and $T \geq 10$ are new (see Appendix A of [21]). In the remaining columns we state properties of what we considered to be the "best" one of the broadcast graphs obtaining the broadcast time T. This is often a *symmetric* graph, that is, $\mathrm{Aut}(G)$ is transitive on the set of directed edges. In addition, each best broadcast graph is bipartite (many of the others are not).

We now discuss our new entries in Table 1 in more detail (see [4] for previous details).

$(3, 5)$: There are exactly four cubic transitive graphs with 24 vertices and broadcast time 5, all of which are Cayley graphs. Of these we discuss two in more detail. For the first, let G be the symmetric group S_4 and let $S = \{s_0, s_1, s_2\} = \{(13), (14), (12)(34)\} \subset G$. Then the graph $\Gamma = \mathrm{Cay}(G, S)$ has a simple protocol with the generators taken in the given order. The second graph is the unique symmetric graph of this order, and does not have a simple protocol.

$(3, 6)$: There is a unique cubic transitive graph Γ with 40 vertices and broadcast time 6. Γ is also the unique cubic symmetric graph of order 40, and is not a Cayley graph. It differs from the $(3, 6)$-graph of order 40 presented in [1], which is not even transitive.

$(3, 7)$: There are exactly six cubic transitive graphs with 64 vertices and broadcast time 7. One of these has diameter 6. Among these is the unique symmetric graph of this order. This graph Γ occurs as the Cayley graph of four non-isomorphic groups of order 64.

$(3, 8)$: We have found a cubic Cayley graph Γ with 96 vertices and broadcast time 8, and there are at most 14 such graphs. Γ is not symmetric. One description of Γ as $\mathrm{Cay}(G, S)$, is as follows. The group G is the semidirect product $C \rtimes D_{16}$, where D_{16} is generated by involutions a, b subject to $(ab)^{16} = 1$ and their action on a generator t of the cyclic group C of order 3 is given by $ata^{-1} = t, btb^{-1} = t^{-1}$. We can take $S = \{a, b, (ab)^3 at\}$.

$(3, 9)$: We have found three cubic Cayley graphs with 144 vertices and broadcast time 9. One such is the Cayley graph of the group $G = \langle a, b \mid a^2 = ab^4ab^{-4} = (b^2ab)^3 = (abab^2)^2 = 1\rangle$, with respect to $\{a, b, b^{-1}\}$.

$(3, 10)$: We have found three cubic Cayley graphs with 216 vertices and broadcast time 10. Among these is one of the three symmetric graphs of this order, known as F216C in the Foster Census (see [17]).

$(3, 11)$: We have found two cubic Cayley graphs with 324 vertices and broadcast time 11, neither of which is symmetric. One such is the Cayley graph of $G = \langle a, b, c \mid a^2 = b^2 = c^2 = cbacabacabca = 1, (cab)^2 = (bca)^2\rangle$, with respect to $\{a, b, c\}$.

$(3, 12)$: We have found a cubic Cayley graph with 506 vertices and broadcast time 12. One representation is as $\mathrm{Cay}(G, S)$ where G is the semidirect product $\mathbb{Z}_{23} \rtimes \mathbb{Z}_{22}$. The action is determined by the requirement that the generator $1 \in \mathbb{Z}_{22}$ maps to the generator $5 \in \mathbb{Z}_{23}^* \cong \mathrm{Aut}(Z_{23})$. We can take S to be the set $\{(1, 21), (18, 1), (0, 11)\} \subset \mathbb{Z}_{23} \times \mathbb{Z}_{22}$.

$(4, 6)$: There is a degree 4 transitive graph Γ of order 56 and broadcast time 6 which was presented in [1]. A broadcast protocol is given in that paper. Γ may be represented as a Cayley graph of $D_{28} = \langle a, b \mid a^2 = b^{28} = (ab)^2 = 1\rangle$, with respect to the set of generating involutions $\{a, ba, b^9a, b^{26}a\}$.

All other new entries which we have found are Cayley graphs similar to the description of the $(3, 12)$ entry. The groups are all semidirect products of two cyclic groups $\mathbb{Z}_m \rtimes \mathbb{Z}_n$. In Table 4, the triple (m, n, k) indicates that the homomorphism from $\mathbb{Z}_n$ into $\mathbb{Z}_m^* \cong \mathrm{Aut}(Z_m)$ is determined by the requirement that it map the generator 1 of $\mathbb{Z}_n$ to an element $k \in \mathbb{Z}_m^*$ such that $k^n = 1$. The ordered pairs represent the Δ generators of the Cayley graph in the usual way as elements of the set $\mathbb{Z}_m \times \mathbb{Z}_n$. The group multiplication in $\mathbb{Z}_m \rtimes \mathbb{Z}_n$, which is usually non-commutative, of the elements (m_1, n_1) and (m_2, n_2) is the element $(m_1 + k^{n_1} \cdot m_2, n_1 + n_2)$.

3. Upper bounds

In this section we derive upper bounds on the size of a (Δ, T) graph in terms of easily computable graph-theoretic properties of the graph. These results help us eliminate many potential candidates while searching for large (Δ, T) graphs.

Let $\Gamma(d)$ be the infinite rooted tree in which every vertex has d children. Then $\Gamma(d)$ has an obvious broadcast protocol from the root, in which every vertex sends to its children in turn (in some specified fixed order). For each $t \geq 0$, let $\Gamma(d, t)$ be the subtree of $\Gamma(d)$ consisting of all vertices which have received the message after t time steps.

TABLE 4. Data for Cayley graphs of semidirect products of cyclic groups.

(Δ, T)	(m,n,k)	generators
(4,8)	(13,12,2)	(5,11), (3,1), (1,6), (0,6)
(4,9)	(13,20,8)	(7,17), (4,3), (1,19), (5,1)
(4,10)	(37,12,8)	(23,6), (1,6), (24,5), (7,7)
(4,11)	(71,10,14)	(51,7), (68,3), (42,7), (56,3)
(4,12)	(61,20,8)	(2,2), (40,18), (55,17), (22,3)
(5,10)	(51,12,13)	(5,9), (31,3), (15,7), (9,5), (0,6)
(5,11)	(68,16,3)	(67,10), (49,6), (10,9), (26,7), (0,8)
(5,12)	(89,22,81)	(27,4), (31,18), (35,10), (76,12), (0,11)
(6,11)	(66,20,5)	(27,19), (63,1), (9,18), (39,2), (35,15), (53,5)
(6,12)	(135,18,4)	(46,0), (89,0), (83,1), (13,17), (6,13), (66,5)
(7,12)	(113,24,18)	(60,17), (72,7), (31,11), (106,13), (58,14), (79,10), (0,12)

More immediately relevant to broadcasting is $\Gamma'(d)$, the infinite rooted tree in which every vertex has degree d (so the root has d children and all other vertices have $d-1$ children). Define $\Gamma'(d,t)$ analogously to $\Gamma(d,t)$: the vertices informed after t broadcast steps. Throughout this section we will assume $d \geq 2$ for $\Gamma(d)$ and $d \geq 3$ for $\Gamma'(d)$ in order to avoid trivial cases.

Let $F(d,t)$ be the number of vertices of $\Gamma(d,t)$, let $f(d,t,k)$ be the number of vertices of $\Gamma(d,t)$ of depth at most k, and let $g(d,t,k) = F(d,t) - f(d,t,k)$ be the number of vertices of $\Gamma(d,t)$ of depth greater than k. Define analogous quantities F', f', g' for $\Gamma'(d)$.

The following easily established formulas are useful in calculating the above quantities.

Proposition 3.1. *The following relations hold for the above values of d.*
The function F satisfies the recurrence

$$F(d,0) = 1, \qquad F(d,T) = 1 + \sum_{i=1}^{\min(d,T)} F(d,T-i) \qquad \text{for } T \geq 1.$$

The function F' is given by

$$F'(d,T) = 2F(d-1,T-1).$$

The function f satisfies the recurrence

$$f(d,T,k) = \begin{cases} 1 + \sum_{i=1}^{\min(d,T)} f(d,T-i,k-1), & \text{if } T \geq 1 \text{ and } k \geq 1 \\ 1, & \text{if } T = 0 \text{ or } k = 0. \end{cases}$$

The function f' is given by

$$f'(d,T,k) = f(d-1,T-1,k) + f(d-1,T-1,k-1).$$

□

If a graph G has maximum degree Δ and a broadcast protocol of time T originating from a vertex v_0, then this protocol induces a *broadcast tree* (the subgraph S of G on the same vertex set, incorporating only those edges used in the broadcast). Of course S is a tree rooted at v_0. We may also view S as a subtree of $\Gamma'(\Delta, T)$ in an obvious way. Thus, $F'(\Delta, T)$ provides an upper bound for $B(\Delta, T)$. This argument is the basis of the table of upper bounds for $B(\Delta, T)$ given earlier.

The next result extends this kind of counting argument still further, to obtain a useful method for bounding the broadcast times of particular graphs.

Proposition 3.2. *Let G be a graph with maximum degree Δ and broadcast time T. Let v_0 be a vertex of G. Then for $0 \leq k \leq T$,*

$$\#\{\text{vertices } w \text{ of } G \mid \rho_G(v_0, w) > k\} \leq g'(\Delta, T, k).$$

Here ρ_G denotes the usual graph-theoretic distance metric on the vertex set of G, and $\#$ the cardinality of a set.

Proof. Let S be a broadcast tree for G with originator v_0 and time T. Note that for any vertices v, w of G we have $\rho_G(v, w) \leq \rho_S(v, w)$. Then

$$\begin{aligned}
&\#\{\text{vertices } w \text{ of } G \mid \rho_G(v_0, w) > k\} \\
\leq &\#\{\text{vertices } w \text{ of } S \mid \rho_S(v_0, w) > k\} \\
\leq &\#\left\{\text{vertices } w \text{ of } \Gamma'(\Delta, T) \mid \rho_{\Gamma'(\Delta,T)}(v_0, w) > k\right\} \\
= &g'(\Delta, T, k).
\end{aligned}$$

For the last step, we have viewed S as a subtree of $\Gamma'(\Delta, T)$. □

If we apply the above result with $k = T$, we recover the obvious fact that $b(G) \geq \text{diam}(G)$, that is, the diameter of a graph may not exceed its broadcast time.

We now move on to consider what effect the *girth* (the length of the smallest cycle) of a graph has on its broadcast time. Intuitively, for regular graphs of degree d, one expects that large trees $\Gamma'(d, T)$ cannot be embedded in a graph G if the root is to lie in a small cycle of G. The next result makes this idea precise.

Let $\beta(\Delta, g, T)$ be the maximum number of vertices amongst all graphs Γ with maximum degree Δ, girth g and broadcast time T, and let $\beta^{\text{tr}}(\Delta, g, T)$ denote the same function restricted to the transitive graphs.

Proposition 3.3. *The following relations hold.*

$$\beta(3,g,T) \leq \begin{cases} B(3,T), & \text{if } g \geq T \\ g + F(2,T-1) + \sum_{i=3}^{g+1} F(2,T-i), & \text{if } g < T. \end{cases}$$

$$\beta^{\mathrm{tr}}(2,g,T) = \begin{cases} F(2,T), & \text{if } 0 \leq T \leq g-1 \\ g + \sum_{i=2}^{g} \beta^{\mathrm{tr}}(2,g,T-i), & \text{if } g \leq T. \end{cases}$$

$$\beta^{\mathrm{tr}}(3,g,T) = 2\beta^{\mathrm{tr}}(2,g,T-1).$$

□

It follows from the recurrences given in Proposition 3.1 that for fixed d, $F(d,T)$ grows as $(\phi_d)^T$ as $T \to \infty$, where ϕ_d is the unique root in the interval $(1,2)$ of the polynomial $x^{d+1} - 2x^d + 1$. This then gives an upper bound on the exponential rate of growth of $B(\Delta,T)$. Note that as d increases, ϕ_d increases with limit 2.

4. Lower bounds

In this section we present graph-theoretic constructions which provide general lower bounds for $B(\Delta,T)$.

Combination methods. In this subsection we explore some ways of constructing graphs with good broadcast times out of smaller graphs with good broadcast times. Initially we consider the possibility of compounding two graphs.

Definition 4.1. *Given two graphs G and H, the* compound product $G \otimes H$ *has vertex set $V(G) \times V(H)$, and edges:*

1. $\{(u,w),(v,w)\}$ *whenever* $\{u,v\}$ *is an edge of* G, *and*
2. $\{(u,v),(u,w)\}$ *whenever* $\{v,w\}$ *is an edge of* H.

Proposition 4.2. $B(\Delta_1+\Delta_2, T_1+T_2) \geq B(\Delta_1,T_1)B(\Delta_2,T_2)$.

Proof. Let G_1, G_2 be two optimal broadcast graphs for (Δ_1,T_1) and (Δ_2,T_2) respectively. Consider $G = G_1 \otimes G_2$. It is clear that G has maximum degree at most $\Delta_1+\Delta_2$. To broadcast in G in time T_1+T_2 from an originator (u_0,v_0), we first take T_1 steps to inform all vertices of form (u,v_0) where $u \in V(G_1)$, using any broadcast protocol which works for G_1. Then, beginning from each (u,v_0), take T_2 steps to inform all vertices (u,v), where $v \in V(G_2)$, using any protocol which works for G_2. □

Corollary 4.3. $B(\Delta+1,T+1) \geq 2B(\Delta,T)$.

Proof. In Proposition 4.2, take one of the graphs to be K_2, the graph with two vertices and one edge. □

Corollary 4.4. *For $k \geq 2$, $B(\Delta+2, T+k) \geq 2kB(\Delta, T)$.*

Proof. In Proposition 4.2, take one of the graphs to be C_{2k}, the cycle of length $2k$. □

Proposition 4.5. $B(\Delta+1, T+3) \geq 4B(\Delta, T)$.

Proof. Let G be an optimal broadcast graph for (Δ, T). Since G is connected we may take a spanning tree of G and use it to characterize every vertex of G as even or odd, according to its distance from the root. Let $G' = G \otimes C_4$. (As usual, the vertex set of the cycle C_4 is taken to be $\mathbb{Z}_4$ and the edge set $\{\{x, y\} \mid y = x+1\}$.) For each even vertex v of G, delete from G' the edges between $(v, 0)$ and $(v, 1)$ and between $(v, 2)$ and $(v, 3)$. For each odd vertex v of G, delete from G' the edges between $(v, 1)$ and $(v, 2)$ and between $(v, 3)$ and $(v, 0)$. Thus, G' has maximum degree at most $\Delta+1$.

To broadcast in G' from an originator (v, x), proceed as follows. At the first step, inform (v, y) where $y = x \pm 1$ depending on the parity of v. At the second step, inform (w, x) and (w, y) where w is a neighbour of v with the opposite parity to v. At the third step, we can inform (w, z_1) and (w, z_2), where z_1 and z_2 are such that $\{x, y, z_1, z_2\} = \mathbb{Z}_4$. The remainder of the broadcast can be accomplished by applying the original protocol for G to the sets $\{(v, t) \mid v \in v(G)\}$, where $t = x, y, z_1$ or z_2. □

We conjecture that $B(\Delta+1, T+2) \geq 3B(\Delta, T)$. This is almost shown by the next result, which requires one extra hypothesis.

Definition 4.6. *A graph G is* pairable *if it has a 1–regular subgraph which includes all the original vertices. Such a subgraph connects the vertices of G into pairs (such pairings are also called* 1-factorizations *or* perfect matchings*).*

Proposition 4.7. *Let G be a pairable graph with maximum degree at most Δ and broadcast time T. Then there exists a pairable graph G' with maximum degree at most $\Delta+1$ and broadcast time at most $T+2$, and $\#V(G') = 3\#V(G)$.*

Proof. Let G' be the disjoint union of three copies of G. If $\{u, v\}$ is a pair in G, then $\{(u, i), (v, i)\}$ is a pair in G' for $i = 1, 2, 3$. For each such pair, add edges $\{(u, 1), (v, 2)\}$, $\{(u, 2), (v, 3)\}$, and $\{(u, 3), (v, 1)\}$.

Now if the originator is, say, $(u, 1)$, we inform $(v, 1)$ at the first time step and $(v, 2)$ and $(u, 3)$ at the second. The remainder of the broadcast proceeds separately in each of the three copies of G. □

The maximum degree has increased for all of the methods mentioned so far. To complete this subsection, we give a way of constructing broadcast graphs with lower degree.

Definition 4.8. *From an adjacency list A the partial function $f_A(u,v)$ is defined to be i if v is the i-th neighbor of u. A* 2-way split *of a graph $G=(V,E)$, with respect to an adjacency list A representation, is a graph $H=(V',E')$ where $V'=V\times\{0,1\}$ and $E'=E_1\cup E_2$ as defined below:*
$E_1=\{\{(v,0),(v,1)\}\mid v\in V\}$,
$E_2=\{\{(u,b),(v,c)\}\mid \{u,v\}\in E, b=(f_A(v,u)\le \deg(u)/2)$ *and* $c=(f_A(u,v)\le\deg(v)/2)\}$.

This splicing idea may be generalized by replacing each vertex with k vertices and partitioning the neighbors evenly into k parts. Instead of using a clique (as was done in the 2-way split) the k copies of each of V are connected with a broadcast graph of low degree and small broadcast time.

Proposition 4.9. $B(\lceil\Delta/2\rceil+1,2T)\ge 2B(\Delta,T)$.

Proof. From a (Δ,T) broadcast graph G of order n we create a 2-way split H of order $2n$. The graph H has broadcast time at most $2T$ by following the broadcast protocols of G. Here, whenever a vertex (v,b) is informed from a vertex (u,c), $u\ne v$, a single time step delay is used to inform $(v,1-b)$ before proceeding. □

A direct construction. The cube-connected cycles, introduced by Preparata and Vuillemin [15], are a well-known family of cubic graphs with an underlying hypercube-like structure. Below we provide a lower bound on the broadcast time of these networks. An immediate consequence of this result is that for all $\Delta\ge 3$, $B(\Delta,T)$ grows exponentially with T.

The cube-connected cycles, n-CCC, are similar to the n-cubes. The vertices are given as pairs (i,V) where i ranges between 0 and $n-1$ and V is a bit vector of length n. For edges, vertex (i,V) is connected to vertex (i',V') if and only if $i=i'$ and V' differs in only the i-th bit from V, or $|i-i'|=1$ and $V=V'$.

The cube-connected cycles were shown to be transitive by Carlsson *et al.* [3]. In fact, they explicitly presented a larger family, the generalized cube-connected cycles, as Cayley graphs.

Theorem 4.10. *The broadcast time of the cube-connected cycle(s) d-CCC is at most* $\left\lceil\frac{5d-2}{2}\right\rceil$.

Proof. Let G be the graph d-CCC. Since G is transitive we only need to provide one broadcast protocol. We will use an optimal underlying broadcast protocol for the hypercube H of dimension d to construct a broadcast protocol for G. Let $C(v)=\{(i,v)\mid i=0,\dots,d-1\}$ represent the set of vertices of (cycle of) G that corresponds to a vertex v of H. Note that the set $\{C(v)\mid v\in V(H)\}$ partitions the vertices of G into equivalence classes.

We now describe the broadcast protocol. First note that we can optimally broadcast in H by using a simple protocol (sending messages to neighbors at dimension t at time t). With

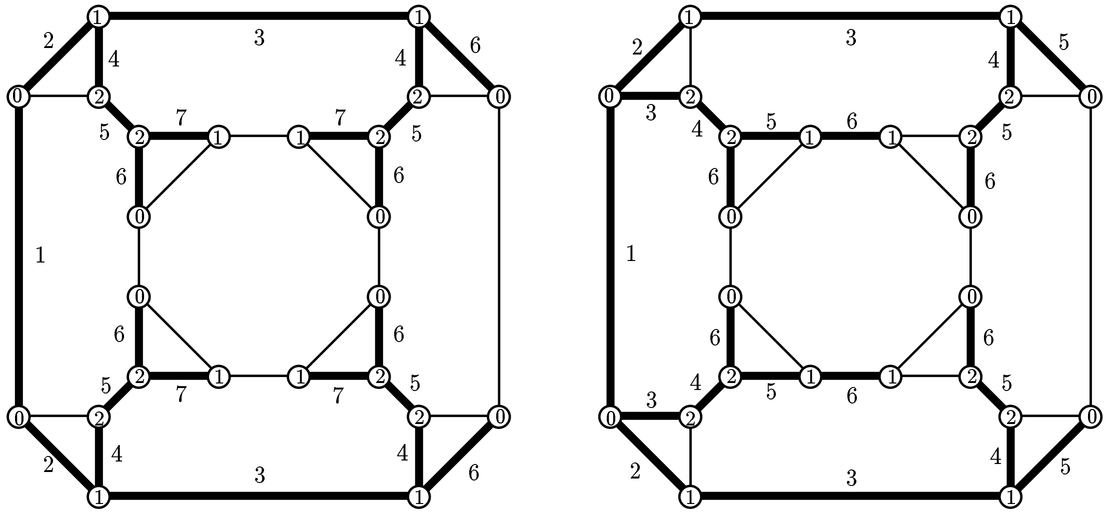

FIGURE 2. The cube-connected cycle(s) 3-CCC and two broadcast protocols: (1) via Theorem 4.10 and (2) via a minimal broadcast tree.

vertex $(0, 00\ldots00)$ as originator in G, the first broadcast is to vertex $(0, 00\ldots01)$, that is, we use dimension 1. At time 2 both these vertices send to their first neighbor on the cycle $C(v)$. At time 3 the informed vertices $(1, 00\ldots00)$ and $(1, 00\ldots01)$ send to their neighbors in dimension 2. Continue the process as follows: at time step $2t-1$ an informed vertex $(t-1, V)$ broadcasts in dimension t to its neighbor $(t-1, V+2^t)$.

There is a transmission delay of one time step after the first vertex of $C(v)$ is informed and before the next neighbor outside of $C(v)$ is informed. Since it takes d time steps to broadcast in the d-cube H, plus $d-1$ delays, at least one vertex in each $C(v)$ is informed by time $2d-1$.

To finish off the broadcasting in G we need at most $\lceil d/2 \rceil$ time steps for a representative v of $C(v)$ to inform any remaining vertices of the cycle $C(v)$. Thus we can broadcast in at most $2d-1+\lceil d/2 \rceil = \lceil \frac{5d-2}{2} \rceil$ time steps. □

Corollary 4.11. $B(3, \lceil \frac{5d-2}{2} \rceil) \geq d2^d$.

Proof. This result follows from Theorem 4.10 and the fact that d-CCC has $d2^d$ vertices. □

The broadcast bounds given in the previous theorem are not sharp. We have found broadcast protocols for the cube-connected cycles 3-CCC and 5-CCC with broadcast times 6 and 11, respectively (one less than our general bound). However, the actual best broadcast time for 4-CCC matches our general bound of 9. On the right of Figure 2 we show a nice broadcast protocol of minimum time for 3-CCC (here one simply broadcasts clockwise or counter-clockwise around each 3-cycle depending on the parity of the time that the first vertex in the cycle receives the message).

5. Comments on our Computation

The examples in Section 2 were generated by examining known graphs with a high degree of symmetry. In particular, the authors found the online database [17] maintained by Gordon Royle to be invaluable. The enumeration of transitive cubic graphs in that database was the raw material for Table 3. The generation of random Cayley graphs, based on semidirect products of cycles (see [5]), was the source of the other Δ–regular graphs that yield new lower bounds in Table 1.

Once we have a list of potential graphs, the next requirement is to know their broadcast times. As mentioned in Section 1, finding the broadcast time is very difficult. We were comfortably able to compute broadcast times of cubic graphs with up to about 80 vertices (time $T \leq 8$). For graphs of higher degree ($\Delta \geq 4$) our current tractable range drops down to graphs with fewer than 50 vertices.

It is possible to partially overcome this difficulty by using a stochastic search algorithm to find broadcast protocols. We used the following simple rule: at each time step, each informed vertex selects one of its uninformed neighbours at random to inform. This generates a random protocol which will inform the whole graph in some finite time. The process may be repeated as often as desired; the smallest of the times found is an upper bound for the broadcast time of the graph. If this upper bound matches a known lower bound, for a given number of vertices, then we have found the broadcast time of the graph. For the examples given in Section 2, the number of attempted random protocols ranged from a few hundred to a few hundred thousand. Since all of our input graphs were transitive, it was sufficient for our C++ implementation to search for broadcast protocols originating from a single vertex (e.g., in the case of Cayley graphs we started from the identity vertex).

To lessen our computational effort we explored several results which bound the broadcast time of a graph in terms of easily computable properties, such as the girth and the diameter. The results mentioned earlier in Section 3 were helpful. Proposition 3.2 proved to be an especially sharp test. For computing many of these graph bounds (on a sequence of graphs) we used the Magma program [2] (called from Perl scripts [22]).

We observed that graphs with large girth and small diameter often have small broadcast times. This suggests that we should look for graphs with a high girth/diameter ratio. In the cases we examined, we found that among all transitive cubic graphs on n vertices which pass the tests in Section 3, the minimal broadcast time always occurs for a graph whose girth/diameter ratio is maximal. Another simple heuristic which should work well in practice is to consider graphs with large automorphism groups. We did not use these non-rigorous ideas to eliminate any graphs in our search, but found them accurate enough to mention.

6. Some conjectures and open problems

Many problems and conjectures arose in the course of this work. We state only a few of them below.

- We know now that $B(\Delta, T)$ grows exponentially with T for $\Delta \geq 3$. It is natural to wonder whether this growth has a limiting exponential rate, i.e. whether the quantity
$$f(\Delta) = \lim_{T\to\infty} \frac{\ln B(\Delta, T)}{T}$$
exists. One could also ask what value it takes. The answer might give a succinct, quantitative description of the benefits of higher connectivity (i.e. higher degree). Assume the limit exists. It is trivial to see that $f(2) = 0$, and that $f(\Delta)$ is an increasing function of Δ. We have seen in this paper that $f(\Delta) > 0$ for $\Delta \geq 3$. The simple estimate $B(\Delta, T) \leq B(T, T) = 2^T$ gives the upper bound $f(\Delta) \leq \ln 2$ for all Δ. The estimates in Section 3 give more precise upper bounds; in particular $f(3) \leq \ln((1+\sqrt{5})/2) \approx 0.4812$.
- All known examples suggest that $B(\Delta, T+1) \geq (3/2)B(\Delta, T)$. This, if true, would of course be a strong lower bound on the actual growth rate of $B(\Delta, T)$.
- Is it true that for all $T \geq 2$, there is an optimal $(3, T)$ broadcast graph with girth $T+2$ if T is even and $T+1$ if T is odd?
- Besides our diameter and girth bounds, does there exist a good polynomial-time algorithm that predicts whether a graph has a small broadcast time?

Acknowledgement

We thank Golbon Zakeri for inspiring us to continue doing research on the (Δ, T) broadcast problem. We also thank Paul Hafner for teaching us how to investigate Cayley graphs with the Magma system and for his comments on an earlier version of this paper.

References

[1] J-C. Bermond, P. Hell, A. L. Liestman and J. G. Peters. Sparse broadcast graphs, *Discrete Appl. Math.* **36** (1992), no. 2, 97–130.
[2] W. Bosma and J. Cannon, *Handbook of Magma Functions*, University of Sydney, 1994.
[3] G. E. Carlsson, J. E. Cruthirds, H. B. Sexton and C. G. Wright. Interconnection networks based on a generalization of cube-connected cycles, *IEEE Trans. Comput.* **34** (1985), 769–777.
[4] M. J. Dinneen, M. R. Fellows and V. Faber. Algebraic constructions of efficient broadcast networks, *Applied algebra, algebraic algorithms and error-correcting codes* (New Orleans, LA, 1991), Lecture Notes in Comput. Sci., vol. 539, Springer, Berlin, 1991, pp. 152–158.
[5] M. J. Dinneen and P. R. Hafner. New results for the degree/diameter problem, *Networks* **24** (1994), 359–367.
[6] M. J. Dinneen. The complexity of broadcasting in bounded-degree networks, Tech. Report LACES-05C-94-31, Los Alamos National Laboratory, 1994.
[7] M. J. Dinneen, J. A. Ventura, M. C. Wilson and G. Zakeri. Compound constructions of minimal broadcast networks, CDMTCS Research Report 026, 1997.
[8] A. M. Farley. Minimal broadcast networks, *Networks* **9** (1979), 313–332.
[9] M. R. Garey and D. S. Johnson. *Computers and intractability. A guide to the theory of NP-completeness*, W. H. Freeman and Co., San Francisco, 1979.

[10] C. GowriSankaran. Broadcasting on recursively decomposable Cayley graphs, Proc. International Workshop on Broadcasting and Gossiping 1990 (Sechelt, BC), in *Discrete Appl. Math.*, **53** (1994), pp. 171–182.
[11] S. M. Hedetniemi, S. T. Hedetniemi and A. L. Liestman. A survey of gossiping and broadcasting in communication networks, *Networks* **18** (1988), 319–349.
[12] G. Kortsarz and D. Peleg. Approximation algorithms for minimum-time broadcast, *SIAM J. Discrete Math.* **8** (1995), 401–427.
[13] M. Middendorf. Minimum broadcast time is NP-complete for 3–regular planar graphs and deadline 2, *Inform. Process. Lett.* **46** (1993), 281–287.
[14] D. B. Peters and J. G. Peters. Bounded depth broadcasting, *Discrete Appl. Math.* **66** (1996), 255–270.
[15] F. P. Preparata and J. Vuillemin. The cube-connected cycles: a versatile network for parallel computation, *Comm. ACM* **24** (1981), 300–309.
[16] R. Ravi. Rapid rumor ramification: Approximating the minimum broadcast time, Proc. 35th IEEE Symp. on Foundations of Computer Science, 1994, pp. 202–213.
[17] G. Royle. Online graph database at `http://www.cs.uwa.edu/~gordon/`.
[18] M. Sampels. Large networks with small diameter, Proc. 23rd Workshop on Graph-Theoretic Concepts in Computer Science (WG '97), Lecture Notes in Comput. Sci., vol. 1335, Springer-Verlag, 1997, pp. 288–302.
[19] A. Shastri. Broadcasting in general networks. I. Trees, *Computing and combinatorics* (Xi'an, 1995), Lecture Notes in Comput. Sci., vol. 959, Springer, Berlin, 1995, pp. 482–489.
[20] P. Scheuerman and G. Wu. Heuristic algorithms for broadcasting in point-to-point computer networks, *IEEE Trans. Comput.* **33** (1984), 804–811.
[21] L. Twele. *Effiziente Implementierung des Todd-Coxeter Algorithmus im Hinblick auf Grad / Durchmesser-Optimierung von knotentransitiven Graphen*, Diplomarbeit Universität Oldenburg, 1997.
[22] L. Wall and R. L. Schwartz. *Programming Perl*, O'Reilly & Associates, Inc., 1991.

Department of Computer Science, University of Auckland, Auckland, New Zealand
E-mail address: `mjd@cs.auckland.ac.nz`

Department of Statistics, University of Auckland, Auckland, New Zealand
E-mail address: `pritchar@scitec.auckland.ac.nz`

Department of Mathematics, University of Auckland, Auckland, New Zealand
E-mail address: `wilson@math.auckland.ac.nz`