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Abstract

On countable structures computability is usually introduced via numberings. For uncountable structures whose cardinality does not exceed the cardinality of the continuum the same can be done via representations. Which representations are appropriate for doing real number computations? We show that with respect to computable equivalence there is one and only one equivalence class of representations of the real numbers which make the basic operations computable. This characterizes the real numbers in terms of the theory of effective algebras or computable structures, and is reflected by observations made in real number computer arithmetic. We also give further evidence for the well-known non-appropriateness of the representation to some base b by proving that strictly less functions are computable with respect to these representations than with respect to a standard representation of the real numbers.

Furthermore we consider basic constructions of representations and the countable substructure consisting of the computable elements of a represented, possibly uncountable structure. For countable structures we compare effectivity with respect to a numbering and effectivity with respect to a representation. Special attention is paid to the countable structure of the computable real numbers.

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1 Introduction

On the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ or the set of finite words over a finite alphabet one has the classical equivalent computability notions by the Church-Turing thesis, defined e.g. via μ-recursive functions or Turing machines. If one wishes to introduce computability on other structures, for example on the set $\mathbb{Q}$ of rational numbers, one can either consider a special machine model for this dataset or try to refer to existing computability notions. The last idea seems to be preferable since by this idea one refers

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to computations as they are carried out by actual digital computers. For countable data sets or structures S this can be done by (partial) notations $\nu :\subseteq \Sigma^* \to S$ (where Σ is a finite alphabet) or numberings $\nu :\subseteq \mathbb{N} \to S$ where the elements n in $\mathrm{dom}\,(\nu)$ are considered to be names for the objects $\nu(n)$ in S. Then the computation is carried out not directly on the objects but on their names. The concept of numbered structures has been treated in great generality by Maltsev [24]. For the theory of total numberings (numberings where all natural numbers are names for objects) see Ershov [8, 9, 10]. For example it is easy to see that there is one and except for computable equivalence only one numbering of the rational numbers $\mathbb{Q}$ which makes the four basic arithmetic field operations computable and the equality relation enumerable. Altogether, structures with this property, that is, with the property that they have one and only one numbering except for equivalence, are especially interesting because their structure already determines the computability notion on them. If one considers special numberings with decidable domains such structures are called "computably stable" by Stoltenberg-Hansen and Tucker [31], or, for one-to-one numberings with decidable domains they are called "computably categorical" in computable model theory, see e.g. Khoussainov and Shore [18].

In this paper we are especially interested in the real numbers. The problem to introduce computability on the real numbers leads to the problem how to represent real numbers on a digital computer or its theoretical model, the Turing machine. Of course, one cannot represent every real number by a finite word or a natural number. Hence, one possibility is to restrict oneself to a countable substructure on which computability can be introduced via a numbering as above. In this more constructive approach, represented in the Russian school of recursive analysis, see e.g. Kushner [22], also reflected in constructive analysis, see Bishop and Bridges [2], one considers only computable real numbers, that is, numbers which can be approximated algorithmically with arbitrary precision. These real numbers can be represented by programs for these algorithms, and hence, can be numbered. We shall consider this approach in Section 5.

The other approach is based on the idea that real numbers can be represented by infinite sequences of digits where larger portions of such a sequence contain more and more precise information about the real number. This leads to a computability notion based on approximations. This computability notion for real functions was studied by Grzegorczyk [12, 13], Lacombe [23], Hauck [14, 15, 16], Pour-El and Richards [29], Kreitz and Weihrauch [33, 21, 38, 34], Ko [19], and others. Kreitz and Weihrauch [33, 21, 34], see also Hauck [16], developed a general theory of represented topological spaces, which allows the introduction of computability on a large variety of spaces. The basic idea is the same as with numberings: one considers a mapping from a name space to the space in question, called a representation, and performs computations not directly on the objects but on their names.

For the real numbers the question arises which representations are suitable for real number computations. We answer this question by proving that there is one and except for computably equivalent ones only one representation of the real numbers such that certain basic operations on the real numbers are effective. These operations are determined by the fact that the set of real numbers $\mathbb{R}$ is a complete archimedean ordered field and uniquely determined by these conditions. Our result has several consequences. On the one hand it gives an algebraic characterization of the computability notion over the real numbers: the notion of a computable real function is intrinsically determined by the structure of the real numbers themselves and by the general approach to define computability using representations and the Turing machine model. An ad hoc definition is not necessary, though often it is very useful. On the other hand it explains the experience made in practice that certain representations of the real numbers are favourable for computational purposes while others are not. For example, computer arithmetic for real number computations, i.e. the hardware and software implementation of algorithms for fast and efficient real number computations makes essential use of *redundant number systems* which stem from representations lying in the class of standard representation of the real numbers, see Muller [28] and Section 4. Note that Kreitz and Weihrauch [21, 38] defined the notion of an *admissible* representation which is defined mainly in topological terms, but still relies on the choice of a numbering of a base of the topology. If this numbering is chosen appropriately, then the admissible representations of the reals are standard representations.

Before we analyze the real number structure we provide a general framework of effectivity for countable structures and for uncountable structures. In the first section we introduce structures, for countable structures effectivity and effective categoricity with respect to numberings, and for structures whose cardinality does not exceed the cardinality of the continuum effectivity and effective categoricity with respect to representations. It is essential that we allow also infinitary operations.

In the following section about basic constructions we show that the classical result (Maltsev [24]) about the minimal (in terms of reduction) numbering of a finitely generated algebra can be transferred to these cases. Furthermore we show that a structure which is effectively categorical with respect to representations has a canonical countable substructure of computable elements. For countable structures we compare the notions of effectivity with respect to a numbering and of effectivity with respect to a representation. These notions are equivalent if only finitary operations are allowed and the structure is generated by its constants and functions, but they turn out to be not equivalent in general.

In the subsequent section the full structure of the real numbers is considered, and it is shown to be effectively categorical. We complement this result by giving further evidence that the representation to a base b is not appropriate for computational purposes: we prove that the class of functions which are computable with respect to any of these representations is always a strict subset of the class of functions computable with respect to a standard representation of the real numbers.

Then we discuss various results connected with the countable structure of the computable real numbers. In analogy to the effective categoricity of the full real number structure with respect to representations, the countable structure of the computable real numbers is effectively categorical with respect to numberings. This follows immediately from a theorem of Moschovakis [26].

The paper ends with a discussion of metric spaces and with open problems.

2 Effectivity for Countable and for Uncountable Structures

The section starts with the definition of the notion of a structure in the form in which we will use it. Then we introduce numberings and representations, define computability for functions and relations with respect to them, and explain when a structure may be called effective or effectively categorical with respect to numberings or representations.

By $f :\subseteq X \to Y$ we denote a partial function with domain $\operatorname{dom} f \subseteq X$ and range $f \subseteq Y$. We denote $x \in \operatorname{dom} f$ also by $f(x) =\downarrow$ or by $f(x) \downarrow$ while $x \notin \operatorname{dom} f$ is denoted by $f(x) =\uparrow$ or $f(x) \uparrow$. If S is a set, then $S^\omega = \{p \mid p : \mathbb{N} \to S\}$ is the set of all infinite sequences over S. A set $A \subseteq \mathbb{N}$ is called an *initial segment of* $\mathbb{N}$ if there is an $N \in \mathbb{N} \cup \{\omega\}$ with $A = \{n \in \mathbb{N} \mid n < N\}$ where $\{n \in \mathbb{N} \mid n < \omega\} = \mathbb{N}$. We call a function $m :\subseteq \mathbb{N} \to \{1, 2, \ldots\} \cup \{\omega\}$ *computable* iff the function $\tilde{m} :\subseteq \mathbb{N} \to \mathbb{N}$ with $\operatorname{dom} \tilde{m} = \operatorname{dom} m$ and $\tilde{m}_i = m_i$ if $m_i < \omega$ and $\tilde{m}_i = 0$ if $m_i = \omega$, is computable.

Definition 2.1 1. A *signature* is a triple (N_c, m, n) consisting of an element $N_c \in \mathbb{N} \cup \{\omega\}$ and two computable functions $m :\subseteq \mathbb{N} \to \{1, 2, \ldots\} \cup \{\omega\}$ and $n :\subseteq \mathbb{N} \to \{1, 2, \ldots\} \cup \{\omega\}$ whose domains are initial segments of $\mathbb{N}$.

2. A *structure* $\mathcal{S}$ *of signature* (N_c, m, n) is a quadruple

$$\mathcal{S} = (S, c, f, P)$$

consisting of

(a) a set S, the *universe*,

(b) a function $c :\subseteq \mathbb{N} \to S$ with $\operatorname{dom} c = \{i \in \mathbb{N} \mid i < N_c\}$, the list of *constants*,

(c) a function $f :\subseteq \bigoplus_{j \in \operatorname{dom} m} S^{m_j} \to S$ defined on a subset of the direct sum $\bigoplus_{j \in \operatorname{dom} m} S^{m_j}$ of the spaces S^{m_j},

(d) and a subset $P \subseteq \bigoplus_{k \in \operatorname{dom} n} S^{n_k}$.

3. For $j \in \operatorname{dom} m$ we define the *function* f_j to be the restriction of f to the j-th component S^{m_j} of the direct sum $\bigoplus_{j \in \operatorname{dom} m} S^{m_j}$, and analogously for $k \in \operatorname{dom} n$ the *predicate* P_k to be the intersection of P with the k-th component S^{n_k} of the direct sum $\bigoplus_{k \in \operatorname{dom} n} S^{n_k}$. The numbers m_j and n_k are called the *arities* of f_j and P_k, respectively. Usually a structure will be written in the form

$$(S, c_0, c_1, c_2, \ldots, f_0, f_1, f_2, \ldots, P_0, P_1, P_2, \ldots)$$

where the signature is clear from the context.

Note that we consider not only functions and relations on finite vectors but also *infinitary* functions and relations which are defined on infinite sequences $s \in S^\omega$.

Remarks 2.2 Without problems the definition could be changed in the following ways.

1. One could consider constants as functions on $S^0 = \emptyset$ instead of treating constants and functions separately.

2. Sometimes it is advantageous to consider not only functions $f :\subseteq S^m \to S$ but also functions $f :\subseteq S^m \times T^n \to R$ where T and R might be other structures. The same applies to predicates. We shall consider an example in the last section.

3. In many applications, especially in computable analysis it is often desirable to consider also operators which do not produce one specific value on one argument but one value out of a set of values. This could be modeled by set-valued functions but often the notion of a correspondence (see Bourbaki [3]) is more appropriate, compare Weihrauch and Kreitz [21, 34], Brattka and Hertling [5], and Brattka [4].

Examples 2.3 1. The rational number structure $(\mathbb{Q}, 0, 1, +, -, *, 1/., =)$ consisting of the rational numbers, the constants 0 and 1, the field operations addition "+": $\mathbb{Q}^2 \to \mathbb{Q}$, additive inverse "−": $\mathbb{Q} \to \mathbb{Q}$, multiplication "∗": $\mathbb{Q}^2 \to \mathbb{Q}$, multiplicative inverse "1/.": $\mathbb{Q} \setminus \{0\} \to \mathbb{Q}$, and the predicate "=" $= \{(q,q) \in \mathbb{Q}^2 \mid q \in \mathbb{Q}\}$. This structure has the signature $(2, m, n)$ with $\operatorname{dom} m = \{0, 1, 2, 3\}$, and $m_0 = m_2 = 2$, $m_1 = m_3 = 1$, and $\operatorname{dom} n = \{0\}$, $n_0 = 2$.

2. The real number structure $(\mathbb{R}, 0, 1, +, -, *, 1/., \mathrm{Cauchylim}, <)$ consisting of the real numbers, the constants 0 and 1, the field operations addition "+": $\mathbb{R}^2 \to \mathbb{R}$, additive inverse "−": $\mathbb{R} \to \mathbb{R}$, multiplication "∗": $\mathbb{R}^2 \to \mathbb{R}$, multiplicative inverse "1/.": $\mathbb{R} \setminus \{0\} \to \mathbb{R}$, the order relation "<" $= \{(x,y) \in \mathbb{R}^2 \mid x < y\}$, and the limit operator $\mathrm{Cauchylim} :\subseteq \mathbb{R}^\omega \to \mathbb{R}$ which maps each Cauchy sequence of reals to its limit, that is

$$\operatorname{dom}(\mathrm{Cauchylim}) = \{(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^\omega \mid (\forall \varepsilon > 0)(\exists N)(\forall n, m > N)\ |x_n - x_m| \leq \varepsilon\},$$

$$\mathrm{Cauchylim}((x_n)_n) = \lim_{n \to \infty} x_n \quad \text{for all } (x_n)_n \in \operatorname{dom}(\mathrm{Cauchylim}).$$

This structure has the signature $(2, m, n)$ with $\operatorname{dom} m = \{0, 1, 2, 3, 4\}$, and $m_0 = m_2 = 2$, $m_1 = m_3 = 1$, $m_4 = \omega$, and $\operatorname{dom} n = \{0\}$, $n_0 = 2$.

We wish to introduce computability on structures and to make precise what it means to say that an element, a function, or a predicate of a structure is computable. For a countable structure S this can be done by using numberings or notations, which is equivalent. We shall use numberings.

Definition 2.4 A *numbering* of a countable set S is a surjective function $\nu :\subseteq \mathbb{N} \to S$.

Remark 2.5 We do not impose any restriction on the domain of definition of the numbering, thereby following e.g. Maltsev [24], Moschovakis [26, 27], and Weihrauch [34], though in many references in the literature mainly numberings with computably enumerable or decidable domains are considered (see also Maltsev [24] or Stoltenberg-Hansen and Tucker [31]; Ershov [8, 9, 10] treats total numberings). The reason for our choice is the fact that we also wish to capture the structure of the computable real numbers, which does not have a standard numbering with a nice domain, see Section 5. It seems that the introduction of computability on structures via naming systems gives rise to two aims: (a) to be able to perform the desired operations in an effective way on names of objects, (b) to be able to recognize whether a natural number or string is a name of an object or not. With our definition above we address the first aim, neglecting the second. The second is infeasible already for the computable real numbers.

For uncountable structures whose cardinality is at most the cardinality of the continuum one can proceed in the same manner using representations, see Hauck [16] and Weihrauch and Kreitz [33, 21, 34]. Therefore one must have a natural computability notion on the space of names. Possible spaces of names are the Cantor space $\Sigma^\omega = \{p \mid p : \mathbb{N} \to \Sigma\}$ (i.e. the space of all infinite sequences over a finite alphabet Σ; computable functions are defined via Oracle Turing machines or Type 2 Turing machines) or the Baire space $\mathbb{F} := \{p \mid p : \mathbb{N} \to \mathbb{N}\}$ (the space of all total functions mapping natural numbers to natural numbers; computable functions will be defined below). For complexity theoretic considerations one should use the Cantor space because its computability is based directly on the Turing machine model, see e.g. Weihrauch [36, 37]. We shall use the Baire space $\mathbb{F}$, mainly because of notational simplicity. Most of the following notions are taken from Weihrauch and Kreitz [33, 21, 34].

Definition 2.6 A *representation* of a set S is a surjective function $\rho :\subseteq \mathbb{F} \to S$.

We summarize numberings and representations by saying that a function $\delta :\subseteq X \to S$ is a *naming system* if X is either equal to $\mathbb{N}$ or $\mathbb{F}$ and δ is either a numbering or a representation.

In order to introduce computability on a numbered or represented set we need computability notions on $\mathbb{N}$ and $\mathbb{F}$. We use the usual computable functions and functionals, compare Rogers [30], Weihrauch [34]. For the convenience of the reader we give the definitions.

Computable functions $f :\subseteq \mathbb{N}^k \to \mathbb{N}$ and *computably enumerable (c.e.)* sets $A \subseteq \mathbb{N}^k$ ($k \geq 1$ arbitrary) are defined in the usual way (also called partial recursive, recursively enumerable).

A function $f :\subseteq \mathbb{N} \to \mathbb{F}$ is called *computable* iff there is a computable function $g :\subseteq \mathbb{N}^2 \to \mathbb{N}$ with $\mathrm{dom}\, f = \{n \in \mathbb{N} \mid (\forall i)\ g(i,n) \downarrow\}$ and $f(n)(i) = g(i,n)$, for all $n \in \mathrm{dom}\, f$ and $i \in \mathbb{N}$.

A function $f :\subseteq \mathbb{F} \to \mathbb{N}$ is called *computable* iff there is a computable[1] function $g :\subseteq \mathbb{N}^* \to \mathbb{N}$ satisfying the following three conditions: (1) $g(v) = g(vw)$ for all v, $vw \in \mathrm{dom}\, g$, (2) $\mathrm{dom}\, f = (\mathrm{dom}\, g)\mathbb{F} = \{p \in \mathbb{F} \mid \text{a prefix of } p \text{ lies in } \mathrm{dom}\, g\}$, (3) $f(p) = g(v)$ for any $p \in \mathrm{dom}\, f$ and any prefix $v \in \mathrm{dom}\, g$ of p.

A function $g :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ is called *monotone* iff $g(vw) \in g(v)\mathbb{N}^*$ for all v, $vw \in \mathrm{dom}\, g$. If $g :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ is monotone, then the function $f :\subseteq \mathbb{F} \to \mathbb{F}$ *induced* by g is defined by $\mathrm{dom}\, f = \bigcap_n (g^{-1}(\mathbb{N}^n\mathbb{N}^*)\mathbb{F})$ and $f(p) \in g(v)\mathbb{F}$ for all prefixes $v \in \mathrm{dom}\, g$ of p. A function $f :\subseteq \mathbb{F} \to \mathbb{F}$ is called *computable* iff there is a monotone, computable[2] function $g :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ which induces f. It is well-known and easy to prove that a function $f :\subseteq \mathbb{F} \to \mathbb{F}$ is computable if and only if there is a total (!), monotone, computable function $g : \mathbb{N}^* \to \mathbb{N}^*$ which induces f.

A function $f :\subseteq \mathbb{N} \times \mathbb{F} \to X$ (where $X \in \{\mathbb{N}, \mathbb{F}\}$) is called *computable* iff the function $g :\subseteq \mathbb{F} \to X$ with $g(p) := f(p(0), (p(1), p(2), \ldots))$ is computable. Note that a function $g :\subseteq \mathbb{F} \to \mathbb{F}$ is computable if and only if there is a computable function $h :\subseteq \mathbb{N} \times \mathbb{F} \to \mathbb{N}$ with $\mathrm{dom}\, g = \{p \in \mathbb{F} \mid (\forall n)\ h(n,p) \downarrow\}$ and $g(p)(n) = h(n,p)$ for all $p \in \mathrm{dom}\, g$ and $n \in \mathbb{N}$.

A subset $U \subseteq \mathbb{F}$ is called *computably enumerable (c.e.)* iff there is a computable function $f :\subseteq \mathbb{F} \to \mathbb{N}$ with $\mathrm{dom}\, f = U$. This is equivalent to saying that there is a computably enumerable[3] subset $A \subseteq \mathbb{N}^*$ with $U = A\mathbb{F} = \bigcup_{w \in A} w\mathbb{F} = \{p \in \mathbb{F} \mid \text{a prefix of } p \text{ lies in } A\}$. A subset $U \subseteq \mathbb{N} \times \mathbb{F}$ is called *c.e.* iff the set $\{p \in \mathbb{F} \mid (p(0), (p(1), p(2), \ldots)) \in U\}$ is c.e.

We denote the set of partial computable functions $f :\subseteq \mathbb{N} \to \mathbb{N}$ by $\mathrm{P}^{(1)}$. Often we shall use a total standard numbering $\varphi : \mathbb{N} \to \mathrm{P}^{(1)}$ with the properties:

[1] To be precise: this means that there is a computable function $\tilde{g} :\subseteq \mathbb{N} \to \mathbb{N}$ with $g(\nu_{\mathbb{N}^*}(i)) = \tilde{g}(i)$ for all $i \in \mathbb{N}$, where $\nu_{\mathbb{N}^*} : \mathbb{N} \to \mathbb{N}^*$ is the bijection with $\nu_{\mathbb{N}^*}(0) = \varepsilon$ and $\nu_{\mathbb{N}^*}(1 + \langle k, \langle n_0, \ldots, n_k \rangle\rangle) = (n_0, \ldots, n_k)$ for $k \in \mathbb{N}$, $n_0, \ldots, n_k \in \mathbb{N}$. For the standard tupling function $\langle , \rangle$ see after Corollary 2.12.

[2] This is also meant relatively to the numbering $\nu_{\mathbb{N}^*}$: there is a computable function $\tilde{g} :\subseteq \mathbb{N} \to \mathbb{N}$ with $g(\nu_{\mathbb{N}^*}(i)) = \nu_{\mathbb{N}^*}\tilde{g}(i)$ for all $i \in \mathbb{N}$.

[3] This means: the set $\nu_{\mathbb{N}^*}^{-1}(A)$ is c.e.

1. (*utm Theorem*) the function $u_\varphi :\subseteq \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $u_\varphi(n,i) = \varphi_n(i)$ for all $n, i \in \mathbb{N}$ is computable,

2. (*smn Theorem*) for any computable $H :\subseteq \mathbb{N}^2 \to \mathbb{N}$ there is a total computable function $r : \mathbb{N} \to \mathbb{N}$ with $\varphi_{r(n)}(i) = H(n,i)$ for all $n, i \in \mathbb{N}$.

The total computable functions, i.e. the elements of the set $\mathrm{R}^{(1)} = \{f : \mathbb{N} \to \mathbb{N} \mid f$ is computable$\}$, are the computable elements of $\mathbb{F}$.

Definition 2.7 Let $X, Y \in \{\mathbb{N}, \mathbb{F}\}$, and $\gamma :\subseteq X \to S$ and $\delta :\subseteq Y \to T$ be naming systems of sets S and T.

1. An element $s \in S$ is called *γ-computable* iff it has a computable γ-name, i.e. if there is a computable element $p \in X \cap \operatorname{dom} \gamma$ with $\gamma(p) = s$.

2. A function $f :\subseteq S \to T$ is called *(γ, δ)-computable* iff there is a computable function $g :\subseteq X \to Y$ with $\operatorname{dom} f\gamma \subseteq \operatorname{dom} g$ and
$$f\gamma(p) = \delta g(p)$$
for all $p \in \operatorname{dom} f\gamma$. This is expressed by saying that g *(γ, δ)-tracks* f.

3. A subset $P \subseteq S$ is called *γ-enumerable* iff there is a computably enumerable set $A \subseteq X$ with
$$A \cap \operatorname{dom} \gamma = \gamma^{-1}(P)\,.$$
Then we say that A *γ-enumerates* P.

The definition of a γ-computable element is void if γ is a numbering since then all elements are γ-computable.

Remarks 2.8 1. By not imposing any restriction on the domain of a (γ, δ)-computable function we follow Maltsev [24] and Weihrauch [34]. Often only functions which are strongly computable in the following sense are considered: a function $f :\subseteq S \to T$ is called *strongly (γ, δ)-computable* iff there is a computable function $g :\subseteq X \to Y$ which (γ, δ)-tracks f and satisfies the condition $\operatorname{dom} f\gamma = \operatorname{dom} g \cap \operatorname{dom} \gamma$. It is easy to see that this is equivalent to demanding that f is (γ, δ)-computable and its domain $\operatorname{dom} f$ is γ-enumerable if δ is a numbering. But if δ is a representation the domain of f can be more complicated. For our purposes the simple computability definition of a function turns out to be sufficient. A further reason will be given in Section 5.

2. Often a subset $P \subseteq S$ is called γ-decidable if its characteristic function $c_A : S \to \mathbb{N}$, $c_A(s) = (1$ if $s \in A$, $= 0$ otherwise$)$ is $(\gamma, \mathrm{id}_{\mathbb{N}})$-computable. It is easy to see that this is equivalent to A and $S \setminus A$ being γ-enumerable. Therefore we will only use the notion of γ-enumerability.

3. For the definition of (γ, δ)-computability for correspondences (see Remark 2.2) the reader is referred to Weihrauch and Kreitz [21, 34].

We show that the computability induced by a naming system depends only on its equivalence class.

Definition 2.9 Let S, T be two sets, and $\gamma :\subseteq X \to S$, $\delta :\subseteq Y \to T$ be naming systems. We say that γ *can be reduced* to δ and write $\gamma \le \delta$ iff there is a computable function $g :\subseteq X \to Y$ with

$$\operatorname{dom}\gamma \subseteq \operatorname{dom} g \quad \text{and} \quad \gamma(p) = \delta g(p) \quad \text{for all} \ \ p \in \operatorname{dom}\gamma$$

(this implies $S \subseteq T$). In that case we say that g *proves* $\gamma \le \delta$. The naming systems γ and δ are *equivalent* (written: $\gamma \equiv \delta$) iff $\gamma \le \delta$ and $\delta \le \gamma$.

We leave the proof of the following lemma to the reader.

Lemma 2.10 *Let $X, Y \subseteq \{\mathbb{N}, \mathbb{F}\}$. Let $A \subseteq Y$ be c.e. and $f :\subseteq X \to Y$ be computable.*

1. *If $Y = \mathbb{N}$, then also $f^{-1}(A)$ is c.e.*
2. *If $Y = \mathbb{F}$, then there is a c.e. set $B \subseteq X$ with $B \cap \operatorname{dom} f = f^{-1}(A)$.*

Proposition 2.11 *Let S, T be sets, and $\gamma :\subseteq X \to S$, $\gamma' :\subseteq X' \to S$ and $\delta :\subseteq Y \to T$, $\delta' :\subseteq Y' \to T$ be naming systems with $\gamma \le \gamma'$ and $\delta \le \delta'$. Then:*

1. *If an element $s \in S$ is γ-computable it is also γ'-computable.*
2. *If a function $f :\subseteq S \to T$ is (γ', δ)-computable it is also (γ, δ')-computable.*
3. *If a set $P \subseteq S$ is γ'-enumerable it is also γ-enumerable.*

Proof. 1. If $p \in \operatorname{dom}\gamma$ is a computable γ-name for an element $s \in S$ and a computable function g proves $\gamma \le \gamma'$, then $g(p)$ is a computable γ'-name for s.

2. If g_1 proves $\gamma \le \gamma'$, g_2 proves $\delta \le \delta'$, and h (γ, δ)-tracks f, then the function $g_2 \circ h \circ g_1$ is computable itself or has a computable extension. This extension (γ, δ')-tracks f.

3. Assume that g proves $\gamma \le \gamma'$ and A γ'-enumerates P. By Lemma 2.10 there is a c.e. set $B \subseteq X$ with $B \cap \operatorname{dom} g = g^{-1}(A)$. One checks that this set γ-enumerates P. □

Corollary 2.12 *Let S, T be sets, and $\gamma :\subseteq X \to S$, $\gamma' :\subseteq X' \to S$ and $\delta :\subseteq Y \to T$, $\delta' :\subseteq Y' \to T$ be naming systems with $\gamma \equiv \gamma'$ and $\delta \equiv \delta'$. Then:*

1. *Any element $s \in S$ is γ-computable, iff it is γ'-computable.*
2. *Any function $f :\subseteq S \to T$ is (γ, δ)-computable, iff it is (γ', δ')-computable.*
3. *Any set $P \subseteq S$ is γ-enumerable, iff it is γ'-enumerable.*

Before we define effectivity of structures we need simple constructions of new naming systems out of given ones. Given a naming system $\gamma :\subseteq X \to S$ we wish to construct canonical naming systems γ^k for S^k, γ^ω for a subset of S^ω, and naming systems for direct sums of named sets.

First we treat the case of numberings. We need the standard bijection $\langle , \rangle : \mathbb{N}^2 \to \mathbb{N}$ defined by $\langle i, j \rangle := \frac{1}{2}(i+j)(i+j+1) + j$ and the derived bijections $\langle \ldots \rangle : \mathbb{N}^{k+1} \to \mathbb{N}$ defined by $\langle i \rangle := i$ for $k = 0$ and $\langle i_1, i_2, \ldots, i_{k+1} \rangle := \langle i_1, \langle i_2, \ldots, i_{k+1} \rangle \rangle$ for $k \ge 2$. We also use the projections $\pi^k_l : \mathbb{N} \to \mathbb{N}$ defined by $\pi^k_l \langle i_1, \ldots, i_k \rangle := i_l$ for $1 \le l \le k$. Let

$\nu_i :\subseteq \mathbb{N} \to S_i$ for $0 \le i \le k$ for some $k \in \mathbb{N}$ be numberings of sets S_i. We define the numbering $(\nu_0, \ldots, \nu_k) :\subseteq \mathbb{N} \to S_0 \times \cdots \times S_k$ by

$$(\nu_0, \ldots, \nu_k)\langle n_0, \ldots, n_k\rangle := \begin{cases} (\nu_0(n_0), \ldots, \nu_k(n_k)) & \text{if } n_0 \in \operatorname{dom}\nu_0, \ldots, n_k \in \operatorname{dom}\nu_k \\ \uparrow & \text{otherwise.} \end{cases}$$

Let $\nu :\subseteq \mathbb{N} \to S$ be a numbering. For $1 \le k < \omega$ the numbering $\nu^k :\subseteq \mathbb{N} \to S^k$ is defined by $\nu^k := (\nu, \ldots, \nu)$ (k times). The set of all ν-*computable* sequences is defined as

$$S^{\omega,\nu-\mathrm{comp}} := \{s = (s_0, s_1, s_2 \ldots) \in S^\omega \mid (\exists p \in \mathrm{R}^{(1)})(\forall i)\ s_i = \nu(p(i))\}.$$

The numbering $\nu^\omega :\subseteq \mathbb{N} \to S^{\omega,\nu-\mathrm{comp}}$ is defined via a total standard numbering $\varphi : \mathbb{N} \to \mathrm{P}^{(1)}$ by

$$\nu^\omega(n) := \begin{cases} (\nu\varphi_n(0), \nu\varphi_n(1), \ldots) & \text{if } \varphi_n \in \mathrm{R}^{(1)} \text{ and } \varphi_n(i) \in \operatorname{dom}\nu, \text{ for all } i \\ \uparrow & \text{otherwise.} \end{cases}$$

Finally, for an arbitrary set $A \subseteq \mathbb{N}$ and function $m : A \to \{1, 2, \ldots\} \cup \{\omega\}$ we define the numbering $\bigoplus_{j\in A} \nu^{m_j} :\subseteq \mathbb{N} \to \bigoplus_{j\in A} S^{m_j}$ by

$$\bigoplus_{j\in A} \nu^{m_j}\langle i, k\rangle := \text{ the element } \nu^{m_i}(k) \text{ in the } i\text{-th component of } \bigoplus_{j\in A} S^{m_j}$$

for each $i \in A$ and $k \in \operatorname{dom}\nu^{m_i}$, and $\bigoplus_{j\in A} \nu^{m_j}\langle i, k\rangle :=\uparrow$ otherwise. Obviously, if $\nu_i \le \nu_i'$ and $\nu \le \nu'$, then $(\nu_0, \ldots, \nu_k) \le (\nu_0', \ldots, \nu_k')$, $\nu^k \le (\nu')^k$ for $1 \le k < \omega$, and (by an application of the smn Theorem) $\nu^\omega \le (\nu')^\omega$. This is true also for $\bigoplus_{j\in A} \nu^{m_j}$ if A is an initial segment of $\mathbb{N}$ and the arity function m is a computable function. Also, replacing φ by another total standard numbering of $\mathrm{P}^{(1)}$ does not change the equivalence class of ν^ω (or of $\bigoplus_{j\in A} \nu^{m_j}$, if A is an initial segment of $\mathbb{N}$ and the arity function m is a computable function).

We do the same for representations, see Weihrauch and Kreitz [33, 21, 34]. We need standard tupling functions on $\mathbb{F}$. The empty word in $\mathbb{N}^*$ is denoted by ε. For $p = p(0)p(1)p(2)\ldots \in \mathbb{F}$ and $q = q(0)q(1)q(2)\ldots \in \mathbb{F}$ we define $\langle p, q\rangle := p(0)q(0)p(1)q(1)\ldots \in \mathbb{F}$. For $p_0, p_1, \ldots, p_{n+1}, \ldots \in \mathbb{F}$ we define recursively: $\langle p_0\rangle := p_0$ and $\langle p_0, \ldots, p_{n+1}\rangle := \langle p_0, \langle p_1, \ldots, p_{n+1}\rangle\rangle$, and a coding of infinite sequences over $\mathbb{F}$ by: $\langle p_0, p_1, p_2, \ldots\rangle(\langle i, j\rangle) := p_i(j)$.

Let $\rho_i :\subseteq \mathbb{F} \to S_i$ for $0 \le i \le k$ for some $k \in \mathbb{N}$ be representations of sets S_i. We define the representation $(\rho_0, \ldots, \rho_k) :\subseteq \mathbb{F} \to S_0 \times \cdots \times S_k$ by

$$(\rho_0, \ldots, \rho_k)\langle p_0, \ldots, p_k\rangle := \begin{cases} (\rho_0(p_0), \ldots, \rho_k(p_k)) & \text{if } p_0 \in \operatorname{dom}\rho_0, \ldots, p_k \in \operatorname{dom}\rho_k \\ \uparrow & \text{otherwise.} \end{cases}$$

Let $\rho :\subseteq \mathbb{F} \to S$ be a representation. For $1 \le k < \omega$ the representation $\rho^k :\subseteq \mathbb{F} \to S^k$ is defined by $\rho^k := (\rho, \ldots, \rho)$ (k times). The representation $\rho^\omega :\subseteq \mathbb{F} \to S^\omega$ is defined by

$$\rho^\omega(\langle p_0, p_1, p_2, \ldots\rangle) := \begin{cases} (\rho(p_0), \rho(p_1), \ldots) & \text{if } p_i \in \operatorname{dom}\rho, \text{ for all } i \\ \uparrow & \text{otherwise.} \end{cases}$$

In order to define a representation of a direct sum of sets S^{m_j} we use the pairing function $\langle,\rangle : \mathbb{N} \times \mathbb{F} \to \mathbb{F}$ with $\langle i, p\rangle := (i, p(0), p(1), p(2), \ldots)$ for $i \in \mathbb{N}$ and $p \in \mathbb{F}$. For an arbitrary set $A \subseteq \mathbb{N}$ and function $m : A \to \{1, 2, \ldots\} \cup \{\omega\}$ we define the representation $\bigoplus_{j\in A} \rho^{m_j} :\subseteq \mathbb{F} \to \bigoplus_{j\in A} S^{m_j}$ by

$$\bigoplus_{j\in A} \rho^{m_j}\langle i, p\rangle := \text{ the element } \rho^{m_i}(p) \text{ in the } i\text{-th component of } \bigoplus_{j\in A} S^{m_j}$$

for each $i \in A$ and $p \in \operatorname{dom}\rho^{m_i}$, and $\bigoplus_{j\in A} \rho^{m_j}\langle i, p\rangle :=\uparrow$ otherwise. As in the case of numberings, if $\rho_i \leq \rho'_i$ or $\rho \leq \rho'$, then also $(\rho_0, \ldots, \rho_k) \leq (\rho'_0, \ldots, \rho'_k)$, $\rho^k \leq (\rho')^k$ for $1 \leq k < \omega$, and $\rho^\omega \leq (\rho')^\omega$. The analogous statement is true also for $\bigoplus_{j\in A} \rho^{m_j}$ if A is an initial segment of $\mathbb{N}$ and the arity function m is a computable function.

We can also compare numberings and representations. If $\nu_i :\subseteq \mathbb{N} \to S_i$ and $\rho_i :\subseteq \mathbb{F} \to S_i$ are numberings and representations of countable sets S_i and $\nu_i \equiv \rho_i$ for all i, then $(\nu_0, \ldots \nu_k) \equiv (\rho_0, \ldots, \rho_k)$ for any $k \in \mathbb{N}$. Especially, $\nu \equiv \rho$ implies $\nu^k \equiv \rho^k$ for any finite $k \geq 1$ and $\bigoplus_{j\in A} \nu^{m_j} \equiv \bigoplus_{j\in A} \rho^{m_j}$ for any initial segment $A \subseteq \mathbb{N}$ and any computable function $m : A \to \{1, 2, \ldots\}$. But for $k = \omega$ this cannot be true since ν^ω denotes only ν-computable sequences (elements of $S^{\omega,\nu-\mathrm{comp}}$) while ρ^ω denotes arbitrary sequences in S^ω. But even the restricted representation $\rho^\omega|_{(\rho^\omega)^{-1}(S^{\omega,\nu-\mathrm{comp}})} :\subseteq \mathbb{F} \to S^{\omega,\nu-\mathrm{comp}}$ is not equivalent to ν^ω in general. (Take for example $S = \mathbb{N}$, $\nu = \mathrm{id}_{\mathbb{N}}$, and $\rho : \mathbb{N}\{0^\omega\} \subseteq \mathbb{F} \to \mathbb{N}$ with $\rho((n, 0, 0, 0, \ldots)) := n$. Obviously $\nu \equiv \rho$, $S^{\omega,\nu-\mathrm{comp}} = \mathrm{R}^{(1)}$, and $\nu^\omega \leq \rho^\omega|_{(\rho^\omega)^{-1}(\mathrm{R}^{(1)})}$, but for continuity reasons $\rho^\omega|_{(\rho^\omega)^{-1}(\mathrm{R}^{(1)})} \not\leq \nu^\omega$). This fact has consequences as we shall see.

We are ready to define effectivity and effective categoricity for structures.

Definition 2.13 A structure $\mathcal{S} = (S, c, f, P) = (S, c_0, c_1, \ldots, f_0, f_1, \ldots, P_0, P_1 \ldots)$ with arities m_j for f_j and n_k for P_k is called *n-effective* (respectively *r-effective*) iff there is a numbering $\gamma :\subseteq \mathbb{N} \to S$ (resp. a representation $\gamma :\subseteq \mathbb{F} \to S$) with the following properties:

1. The numbering c of the constants is reducible to γ.
2. The function $f :\subseteq \bigoplus_{j\in\operatorname{dom} m} S^{m_j} \to S$ is $(\bigoplus_{j\in\operatorname{dom} m} \gamma^{m_j}, \gamma)$-computable.
3. The set $P \subseteq \bigoplus_{k\in\operatorname{dom} n} S^{n_k}$ is $\bigoplus_{k\in\operatorname{dom} n} \gamma^{n_k}$-enumerable.

In that case we say that the naming system γ *makes $\mathcal{S}$ effective.* If $\mathcal{S}$ is n-effective (r-effective) and any two numberings (representations) which make $\mathcal{S}$ effective are equivalent, then we call $\mathcal{S}$ *n-effectively categorical* (*r-effectively categorical*). In that case we call any such numbering a *standard numbering* of $\mathcal{S}$ (such representation a *standard representation* of $\mathcal{S}$).

If the list $f_0, f_1, f_2, \ldots$ is finite then one can consider each function separately, that is, the second condition can be replaced by the condition that f_j is (γ^{m_j}, γ)-computable, for each j. The same applies to the list of predicates $P_0, P_1, P_2, \ldots$. The prefixes "n" and "r" indicate that the notions are defined with respect to numberings and with respect to representations. The following lemma follows immediately from Corollary 2.12 and from the remarks about the naming systems γ^k for $k \in \{1, 2, \ldots\} \cup \{\omega\}$ and about $\bigoplus_{j\in A} \gamma^{m_j}$.

Lemma 2.14 *Let $\mathcal{S}$ be a structure.*

1. *If a numbering ν makes $\mathcal{S}$ effective and μ is an equivalent numbering, then also μ makes $\mathcal{S}$ effective.*

2. *If a representation ρ makes $\mathcal{S}$ effective and σ is an equivalent representation, then also σ makes $\mathcal{S}$ effective.*

By Corollary 2.12 effectively categorical structures are of special interest because one has canonical notions for computable elements, computable functions and enumerable predicates on them.

In the terminology of Maltsev [24], an algebraic system has a "partial recursive"-numbering if and only it is n-effective according to the definition above. Related notions have been studied extensively, *computable structures* in computable model theory, see e.g. Khoussainov and Shore [18], or *effective numberings* and *computable algebras* in the theory of effective algebras, see Stoltenberg-Hansen and Tucker [31].

Examples 2.15 1. It is easy to check that the rational number structure introduced in Example 2.3.1 is n-effectively categorical and r-effectively categorical. A standard numbering of $\mathbb{Q}$ is for example the total mapping $\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}$ with $\nu_{\mathbb{Q}}\langle i,j,k\rangle := (i-j)/(k+1)$. The proof of the n-effective categoricity and of the r-effective categoricity is based on the same ideas as used in the proof of Theorem 4.1 below.

2. The structure of the real numbers in Example 2.3.2 is not n-effective since it is not countable. It is also not r-effective. To see this assume that there is a representation $\rho :\subseteq \mathbb{F} \to \mathbb{R}$ which makes the structure effective. Fix a Cauchy sequence $(x_n)_n$ of reals converging to 0 and let p_n be a ρ-name for x_n, for each n. Let $p := F_{CL}(\langle p_0, p_1, p_2, \ldots\rangle)$ be the name for 0 which is produced on input $\langle p_0, p_1, p_2, \ldots\rangle$ by a computable function F_{CL} which (ρ^ω, ρ)-tracks the limit operator CauchyLim. Furthermore let q be a ρ-name for 1. Since the relation "$<$" is ρ^2-enumerable there is a prefix w of $\langle p, q\rangle$ such that all $p' \in \operatorname{dom}\rho$ which satisfy $\langle p', q\rangle \in w\mathbb{F}$ also satisfy $\rho(p') < 1$. Hence there is a prefix v of p with $\rho(\operatorname{dom}\rho \cap v\mathbb{F}) \subseteq \{x \in \mathbb{R} \mid x < 1\}$. Since the function F_{CL} is continuous there is a prefix v' of $\langle p_0, p_1, \ldots\rangle$ with $F_{CL}(v'\mathbb{F} \cap \operatorname{dom}\rho^\omega) \subseteq v\mathbb{F}$. But by changing the names p_i with $i \geq |v'|$ we can change $\langle p_0, p_1, \ldots\rangle$ to a ρ^ω-name r lying in $v'\mathbb{F}$ of a Cauchy sequence converging to a number greater than 1, e.g. to 2. Then $s := F_{CL}(r)$ is a ρ-name for 2 and lies in $\operatorname{dom}\rho \cap v\mathbb{F}$. This is a contradiction. Hence the real number structure of Example 2 is not r-effective. In fact, the limit operator CauchyLim is too strong. One cannot expect to be able to compute the limit of an arbitrary Cauchy sequence without knowledge about its convergence rate. A natural r-effectively categorical real number structure will be considered in Section 4.

3 Basic constructions

We show that the classical result about the existence of a minimal (with respect to reduction) numbering of a finitely generated algebra can be transferred to our settings, i.e. to numberings and representations of structures with infinitely many possibly infinitary functions. Then we make basic observations about the substructure of computable elements of an r-effective, represented structure and start with a comparison of n-effectivity and r-effectivity for countable structures.

Let $\mathcal{S} = (S, c, f) = (S, c_0, c_1, \ldots, f_0, f_1, \ldots)$ be a structure without predicates (that means: the signature (N_c, m, n) of $\mathcal{S}$ satisfies $\operatorname{dom} n = \emptyset$) where m_j is the arity of f_j, for each j. We define the *closure* S_g *of* $\{c_0, c_1, c_2, \ldots\}$ *under* $\{f_0, f_1, f_2, \ldots\}$ to be the smallest subset R of S with

1. $\{c_0, c_1, c_2, \ldots\} \subseteq R$,
2. $(\forall j)\ f_j(\operatorname{dom} f_j \cap R^{m_j}) \subseteq R$.

One observes that $S_g = \bigcup_\alpha R_\alpha$ (union over ordinals α out of the second number class, compare e.g. Weihrauch [34]) where

$$\begin{aligned} R_0 &:= \{c_0, c_1, c_2, \ldots\}, \\ R_{<\alpha} &:= \bigcup_{\beta<\alpha} R_\beta, \\ R_\alpha &:= R_{<\alpha} \cup \{s \in S \mid (\exists j \in \operatorname{dom} m)\, (\exists \overline{s} \in (R_{<\alpha})^{m_j})\ s = f_j(\overline{s})\} \end{aligned}$$

for each ordinal α out of the second number class. It is obvious that

$$\mathcal{S}_g = (S_g, c_0, c_1, \ldots, f_0|_{S_g^{m_0}}, f_1|_{S_g^{m_1}}, \ldots)$$

is a structure itself. If all functions are finitary, then S_g is countable (then one has already $S_g = R_{<\omega}$). The following result was formulated by Maltsev [24] for finitely many functions f_j and by Weihrauch [34] for infinitely many f_j.

Proposition 3.1 *Let $\mathcal{S} = (S, c_0, c_1, \ldots, f_0, f_1, \ldots)$ be a structure without predicates and such that the functions f_j are all finitary. The set of equivalence classes of numberings of S_g which make the structure $(S_g, c_0, c_1, \ldots, f_0|_{S_g^{m_0}}, f_1|_{S_g^{m_1}}, \ldots)$ effective is non-empty and has a $\leq$-minimum.*

We shall show a slightly stronger result which covers also infinitary functions. If the structure $\mathcal{S}$ contains also infinitary functions f_j, then the closure S_g is not necessarily countable. But one can also obtain a numbering of a reasonable countable substructure in such a case if one restricts oneself to the elements which are constructed recursively in a purely computable way. The numbers for elements will encode computable finite path trees with finite or ω-branching, which show how the elements can be constructed out of the constants using the functions.

We define a subset $T \subseteq S$ and a numbering ν_g of T as follows. In parallel we define the domain D of ν_g. We follow Weihrauch [34, Ch. 2.8] closely.

$$\begin{aligned} D_0 &:= \{\langle 0, i\rangle \mid i \in \operatorname{dom} c\}, \\ D_{<\alpha} &:= \bigcup_{\beta<\alpha} D_\beta, \\ D_\alpha &:= D_{<\alpha} \\ &\quad \cup \{\langle 1+j, \langle l_1, \ldots, l_{m_j}\rangle\rangle \mid\ j \in \operatorname{dom} m,\ m_j < \omega,\ l_i \in D_{<\alpha} \text{ for } i = 1, \ldots, m_j, \\ &\qquad\qquad \text{and } (\nu_g(l_1), \ldots, \nu_g(l_{m_j})) \in \operatorname{dom} f_j\} \\ &\quad \cup \{\langle 1+j, l\rangle \mid\ j \in \operatorname{dom} m,\ m_j = \omega,\ \varphi_l \in \mathrm{R}^{(1)},\ (\forall i)\ \varphi_l(i) \in D_{<\alpha}, \\ &\qquad\qquad \text{and } (\nu_g \varphi_l(i))_i \in \operatorname{dom} f_j\}\}, \end{aligned}$$

$D := \bigcup_\alpha D_\alpha$,

$$\nu_g(n) := \begin{cases} c_i & \text{if } n = \langle 0, i\rangle \in D \\ f_j(\nu_g(l_1), \ldots, \nu_g(l_{m_j})) & \text{if } n = \langle 1+j, \langle l_1, \ldots, l_{m_j}\rangle\rangle \in D \text{ and } m_j \text{ is finite} \\ f_j((\nu_g \varphi_l(i))_i) & \text{if } n = \langle 1+j, l\rangle \in D \text{ and } m_j = \omega\,. \end{cases}$$

One proves by transfinite induction that ν_g is well-defined and $\operatorname{dom}\nu_g = D$. Set $T := \operatorname{range}\nu_g$. Then $T \subseteq S_g$. Define the function f' by

$$f'_j := \begin{cases} f_j|_{T^{m_j}} & \text{if } m_j < \omega \\ f_j|_{T^{\omega,\nu_g-\text{comp}}} & \text{if } m_j = \omega \end{cases}$$

for each $j \in \operatorname{dom} m$.

Theorem 3.2 *1. $\mathcal{T} = (S, c, f') = (T, c_0, c_1, \ldots, f'_0, f'_1, \ldots)$ is a countable structure.*

2. The numbering ν_g makes this structure effective.

3. The numbering ν_g is minimal among all numberings which make this structure effective.

In the special case that all functions f_j are finitary one has $T = S_g$. Hence, Theorem 3.2 implies Proposition 3.1. For example the rational number structure considered in Examples 2.3.1 and 2.15.1 without the equality relation has this form, i.e. its universe $\mathbb{Q}$ is the closure of its constants and functions, and all its functions are finitary. The set $\mathbb{R}_c$ of computable real numbers is also the closure of the constants and functions of a natural structure on $\mathbb{R}_c$, which we will consider in Section 5. It has only one infinitary function — a "recursive normed limit operator" —, and the domain of this operator is contained in $\mathbb{R}_c^{\omega,\nu_g-\text{comp}}$.

The proof of Theorem 3.2 goes along the same lines as the proof of Proposition 3.1. We give is nevertheless because the infinitary functions cause an additional difficulty.

Proof of Theorem 3.2. 1. It is obvious that T is countable and easy to check that $\mathcal{T} = (T, c, f') = (T, c_0, c_1, \ldots, f'_0, f'_1, \ldots)$ is a structure.

2. The total computable function $i \mapsto \langle 0, i\rangle$ proves $c \le \nu_g$. And the total computable function $\langle j, k\rangle \mapsto \langle 1+j, k\rangle$ shows that f' is $(\bigoplus_{j\in\operatorname{dom} m} \nu_g^{m_j}, \nu_g)$-computable.

3. Assume that $\mu :\subseteq \mathbb{N} \to T$ is a numbering which makes the structure $\mathcal{T}$ effective. We have to show that there is a computable function $q :\subseteq \mathbb{N} \to \mathbb{N}$ with $\nu_g(n) = \mu q(n)$ for all $n \in \operatorname{dom}\nu_g$.

Let $\tilde{c}$ be a computable function which proves $c \le \mu$ and let $\tilde{f}$ be a computable function which $(\bigoplus_{j\in\operatorname{dom} m} \mu^{m_j}, \mu)$-tracks f'. Let $p \in \mathrm{R}^{(1)}$ be a total computable function with $\varphi_{p\langle z,l\rangle}(x) = \varphi_z(\varphi_l(x))$ for all $z, l, x \in \mathbb{N}$. In order to obtain the desired function q we apply the following version of the recursion theorem:

$$\text{for any } H \in \mathrm{P}^{(1)} \text{ there is a } z \in \mathbb{N} \text{ with } \varphi_z(n) = H\langle z, n\rangle, \text{ for all } z, n,$$

to the function H defined by

$$H\langle z, n\rangle := \begin{cases} \tilde{c}(i) & \text{if } n = \langle 0, i\rangle \\ \tilde{f}\langle j, \langle \varphi_z(l_1), \ldots, \varphi_z(l_{m_j})\rangle\rangle & \text{if } n = \langle 1+j, \langle l_1, \ldots, l_{m_j}\rangle\rangle \text{ and } m_j < \omega \\ \tilde{f}\langle j, p\langle z, l\rangle\rangle & \text{if } n = \langle 1+j, l\rangle \text{ and } m_j = \omega \\ \uparrow & \text{otherwise}. \end{cases}$$

Let z be the obtained number. We claim that the function $q := \varphi_z$ proves $\nu_g \le \mu$. The proof is done by transfinite induction.

Let $n \in \operatorname{dom} \nu_g$. If $n = \langle 0, i \rangle$, then

$$\nu_g(n) = c_i = \mu\tilde{c}(i) = \mu H\langle z, n\rangle = \mu\varphi_z(n) = \mu q(n)\,.$$

If $n = \langle 1+j, \langle l_1, \ldots, l_{m_j}\rangle\rangle$ and $m_j < \omega$, then

$$\begin{array}{rcl}
\nu_g(n) & = & f'_j(\nu_g(l_1), \ldots, \nu_g(l_{m_j})) \\
& = & f'_j(\mu q(l_1), \ldots, \mu q(l_{m_j})) \\
& = & f'_j \mu^{m_j} \langle q(l_1), \ldots, q(l_{m_j})\rangle \\
& = & \mu \tilde{f}\langle j, \langle q(l_1), \ldots, q(l_{m_j})\rangle\rangle \\
& = & \mu H\langle z, n\rangle \\
& = & \mu q(n)
\end{array}$$

where we have used the induction hypothesis in the second step. Finally, in the case $n = \langle 1+j, l\rangle$ and $m_j = \omega$, one obtains analogously

$$\begin{array}{rcl}
\nu_g(n) & = & f'_j((\nu_g \varphi_l(i))_i) \\
& = & f'_j((\mu\varphi_z\varphi_l(i))_i) \\
& = & f'_j((\mu\varphi_{p\langle z,l\rangle}(i))_i) \\
& = & f'_j \mu^{\omega} p\langle z, l\rangle \\
& = & \mu\tilde{f}\langle j, p\langle z, l\rangle\rangle \\
& = & \mu H\langle z, n\rangle \\
& = & \mu q(n)\,.
\end{array}$$

This ends the proof. □

The last result can also be transferred to the case of representations instead of numberings. Then the infinitary functions do not cause any problems.

Theorem 3.3 *Let $\mathcal{S} = (S, c, f) = (S, c_0, c_1, \ldots, f_0, f_1, \ldots)$ be a structure without predicates. The set of equivalence classes of representations of the closure S_g (of the set of constants $\{c_i\}$ under the functions $\{f_j\}$) which make the structure*

$$\mathcal{S}_g = (S_g, c_0, c_1, \ldots, f_0|_{S_g^{m_0}}, f_1|_{S_g^{m_1}}, \ldots)$$

effective is non-empty and has a $\leq$-minimum.

We remark that for an r-effectively categorical structure $\mathcal{S}$ whose universe S is the closure of its constants and functions the unique equivalence class of representations which make the structure effective is identical with the minimum equivalence class of representations which make the structure without the predicates effective. This is for example the case for the real number structure which we will consider in the following section.

In order to prove Theorem 3.3 we need a standard numbering of $\{F \mid F :\subseteq \mathbb{F} \to \mathbb{F}$ is computable$\}$.

Lemma 3.4 *There is a total numbering $\psi : \mathbb{N} \to \{F \mid F :\subseteq \mathbb{F} \to \mathbb{F}$ is computable$\}$ with the following properties:*

1. *(utm Theorem) The universal function $u_\psi :\subseteq \mathbb{N} \times \mathbb{F} \to \mathbb{F}$ with $u_\psi(n,p) = \psi_n(p)$ for all $n \in \mathbb{N}$ and $p \in \mathbb{F}$ is computable.*

2. *(smn Theorem) For any computable function $H :\subseteq \mathbb{N} \times \mathbb{F} \to \mathbb{F}$ there is a total computable function $r \in \mathrm{R}^{(1)}$ with $\psi_{r(n)}(p) = H(n,p)$ for all $n \in \mathbb{N}$ and $p \in \mathbb{F}$.*

Proof. Each computable function $F :\subseteq \mathbb{F} \to \mathbb{F}$ is induced by a partial computable, monotone function $g :\subseteq \mathbb{N}^* \to \mathbb{N}^*$. The graph of g is $\nu^2_{\mathbb{N}^*}$-enumerable. We shall obtain a total numbering of $\{F \mid F :\subseteq \mathbb{F} \to \mathbb{F}$ is computable$\}$ by defining a total numbering of the graphs of all monotone computable word functions.

Let $\varphi : \mathbb{N} \to \mathrm{P}^{(1)}$ be a total standard numbering. Fix a function $h_1 \in \mathrm{R}^{(1)}$ such that $\varphi_{h_1(n)} \in \mathrm{R}^{(1)}$ and $\mathrm{dom}\,\varphi_n = \{k \in \mathbb{N} \mid 1 + k \in \mathrm{range}\,\varphi_{h_1(n)}\}$, for all n. There is a function $h_2 \in \mathrm{R}^{(1)}$ with

$$\varphi_{h_2(n)}(i) = \begin{cases} \varphi_{h_1(n)}(i) & \text{if the set } \nu^2_{\mathbb{N}^*}\{k \mid 1+k \in \varphi_{h_1(n)}\{0,1,\ldots,i\}\} \\ & \quad \text{is the graph of a monotone function} \\ 0 & \text{otherwise} \end{cases}$$

Then, for each $n \in \mathbb{N}$, the set $\nu^2_{\mathbb{N}^*}\{k \mid 1 + k \in \mathrm{range}\,\varphi_{h_2(n)}\}$ is the graph of a monotone computable function mapping words in $\mathbb{N}^*$ to words in $\mathbb{N}^*$. Let $\psi(n)$ be the function $F :\subseteq \mathbb{F} \to \mathbb{F}$ induced by the monotone function with this graph. It is clear that ψ is a total numbering of $\{F \mid F :\subseteq \mathbb{F} \to \mathbb{F}$ is computable$\}$ and satisfies the first condition (the utm Theorem). We show that is also satisfies the second condition. Let $H :\subseteq \mathbb{N} \times \mathbb{F} \to \mathbb{F}$ be a computable function. Let $\tilde{H} :\subseteq \mathbb{F} \to \mathbb{F}$ be the computable function defined by $\tilde{H}(p) := H(p(0), (p(1), p(2), \ldots))$. Furthermore, let $g :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ be a monotone computable function which induces $\tilde{H}$, and let $\tilde{g} \in \mathrm{R}^{(1)}$ be a total computable function which enumerates the graph of g in the fashion as above, i.e. graph $g = \nu^2_{\mathbb{N}^*}\{k \in \mathbb{N} \mid 1 + k \in \mathrm{range}\,\tilde{g}\}$. Let $r_1 \in \mathrm{R}^{(1)}$ be a function such that $\varphi_{r_1(n)}(i) = 0$ if $\tilde{g}(i) = 0$ or if the first word in the pair of words $\nu^2_{\mathbb{N}^*}(\tilde{g}(i) - 1)$ does not start with n, and such that $\nu^2_{\mathbb{N}^*}(\varphi_{r_1(n)}(i) - 1) = (v, w)$ if $\nu^2_{\mathbb{N}^*}(\tilde{g}(i) - 1) = (nv, w)$ for some $v, w \in \mathbb{N}^*$. Finally fix a function $r_2 \in \mathrm{R}^{(1)}$ with $\mathrm{dom}\,\varphi_{r_2(n)} = \{k \in \mathbb{N} \mid 1 + k \in \mathrm{range}\,\varphi_{r_1(n)}\}$. This function r_2 satisfies the condition $\psi_{r_2(n)} = H(n,p)$ for all $n \in \mathbb{N}$ and $p \in \mathbb{F}$. □

We shall use the following recursion theorem for ψ.

Lemma 3.5 *Let ψ be a total numbering as in the last lemma. Then for any computable function $H :\subseteq \mathbb{N} \times \mathbb{F} \to \mathbb{F}$ there is a number $z \in \mathbb{N}$ with $\psi_z(p) = H(z,p)$ for all $p \in \mathbb{F}$.*

Proof. Let $s \in \mathrm{P}^{(1)}$. By Condition (1) in Lemma 3.4 the function $(n,p) \mapsto \psi_{s(n)}(p)$ is computable. By Condition (2) in Lemma 3.4 there is an $r \in \mathrm{R}^{(1)}$ with $\psi_{r(n)} = \psi_{s(n)}$ for all n. This means that the numbering ψ is precomplete, compare Weihrauch [34]. Let $H :\subseteq \mathbb{N} \times \mathbb{F} \to \mathbb{F}$ be a computable function. Condition (2) in Lemma 3.4 gives a function $r \in \mathrm{R}^{(1)}$ with $\psi_{r(n)}(p) = H(n,p)$ for all $n \in \mathbb{N}$ and $p \in \mathbb{F}$. An application of the recursion theorem, e.g. Corollary 2.5.7(2) in [34], yields the desired number z. □

Proof of Theorem 3.3. The proof follows closely the proof of Theorem 3.2. Compare also Weihrauch [34, Ch. 2.7, 2.8, 3.4].

We define subsets D_α (α ordinals out of the second number class) of $\mathbb{F}$ and a representation $\rho_g : \bigcup_\alpha D_\alpha \to S_g$ as follows.

$$\begin{aligned}
D_0 &:= \{(0,i,0,0,\ldots) \mid i \in \operatorname{dom} c\}\,,\\
D_{<\alpha} &:= \bigcup_{\beta<\alpha} D_\beta\,,\\
D_\alpha &:= D_{<\alpha}\\
&\quad \cup\{\langle 1+j, \langle q_1,\ldots,q_{m_j}\rangle\rangle \mid \begin{array}[t]{l} j \in \operatorname{dom} m,\ m_j < \omega,\ q_i \in D_{<\alpha} \text{ for } i = 1,\ldots,m_j,\\ \text{and } (\rho_g(q_1),\ldots,\rho_g(q_{m_j})) \in \operatorname{dom} f_j\}\end{array}\\
&\quad \cup\{\langle 1+j, \langle q_0,q_1,\ldots\rangle\rangle \mid \begin{array}[t]{l} j \in \operatorname{dom} m,\ m_j = \omega,\ q_i \in D_{<\alpha} \text{ for } i \in \mathbb{N},\\ \text{and } (\rho_g(q_i))_i \in \operatorname{dom} f_j\}\,,\end{array}
\end{aligned}$$

$D := \bigcup_\alpha D_\alpha$,

$$\rho_g(p) := \begin{cases} c_i & \text{if } p = (0,i,0,0,\ldots) \in D\\ f_j(\rho_g(q_1),\ldots,\rho_g(q_{m_j})) & \text{if } p = \langle 1+j, \langle q_1,\ldots,q_{m_j}\rangle\rangle \in D \text{ and } m_j \text{ is finite}\\ f_j((\rho_g(q_i))_i) & \text{if } p = \langle 1+j, \langle q_0,q_1,\ldots\rangle\rangle \in D \text{ and } m_j = \omega\,. \end{cases}$$

As in the case of the numbering ν_g one shows by transfinite induction that the map ρ_g is well-defined with $\operatorname{dom}\rho_g = D$. We claim that ρ_g is a representation of $\mathcal{S}_g$ which makes this structure effective and that ρ_g is minimal among all representations which make $\mathcal{S}_g$ effective. The ρ_g-names for elements encode finite path trees with finite or ω-branching, which show how the elements can be constructed out of the constants c_i using the functions f_j.

By induction one proves $\rho_g(D_\alpha) = R_\alpha$ for all α, hence $\operatorname{range}\rho_g = S_g$. The total computable function from $\mathbb{N}$ to $\mathbb{F}$ with $i \mapsto (0,i,0,0,\ldots)$ proves $c \le \rho_g$. The total computable function $\langle j,p\rangle \mapsto \langle 1+j,p\rangle$ (for $j \in \mathbb{N}$ and $p \in \mathbb{F}$) mapping $\mathbb{F}$ to $\mathbb{F}$ shows that $f|_{\bigoplus_{j\in\operatorname{dom} m} S_g^{m_j}}$ is $(\bigoplus_{j\in\operatorname{dom} m} \rho_g^{m_j}, \rho_g)$-computable. Hence, the representation ρ_g makes the structure $\mathcal{S}_g$ effective.

Let $\sigma :\subseteq \mathbb{F} \to S_g$ be an arbitrary representation which makes S_g effective. We have to show $\rho_g \le \sigma$. Assume that a computable function $\tilde{c} :\subseteq \mathbb{N} \to \mathbb{F}$ proves $c \le \sigma$ and that a computable function $\tilde{f} :\subseteq \mathbb{F} \to \mathbb{F}$ shows that the function $f|_{\bigoplus_{j\in\operatorname{dom} m} S_g^{m_j}}$ is $(\bigoplus_{j\in\operatorname{dom} m} \sigma^{m_j}, \sigma)$-computable. Let ψ be a numbering as in Lemma 3.4. We apply Lemma 3.5 to the following computable function $H :\subseteq \mathbb{N}\times\mathbb{F} \to \mathbb{F}$:

$$f(p) = \begin{cases} \tilde{c}(p(1)) & \text{if } p(0) = 0\\ \tilde{f}\langle j, \langle \psi_z(q_1),\ldots,\psi_z(q_{m_j})\rangle\rangle & \text{if } p(0) = 1+j,\ m_j < \omega,\ p = \langle 1+j, \langle q_1,\ldots,q_{m_j}\rangle\rangle\\ & \quad \text{for some } j \in \mathbb{N} \text{ and } q_i \in \mathbb{F} \text{ for } i = 1,\ldots,m_j\\ \tilde{f}\langle j, \langle \psi_z(q_0),\psi_z(q_1),\ldots\rangle\rangle & \text{if } p(0) = 1+j,\ m_j = \omega,\ p = \langle 1+j, \langle q_0,q_1,\ldots\rangle\rangle\\ & \quad \text{for some } j \in \mathbb{N} \text{ and } q_i \in \mathbb{F} \text{ for } i \in \mathbb{N}\\ \uparrow & \text{otherwise}\,. \end{cases}$$

Let z be the resulting number. It is now straightforward to check by transfinite induction over α (as in the proof of Theorem 3.2) that $\rho_g(p) = \sigma\psi_z(p)$ for each $p \in D_\alpha$. Hence, $\rho_g \le \sigma$ is proved. $\square$

The question arises what the relation between n-effectivity and r-effectivity is. Since r-effectivity also applies to uncountable structures and n-effectivity only to countable structures they cannot be compared completely. We can compare them only for countable structures. First we consider the countable substructure of the computable elements of an r-effective, represented structure.

Definition 3.6 Let $\rho :\subseteq \mathbb{F} \to S$ be a representation of a set S. By $S^{\rho-\mathrm{comp}}$ we denote the set of all ρ-computable elements in S. The *derived numbering* $\nu_\rho :\subseteq \mathbb{N} \to S^{\rho-\mathrm{comp}}$ is defined by

$$\nu_\rho(n) := \begin{cases} \rho\varphi_n & \text{if } \varphi_n \in \mathrm{R}^{(1)} \text{ and } \varphi_n(i) \in \operatorname{dom}\rho, \text{ for all } i \\ \uparrow & \text{otherwise} \end{cases}$$

where $\varphi : \mathbb{N} \to \mathrm{P}^{(1)}$ is a total standard numbering of the partial recursive functions.

The following lemma is a collection of elementary facts about the derived numbering.

Lemma 3.7 *Let $\sigma :\subseteq \mathbb{F} \to S$ and $\tau :\subseteq \mathbb{F} \to T$ be representations.*

1. $\nu_\sigma \leq \sigma$.
2. *If μ is a numbering of a subset of S and $\mu \leq \sigma$, then also $\mu \leq \nu_\sigma$.*
3. *If $f :\subseteq S \to T$ is (σ, τ)-computable, then $f(S^{\sigma-\mathrm{comp}}) \subseteq T^{\tau-\mathrm{comp}}$ and the restriction $f|_{S^{\sigma-\mathrm{comp}}}$ is (ν_σ, ν_τ)-computable.*
4. *If $\sigma \leq \tau$, then also $\nu_\sigma \leq \nu_\tau$.*
5. *If $P \subseteq S$ is σ-enumerable, then the intersection $P \cap S^{\sigma-\mathrm{comp}}$ is ν_σ-enumerable.*
6. *For any $k \in \{1, 2, \ldots\} \cup \{\omega\}$ one has $\nu_{\sigma^k} \equiv \nu_\sigma^k$. If $m :\subseteq \mathbb{N} \to \mathbb{N}$ is a computable function whose domain is an initial segment of $\mathbb{N}$, then also*

$$\nu_{\bigoplus_{j \in \operatorname{dom} m} \sigma^{m_j}} \equiv \bigoplus_{j \in \operatorname{dom} m} \nu_\sigma^{m_j}.$$

Proof. 1. The function $\varphi|_{\varphi^{-1}(\mathrm{R}^{(1)})} :\subseteq \mathbb{N} \to \mathbb{F}$ which maps each $i \in \varphi^{-1}(\mathrm{R}^{(1)})$ to φ_i is computable and proves $\nu_\sigma \leq \sigma$.

2. Let $g :\subseteq \mathbb{N} \to \mathbb{F}$ be a computable function which proves $\mu \leq \sigma$. There is a function $r \in \mathrm{R}^{(1)}$ with $\varphi_{r(n)}(x) = g(n)(x)$ for all $n \in \operatorname{dom} g$ and $x \in \mathbb{N}$. This function r proves $\mu \leq \nu_\sigma$.

3. Assume that $g :\subseteq \mathbb{F} \to \mathbb{F}$ is a computable function which (σ, τ)-tracks f. If $p \in \operatorname{dom}\sigma$ is computable, then also $p \in \operatorname{dom} g$ and $g(p)$ is computable. This shows $f(S^{\sigma-\mathrm{comp}}) \subseteq T^{\tau-\mathrm{comp}}$. There is a function $r \in \mathrm{R}^{(1)}$ with $\varphi_{r(n)}(x) = g(\varphi_n)(x)$ for all $n \in \operatorname{dom} g\varphi$ and $x \in \mathbb{N}$. This function r (ν_σ, ν_τ)-tracks f.

4 This follows directly from (1) and (2) or from (3).

5. Let $A \subseteq \mathbb{N}^*$ be a c.e. set with $A\mathbb{F} \cap \operatorname{dom}\sigma = \sigma^{-1}(P)$. The set

$$B := \{i \in \mathbb{N} \mid (\exists w \in A)\, (\forall j < |w|)\ \varphi_i(j) = w(j)\}$$

is c.e. and satisfies $B \cap \operatorname{dom}\nu_\sigma = \nu_\sigma^{-1}(P)$. This proves the assertion.

6. All of these equivalences are proved by functions not depending on σ. We prove only $\nu_{\sigma^\omega} \equiv \nu_\sigma^\omega$, leaving the other equivalences to the reader. Applying the smn Theorem (for φ) twice yields a function $r \in \mathrm{R}^{(1)}$ with $\varphi_i\langle j, k\rangle = \varphi_{\varphi_{r(i)}(j)}(k)$ for all $i, j, k \in \mathbb{N}$. This function r proves $\nu_{\sigma^\omega} \leq \nu_\sigma^\omega$. One application of the smn Theorem yields a function $s \in \mathrm{R}^{(1)}$ with $\varphi_{s(i)}\langle j, k\rangle = \varphi_{\varphi_i(j)}(k)$ for all $i, j, k \in \mathbb{N}$. This function s proves $\nu_\sigma^\omega \leq \nu_{\sigma^\omega}$. $\square$

Definition 3.8 Let $\mathcal{S} = (S, c, f, P)$ be a structure of signature (N_c, m, n) and $\rho :\subseteq \mathbb{F} \to S$ be a representation which makes $\mathcal{S}$ effective. The *substructure of ρ-computable elements* is the structure $\mathcal{S}^{\rho-\text{comp}} = (S^{\rho-\text{comp}}, c, f', P')$ where f' is the restriction of f to the $(\bigoplus_{j\in\text{dom}\, m} \rho^{m_j})$-computable elements, and P' is the intersection of P with the set of $(\bigoplus_{j\in\text{dom}\, n} \rho^{n_k})$-computable elements.

Part 3 of the last lemma tells us that f' maps $(\bigoplus_{j\in\text{dom}\, m} \rho^{m_j})$-computable elements to ρ-computable elements, and hence, that by $\mathcal{S}^{\rho-\text{comp}}$ we have indeed defined a structure. From the lemma one also deduces the following theorem.

Theorem 3.9 *If a structure $\mathcal{S} = (S, c, f, P)$ is r-effective and a representation $\rho :\subseteq \mathbb{F} \to S$ makes it effective, then the structure $\mathcal{S}^{\rho-\text{comp}}$ is n-effective and the numbering ν_ρ makes it effective.*

It is clear that for two equivalent representations ρ and σ which make a structure S effective the induced substructures $S^{\rho-\text{comp}}$ and $S^{\sigma-\text{comp}}$ are identical. This is especially interesting if the structure S is r-effectively categorical since then it has a canonical substructure of computable elements, which we denote by S_c, namely the substructure $S^{\rho-\text{comp}}$ for any representation ρ which makes S effective. In the following section we will consider the uncountable structure of the real numbers and in the subsequent section its substructure of the computable real numbers.

Finally we compare n-effectivity and r-effectivity for arbitrary countable structures. For a special (classical) case we observe that n-effectivity and r-effectivity are equivalent.

Theorem 3.10 *Let $\mathcal{S} = (S, c_0, c_1, \ldots, f_0, f_1, \ldots, P_0, P_1, \ldots)$ be a structure whose functions f_j and predicates P_k are all finitary and such that the set S is the closure of the constants $\{c_0, c_1, \ldots\}$ under the functions $\{f_0, f_1, \ldots\}$.*

1. *Let $\nu_g :\subseteq \mathbb{N} \to S$ be a minimal numbering as in Theorem 3.2 (or Proposition 3.1), and let $\rho_g :\subseteq \mathbb{F} \to S$ be a minimal representation as in Theorem 3.3. Then $\nu_g \equiv \rho_g$.*
2. *The structure $\mathcal{S}$ is n-effective if and only if it is r-effective.*

Proof. 1. All elements in S are ρ_g-computable. Hence, S is identical with its substructure of ρ_g-computable elements. Both numberings ν_g and ν_{ρ_g} make the structure $(S, c_0, c_1, \ldots, f_0, f_1, \ldots)$ effective. By minimality of ν_g we have $\nu_g \le \nu_{\rho_g}$ and by Lemma 3.7.1 $\nu_{\rho_g} \le \rho_g$. On the other hand, by $\sigma(p) := \nu_g(p(0))$ one defines a representation with $\sigma \equiv \nu_g$. As both σ and ρ_g make the structure $(S, c_0, c_1, \ldots, f_0, f_1, \ldots)$ effective, minimality of ρ_g gives $\rho_g \le \sigma \equiv \nu_g$.

2. If $\mathcal{S}$ is n-effective and μ is a numbering which makes $\mathcal{S}$ effective, then $\nu_g \le \mu$ and $\bigoplus_k \mu^{n_k} \le \bigoplus_k \mu^{n_k}$. Hence, by Proposition 2.11.3, ν_g makes $\mathcal{S}$ effective. The first part of the current theorem shows that $\nu_g \equiv \rho_g$ and, hence, $\bigoplus_j \nu^{m_j} \equiv \bigoplus_j \rho_g^{m_j}$ and $\bigoplus_k \nu^{n_k} \equiv \bigoplus_k \rho_g^{n_k}$. Proposition 2.11 shows that also ρ_g makes $\mathcal{S}$ effective. Hence $\mathcal{S}$ is r-effective. On the other hand, if $\mathcal{S}$ is r-effective, then by the same arguments ρ_g makes $\mathcal{S}$ effective, and finally also ν_g makes $\mathcal{S}$ effective. □

But in general, especially when we allow also infinitary operations, this is not true. The following result will be proved in Section 5.

Theorem 3.11 *There is an n-effectively categorical structure which is not r-effective.*

Many questions on the relation between n-effectivity (n-effective categoricity) and r-effectivity (r-effective categoricity) for countable structures remain open.

4 The Real Number Structure

We are concerned with computability over the real numbers. For a short discussion and a short list of references on this topic the reader is referred to the introduction. Since computability on an uncountable structure cannot be introduced via a numbering we shall use representations. For an overview over many different representations of the real numbers and a comparison of them via reducibility see Deil [7]. First we prove the main result of the section which states that the real number structure is r-effectively categorical. Then we shortly discuss computability of the order relation and give some examples of standard representations. Finally we discuss the b-ary ($b \geq 2$ a natural number) representations. They are well-known since ancient times, though for computational purposes mainly their counterparts for integers, i.e. notations of integers to base b, are used. It is well-known that the b-ary representations are not suitable for real number computations on digital computers or on Turing machines, compare Weihrauch and Kreitz [38]. We shall give further evidence for this by proving that strictly less functions are computable with respect to any b-ary representation than with respect to a standard representation of the real numbers. In the next section we shall consider the countable substructure of the computable real numbers.

The real number structure we consider will reflect the fact that the real numbers are categorical: the set $\mathbb{R}$ of real numbers is a complete archimedean ordered field and uniquely determined by these conditions. We shall use the field structure (constants 0 and 1 and the four basic operations), the ordering, and a condition which expresses effective completeness. Remember that the operator CauchyLim considered in Example 2.3.2 is too strong: one cannot compute the limit of an arbitrary Cauchy sequence without knowledge about its convergence rate, see Example 2.15.2. But if the convergence rate can be bounded computably the limit can be computed. For (countable) recursive metric spaces recursive completeness is defined in the same way, compare Moschovakis [26, 27] and Kushner [22]. In fact, it is sufficient to consider Cauchy sequences with a normed convergence rate. We consider the following *real number structure*

$$(\mathbb{R}, 0, 1, +, -, *, 1/., \mathrm{NormLim}\,, <)$$

consisting of the set $\mathbb{R}$ of real numbers, of the constants 0 and 1, of the arithmetic field operations addition "$+$": $\mathbb{R}^2 \to \mathbb{R}$, additive inverse "$-$": $\mathbb{R} \to \mathbb{R}$, multiplication "$*$": $\mathbb{R}^2 \to \mathbb{R}$, multiplicative inverse "$1/.$": $\mathbb{R} \setminus \{0\} \to \mathbb{R}$, the order relation "$<$"$= \{(x,y) \in \mathbb{R}^2 \mid x < y\}$, and the normed limit operator NormLim $:\subseteq \mathbb{R}^\omega \to \mathbb{R}$ which maps each normed Cauchy sequence of reals to its limit, that is

$$\mathrm{dom}\,(\mathrm{NormLim}\,) := \mathbb{R}^{\omega,\mathrm{normed}} := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\omega \mid (\forall m, n)\ |x_n - x_m| \leq 2^{-\min\{m,n\}}\},$$

$$\mathrm{NormLim}\,((x_n)_n) = \lim_{n\to\infty} x_n \qquad \text{for all } (x_n)_n \in \mathrm{dom}\,(\mathrm{NormLim}\,).$$

Theorem 4.1 *The structure* $(\mathbb{R}, 0, 1, +, -, *, 1/., \mathrm{NormLim}\,, <)$ *is r-effectively categorical.*

Remark 4.2 One could replace the operator NormLim also by other operators expressing effective completeness, for example by the operator which for a monotone, converging sequence of intervals computes its limit, expressed by $(x_0, x_1, x_2, x_3, \ldots) \mapsto \lim_n x_n$, if

$x_0 \le x_2 \le \ldots \le x_3 \le x_1$ and $\bigcap_n [x_{2n}, x_{2n+1}]$ contains only one point. On the other hand, for example the operator which maps each bounded sequence of reals to its supremum is not effective with respect to a standard representation, again because the convergence rate is unknown in general. Demanding effectiveness of this operator leads to a one-sided representation shortly considered in Section 6.

First we shall show that this structure is r-effective. Therefore we need a representation of the real numbers. More examples of standard representations will be given after the proof. We use the total numbering $\nu_{\mathbb{D}} : \mathbb{N} \to \mathbb{D}$ of the dyadic rational numbers:

$$\mathbb{D} := \{x \in \mathbb{R} \mid (\exists i, j, k \in \mathbb{N})\ x = \frac{i-j}{2^k}\}$$

defined by $\nu_{\mathbb{D}}\langle i, j, k\rangle := (i-j)/2^k$. The representation

$$\rho_C :\subseteq \mathbb{F} \to \mathbb{R}, \qquad \mathrm{dom}\,\rho_C := \{p \in \mathbb{F} \mid (\forall m, n)\ |\nu_{\mathbb{D}} p(m) - \nu_{\mathbb{D}} p(n)| \le 2^{-\min\{m,n\}}\},$$

$$\rho_C(p) := \lim_{n\to\infty} \nu_{\mathbb{D}} p(n) \qquad \text{for all } p \in \mathrm{dom}\,\rho_C$$

is called *normed Cauchy representation.*

Proof of Theorem 4.1. First we show that the representation ρ_C makes $\mathbb{R}$ effective. The constants 0 and 1 have computable ρ_C-names, e.g. $\rho_C(p) = 0$ if $p(i) = \langle 0, 0, 0\rangle$ for all i, and $\rho_C(q) = 1$ if $q(i) = \langle 1, 0, 0\rangle$ for all i. The addition "+" is (ρ_C^2, ρ_C)-computable: define the total recursive function $g : \mathbb{N}^2 \to \mathbb{N}$ by

$$g(\langle i, j, k\rangle, \langle i', j', k'\rangle) := \langle i \cdot 2^{k'} + i' \cdot 2^k, j \cdot 2^{k'} + j' \cdot 2^k, k + k'\rangle\,.$$

Then the function $h : \mathbb{F} \to \mathbb{F}$ with

$$h(\langle p, q\rangle)(i) := g(p(i+1), q(i+1))$$

is computable and satisfies $\rho_C(p) + \rho_C(q) = \rho_C(h(\langle p, q\rangle))$ for all $\langle p, q\rangle \in \mathrm{dom}\,\rho_C^2$. We leave it to the reader to check in a similar way that the multiplication "$*$" is (ρ_C^2, ρ_C)-computable, and the additive inverse "$-$" and the multiplicative inverse "$1/.$" are (ρ_C, ρ_C)-computable. The computable function $F : \mathbb{F} \to \mathbb{F}$ defined by

$$F(\langle p_0, p_1, p_2, \ldots\rangle)(i) := p_{i+2}(i+2) \qquad \text{where } p_i \in \mathbb{F} \text{ for all } i$$

satisfies $\mathrm{NormLim}\,\rho_C^\omega(p) = \rho_C F(p)$ for all $p \in \mathrm{dom}\,(\mathrm{NormLim} \circ \rho_C^\omega)$, as can easily be checked by triangle inequalities. Hence NormLim is (ρ_C^ω, ρ_C)-computable. That the order relation "$<$" is ρ_C^2-enumerable, is proved by the c.e. set

$$A := \{w \in \mathbb{N}^* \mid (\exists i < |w|/2)\ \nu_{\mathbb{D}}(w(2i)) + 2^{1-i} < \nu_{\mathbb{D}}(w(2i+1))\}\,,$$

which has the property $A\mathbb{F} \cap \mathrm{dom}\,\rho_C^2 = (\rho_C^2)^{-1}(\text{"}<\text{"})$. We have proved that ρ_C makes $\mathbb{R}$ effective.

In order to prove the r-effective categoricity it is sufficient to consider representations of restricted structures. Assume that $\sigma :\subseteq \mathbb{F} \to \mathbb{R}$ and $\tau :\subseteq \mathbb{F} \to \mathbb{R}$ are representations such that σ makes the structure $(\mathbb{R}, 0, 1, +, -, *, 1/., <)$ effective, and such that τ makes the structure $(\mathbb{R}, 0, 1, +, -, *, 1/., \mathrm{NormLim}\,)$ effective. We shall show that this implies

$\sigma \leq \tau$. This proves that any two representations which make the full real number structure effective are equivalent.

The reduction algorithm works as follows: One analyzes a σ-name of a real number by constructing σ-names of rational numbers and comparing them with the real number using the σ-algorithm for the order relation. Thus one can obtain a normed Cauchy-sequence of rational numbers approximating the real number. Simultaneously one constructs τ-names for these rational numbers and, using these, a τ^ω-name for the whole sequence. Finally one uses the τ-algorithm for NormLim in order to synthesize a τ-name of the limit of this sequence, which is just the real number in question.

Let us do this formally. Let $p_{\sigma,0}$ and $p_{\sigma,1}$ be computable σ-names for 0 and 1, respectively, and $f_{\sigma,+}$, $f_{\sigma,-}$, $f_{\sigma,*}$, $f_{\sigma,1/.}$ be computable functions that (σ^2, σ)-track (respectively (σ,σ)-track) the operations "+", "−", "*", "1/.". We need a total computable function $d_\sigma : \mathbb{N} \to \mathbb{F}$ which $(\nu_\mathbb{D}, \sigma)$-tracks the embedding of the dyadic rationals $\mathbb{D}$ into the reals, that is, it must satisfy $\sigma(d_\sigma\langle i,j,k\rangle) = (i-j)/2^k$, for all $\langle i,j,k\rangle$. Therefore we define in a recursive manner computable functions $h_1, h_2 : \mathbb{N} \to \mathbb{F}$ by

$$\begin{aligned} h_1(0) &:= p_{\sigma,0}, \qquad & h_1(i+1) &:= f_{\sigma,+}(p_{\sigma,1}, h_1(i))\,, \\ h_2(0) &:= p_{\sigma,1}, \qquad & h_2(i+1) &:= f_{\sigma,*}(f_{\sigma,+}(p_{\sigma,1}, p_{\sigma,1}), h_2(i))\,, \end{aligned}$$

and set

$$d_\sigma\langle i,j,k\rangle := f_{\sigma,*}(f_{\sigma,+}(h_1(i), f_{\sigma,-}(h_1(j))), f_{\sigma,1/.}(h_2(k)))\,.$$

In the same way, using computable τ-names for 0 and 1, and computable functions which track the arithmetic operations "+", "−", "*", "1/." with respect to τ, one can define a computable function $d_\tau : \mathbb{N} \to \mathbb{F}$ with $\tau(d_\tau\langle i,j,k\rangle) = (i-j)/2^k$, for all $\langle i,j,k\rangle$. Since the order "<" is σ^2-enumerable there is a c.e. set $A \subseteq \mathbb{N}^*$ such that for all $\langle p,q\rangle \in \operatorname{dom}\sigma^2$:

$$\sigma(p) < \sigma(q) \iff (\exists w \in A)\ w \text{ is prefix of } \langle p,q\rangle\,.$$

Let $h_3 : \mathbb{N} \to \mathbb{N}^*$ be a total computable function which enumerates A, i.e. $\operatorname{range}(h_3) = A$. We define a partial computable function $g :\subseteq \mathbb{N} \times \mathbb{F} \to \mathbb{N}$ by

$$g(n,p) := \pi_1^3 \min\{\langle i,k_1,k_2\rangle \mid \begin{array}[t]{l} h_3(k_1) \text{ is a prefix of } \langle d_\sigma(i), p\rangle \\ \text{and } h_3(k_2) \text{ is a prefix of } \langle p, f_{\sigma,+}(d_\sigma(i), d_\sigma\langle 1,0,n\rangle)\rangle\} \end{array}$$

where $g(n,p)$ is undefined if the minimum does not exist, i.e. if the set if empty. For $p \in \operatorname{dom}\sigma$ and any $n \in \mathbb{N}$ the value $g(n,p)$ is defined and is equal to a number i with

$$\sigma(d_\sigma(i)) < \sigma(p) < \sigma(d_\sigma(i)) + \frac{1}{2^n}\,.$$

Hence, as we have $\sigma d_\sigma(i) = \tau d_\tau(i)$ for all i, for $p \in \operatorname{dom}\sigma$ the sequence

$$\langle d_\tau g(0,p), d_\tau g(1,p), d_\tau g(2,p), \ldots\rangle \in \mathbb{F}$$

is a τ^ω-name for a normed Cauchy sequence converging to $\sigma(p)$. Finally let $f_{\tau,\mathrm{NormLim}}$ be a computable function which (τ^ω, τ)-tracks the limit operator NormLim. The function $F :\subseteq \mathbb{F} \to \mathbb{F}$, defined by

$$F(p) := f_{\tau,\mathrm{NormLim}}(\langle d_\tau g(0,p), d_\tau g(1,p), d_\tau g(2,p), \ldots\rangle)$$

is computable and satisfies $\sigma(p) = \tau F(p)$ for all $p \in \operatorname{dom}\sigma$. Hence, $\sigma \leq \tau$ is proven. $\square$

The last theorem allows us to speak of the *standard representations* of the real numbers, namely the elements of the unique equivalence class of representations which make the above real number structure effective. Weihrauch and Kreitz [38] considered the same class of representations, compared it with other representations and argued that among these it is for topological reasons the only reasonable one. Yet, their argument still relies on a reasonable choice of a numbering of a base of the topology. Theorem 4.1 justifies the notion of a standard representation of the real numbers directly by the structure of the real numbers.

Let ρ be a standard representation of $\mathbb{R}$. We have seen that ρ-computability of real numbers, ρ^ω-computability of sequences of real numbers, (ρ^n, ρ)-computability of functions $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ for $n \in \{1, 2, \ldots\} \cup \{\omega\}$ and ρ^n-enumerability of sets $U \subseteq \mathbb{R}^n$ for $n \in \{1, 2, \ldots\} \cup \{\omega\}$ does not depend on the choice of ρ. Since it seems to be very reasonable to demand that the operations formulated in the real number structure in the theorem are effective, this means that we have a canonical computability notion on the real numbers. It makes sense to call a real object (number, sequence of numbers, function or set) computable (or enumerable) if and only if it is computable (or enumerable) with respect to a standard representation. This is in fact the computability notion considered by the authors cited in the introduction with the exception that some of them impose further restrictions on the domain of definition of the considered functions. It is well-known and easy to check, that all the common functions like rational functions with computable coefficients, exp, log, sin, cos, and so on are computable. It is of fundamental importance that all real number functions which are computable in this sense are continuous. This follows easily by considering the representation ρ_C, compare also Weihrauch [34]. In fact, this computability for functions can be considered as a form of effective continuity, see e.g. Weihrauch [35], Hertling [17].

Often one would like to be able to decide for two arbitrary numbers $x, y \in \mathbb{R}$ whether $x < y$ is true or not. It is clear that this is impossible with respect to ρ_C and hence with respect to any standard representation. One might ask whether it is perhaps possible to choose a non-standard representation which makes this test decidable, even if one looses computability of some of the other operations. But even this is impossible.

Proposition 4.3 (Weihrauch [36, Lemma 4.5]) *There is no representation $\rho :\subseteq \mathbb{F} \to \mathbb{R}$ such that the relation "$\geq$"$= \{(x, y) \in \mathbb{R}^2 \mid x \geq y\}$ is ρ^2-enumerable.*

Proof. Assume there is a representation $\rho :\subseteq \mathbb{F} \to \mathbb{R}$ and a c.e. set $A \subseteq \mathbb{N}^*$ such that $A\mathbb{F} \cap \operatorname{dom}\rho^2 = (\rho^2)^{-1}(\text{"}\geq\text{"})$. Let $p \in \operatorname{dom}\rho$. Because of $\rho(p) \geq \rho(p)$ there is a $w \in A$ which is a prefix of $\langle p, p\rangle$. Let v be a finite prefix of p such that $\langle v\Sigma^\omega, v\Sigma^\omega\rangle \subseteq w\Sigma^\omega$. This implies that for all $q \in v\mathbb{F} \cap \operatorname{dom}\rho$ we have $\rho(p) \geq \rho(q)$ and $\rho(q) \geq \rho(p)$. Hence $\rho(v\mathbb{F}) = \{\rho(p)\}$, and $\operatorname{range}\rho$ is countable in contradiction to $\operatorname{range}\rho = \mathbb{R}$. $\square$

We list a few more standard representations of the real numbers.

1. The *nested interval representation* $\rho_I :\subseteq \mathbb{F} \to \mathbb{R}$. We define a numbering $\nu_I : \mathbb{N} \to \{[x, y] \subseteq \mathbb{R} \mid x \in \mathbb{D}, y \in \mathbb{D}, x \leq y\} \cup \{\emptyset\}$ of closed intervals with dyadic boundaries by $\nu_I\langle i, j\rangle := [\nu_{\mathbb{D}}(i), \nu_{\mathbb{D}}(j)]$ where $[x, y] = \{z \in \mathbb{R} \mid x \leq z \leq y\}$ if $x \leq y$, and $[x, y] = \emptyset$ if $y < x$. Then ρ_I is defined by
$$\rho_I(p) := x \quad \text{iff } \nu_I(p(k)) \supseteq \nu_I(p(k+1)) \text{ for all } k, \text{ and } \bigcap_n \nu_I(p(n)) = x.$$

2. The *open balls representation* $\rho_B :\subseteq \mathbb{F} \to \mathbb{R}$. We define a total numbering ν_B of a base of open balls in $\mathbb{R}$ by $\nu_B\langle i,j\rangle := \{x \in \mathbb{R} \mid |x - \nu_{\mathbb{D}}(i)| < 2^{-j}\}$. Then ρ_B is defined by

$$\rho_B(p) := x \quad \text{iff} \quad \{p(0), p(1), p(2), \ldots\} = \{i \in \mathbb{N} \mid x \in \nu_B(i)\}.$$

3. In the three representations ρ_C, ρ_I, and ρ_B one can replace the numbering $\nu_{\mathbb{D}}$ by the numbering $\nu_{\mathbb{Q}}$ of the rationals, see Example 2.3.1.

4. The *redundant number representations* $\rho_{k,l} :\subseteq \mathbb{F} \to \mathbb{F}$. Consider two numbers $k, l \in \{1, 2, 3, \ldots\}$ and identify the alphabet $\Sigma_{k,l} := \{:, -k, -(k-1), \ldots, 0, \ldots, l\}$ consisting of the colon ":" and of the integers from $-k$ to l with a subset of $\mathbb{N}$. Then define $\rho_{k,l} :\subseteq \Sigma_{k,l}^\omega \to \mathbb{R}$ by $\mathrm{dom}\,(\rho_{k,l}) := \{-k, \ldots, l\}^*\{:\}\{-k, \ldots, l\}^\omega$ and, using $m := \max\{k, l\} + 1$ set

$$\rho_{k,l}(p) := \sum_{i=0}^{n-1} p(i) \cdot m^{n-1-i} + \sum_{i=n+1}^{\infty} p(i) \cdot m^{n-i}$$

for $p \in \{-k, \ldots, l\}^n\{:\}\{-k, \ldots, l\}^\omega$ (i.e. $p(n) =$":") for some $n \in \mathbb{N}$.

The last representations $\rho_{k,l}$ and variations thereof are of great importance in real number computer arithmetic, which is concerned with the hardware and software implementation of real number algorithms, see Muller [28]. This shows that the analysis of effectivity and effective categoricity of structures, that is, the analysis of the question which representations are suitable for certain structures, is not just of academic interest but has direct applications in practice. Of course, besides the fact that these redundant number representations belong to the unique (in terms of computable equivalence) class of representations which make the structure $\mathbb{R}$ effective, these representations also have the special property that they are suitable for complexity theoretic reasons. In fact, a large number of complexity theoretic results in computable analysis (Brent [6], Ko [19]) can be formulated with the *signed-digit representation* $\rho_{1,1}$, which was introduced by Avizienis [1].

These representations are very similarly defined to the *b-ary representations* ρ_b. Identifying the alphabet $\Sigma_b := \{-, :, 0, 1, \ldots, b-1\}$, which consists of the minus symbol "$-$", the colon ":" and of the integers from 0 to $b-1$, with a subset of $\mathbb{N}$, one defines $\rho_b :\subseteq \Sigma_b^\omega \to \mathbb{R}$ by

$$\mathrm{dom}\,\rho_b := \{\varepsilon, -\}\{\text{words in } \{0, \ldots, b-1\}^* \text{ not starting with } 0\}\{:\}\{0, \ldots, b-1\}^\omega$$

and defines

$$\rho_b(p) := \begin{cases} \sum_{i=0}^{n-1} p(i) \cdot b^{n-1-i} + \sum_{i=n+1}^{\infty} p(i) \cdot b^{n-i} & \text{if } p \in \mathrm{dom}\,\rho_b,\ p \text{ does not start} \\ & \text{with "}-\text{", and } p(n) = \text{":"} \\ -(\sum_{i=1}^{n-1} p(i) \cdot b^{n-1-i} + \sum_{i=n+1}^{\infty} p(i) \cdot b^{n-i}) & \text{if } p \in \mathrm{dom}\,\rho_b,\ p \text{ starts with "}-\text{"}, \\ & \text{and } p(n) = \text{":"}. \end{cases}$$

The b-ary representations do not belong to the class of standard representations of $\mathbb{R}$. If ρ denotes a standard representation of $\mathbb{R}$, then obviously $\rho_b \leq \rho$. But already for topological reasons $\rho \not\leq \rho_b$, see e.g. Deil [7], Weihrauch and Kreitz [38]. In fact, it is well-known that not even addition is (ρ_b^2, ρ_b)-computable, for any $b \geq 2$. This can easily be shown in a way similar to the first part of the proof of Theorem 4.4 below. Still, one might hope that by using these very common representations one might gain some computable functions which are not computable with respect to a standard representation. This is not the case, not even when one uses different bases in one computation.

Theorem 4.4 *Let $b \geq 2$, $b' \geq 2$, and let ρ be a standard representation of $\mathbb{R}$. Fix an $n \in \{1, 2, 3, \dots\} \cup \{\omega\}$. Then every $(\rho_b^n, \rho_{b'})$-computable function is (ρ^n, ρ)-computable but not vice versa.*

Proof. First, we show that the function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x_0, x_1, \dots) := (bb' - 1) \cdot x_0$ is (ρ^n, ρ)-computable but not $(\rho_b^n, \rho_{b'})$-computable. It is clear that it is (ρ^n, ρ)-computable. If is sufficient to prove the other statement for $n = 1$ since the projection from $\mathbb{R}^n$ to $\mathbb{R}$ onto the first coordinate is (ρ_b^n, ρ_b)-computable. So we assume $n = 1$. Assume there is a computable function $F :\subseteq \Sigma_b^\omega \to \Sigma_{b'}^\omega$ with $f\rho_b(p) = \rho_{b'}F(p)$ for all $p \in \operatorname{dom}\rho_b$. Let p be a ρ_b-name of $1/(bb' - 1)$. Note that this name is unique and does not end on only 0's or only the digit $(b - 1)$ since $1/(bb' - 1)$ is not of the form z/b^n for some integer z and some $n \geq 0$. The sequence $F(p)$ is a $\rho_{b'}$-name of the number $1 = f(1/(bb' - 1))$. Every $\rho_{b'}$-name of 1 consists of a finite word, the colon and then an infinite sequence either containing only 0's or containing only the digit $(b' - 1)$. Let w be a prefix of $F(p)$ containing the colon. In the first case ($F(p)$ ending on 0's) the set $w\Sigma_{b'}^\omega$ contains $\rho_{b'}$-names only of 1 and numbers greater than 1; in the second case ($F(p)$ ending on $(b' - 1)$'s) it contains $\rho_{b'}$-names only of 1 and of numbers smaller than 1. Since F is computable and hence continuous there is a prefix v of p with $F(v\Sigma_b^\omega) \subseteq w\Sigma_{b'}^\omega$. Since f is increasing, the set $v\Sigma_b^\omega \cap \operatorname{dom}\rho_b$ may contain either only ρ_b-names for numbers $\geq 1/(bb' - 1)$ or only ρ_b-names for numbers $\leq 1/(bb' - 1)$. But this is false since p does not end on 0's or $(b - 1)$'s. By decreasing one of the digits $\neq 0$ of p after the prefix v one obtains a ρ_b-name in $v\Sigma^\omega$ of a real number smaller than $1/(bb' - 1)$. And by increasing a digit $\neq b - 1$ of p after the prefix v one obtains a ρ_b-name in $v\mathbb{F}$ of a real number greater than $1/(bb' - 1)$. Hence, the function f is not $(\rho_b, \rho_{b'})$-computable.

Secondly, we have to prove that every $(\rho_b^n, \rho_{b'})$-computable function is also (ρ^n, ρ)-computable. Since $\rho_{b'} \leq \rho$ any $(\rho_b^n, \rho_{b'})$-computable function is also (ρ_b^n, ρ)-computable. Let $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ be a (ρ_b^n, ρ)-computable function. We show that it is also (ρ^n, ρ)-computable. We will treat the case $n = 1$ in detail and in the end explain why the assertion is true also for the case of larger $n \in \{2, 3, \dots\} \cup \{\omega\}$. So we assume $n = 1$. We can also assume that $\rho = \rho_C$ is the normed Cauchy representation.

The idea is the following. From a ρ_C-name p of a real number x we cannot compute a ρ_b-name. But we can for each t compute two words with t digits after the colon which are prefixes of ρ_b-names of numbers close to the left and right from x such that at least one of these two words is a prefix of a ρ_b-name for x itself. Now we apply the algorithm for the (ρ_b, ρ_C)-computation of f to both words. When we obtain results close to each other, both are good approximations of the correct value, hence one of them can be used for the output. This algorithm will give better and better approximations since for sufficiently large t large prefixes of both words will be prefixes of ρ_b-names for x, in other words, since for t tending to infinity both words converge to ρ_b-names of x.

For the formal proof we proceed as follows. It is clear that there is a computable function $h_1 :\subseteq \mathbb{N} \times \mathbb{F} \to \mathbb{N}$ such that for all $t \in \mathbb{N}$ and $p \in \operatorname{dom}\rho_C$ the value $h_1(t, p)$

exists and is equal to a number $\langle i,j\rangle$ with

$$\frac{i-j}{b^t} < \rho_C(p) < \frac{i-j}{b^t} + \frac{2}{b^t}\,.$$

Let $h_2 : \mathbb{N}^2 \to \Sigma_b^*$ be the computable function such that $\rho_b(h_2(t,\langle i,j\rangle)0^\omega) = (i-j)/b^t$ and $h_2(t,\langle i,j\rangle)$ has t digits after the colon. We define $h_3, h_4 :\subseteq \mathbb{N}\times\mathbb{F} \to \Sigma_b^*$ by $h_3(t,p) := h_2(t, h_1(t,p))$ and $h_4(t,p) := h_2(t, \langle \pi_1^2 h_1(t,p)+1, \pi_2^2 h_1(t,p)\rangle)$. Then for $t\in\mathbb{N}$ and $p \in \operatorname{dom}\rho_C$ we have

$$\rho_b(h_4(t,p)0^\omega) = \rho_b(h_3(t,p)0^\omega) + 1/b^t$$

and

$$\rho_b(h_3(t,p)0^\omega) < \rho_C(p) < \rho_b(h_3(t,p)0^\omega) + \frac{2}{b^t}\,.$$

Hence, both words $h_3(t,p)$ and $h_4(t,p)$ have exactly t digits after the colon, and at least one of them is a prefix of a ρ_b-name for $\rho_C(p)$. Furthermore, both sequences $(h_3(t,p)0^\omega)_t$ and $(h_4(t,p)0^\omega)_t$ converge to ρ_b-names of $\rho_C(p)$: if $\rho_C(p)$ is not a number of the form z/b^l (for some integers z and l), then it has only one ρ_b-name and both sequences converge to it. Otherwise it has exactly two different ρ_b-names and for sufficiently large t the infinite words $h_3(t,p)(b-1)^\omega$ and $h_4(t,p)0^\omega$ will be the ρ_b-names for $\rho_C(p)$.

Let $F :\subseteq \Sigma_b^\omega \to \mathbb{F}$ be a computable function which (ρ_b,ρ_C)-tracks f and let $g :\subseteq \Sigma_b^* \to \mathbb{N}^*$ be a monotone, computable function which induces F (see before Definition 2.7). Let $(v_i)_i$ be a computable list of the words in $\operatorname{dom} g$. We define a computable function $h_5 :\subseteq \mathbb{N}\times\mathbb{F}\to\mathbb{N}$ by

$$\begin{aligned} h_5(m,p) := \pi_1^2 \min\{\langle k,t\rangle \mid\ & (\exists i,j\le t)\ v_i \text{ is prefix of } h_3(t,p) \text{ and } |g(v_i)| \ge m+4 \\ & \&\ v_j \text{ is prefix of } h_4(t,p) \text{ and } |g(v_j)| \ge m+4 \\ & \&\ |\nu_{\mathbb{D}}(g(v_i)(m+3)) - \nu_{\mathbb{D}}(g(v_j)(m+3))| < 3\cdot 2^{-(m+3)} \\ & \&\ g(v_i)(m+3) = k\}\,. \end{aligned}$$

Finally we define a computable function $G :\subseteq \mathbb{F}\to\mathbb{F}$ by

$$G(p) := (h_5(0,p), h_5(1,p), h_5(2,p), \ldots)\,.$$

We claim that this function (ρ_C,ρ_C)-tracks f. Let $m\in\mathbb{N}$ and $p\in\operatorname{dom}\rho_C$, furthermore $x := \rho_C(p)$. It is sufficient to show that

$$|f(x) - \nu_{\mathbb{D}}(h_5(m,p))| < 2^{-(m+1)}\,.$$

First we show that $h_5(m,p)$ exists. Since $(h_3(t,p)0^\omega)_t$ and $(h_4(t,p)0^\omega)_t$ converge to ρ_b-names of x and F maps these names to ρ_C-names of $f(x)$, there are a number t and a prefix v_i of $h_3(t,p)$ which is also a prefix of a ρ_b-name for x with $|g(v_i)|\ge m+4$, as well as a prefix v_j of $h_4(t,p)$ which is also a prefix of a ρ_b-name for x with $|g(v_j)|\ge m+4$. It is clear that for these i,j

$$\begin{aligned} & |\nu_{\mathbb{D}}(g(v_i)(m+3)) - \nu_{\mathbb{D}}(g(v_j)(m+3))| \\ \le\ & |\nu_{\mathbb{D}}(g(v_i)(m+3)) - f(x)| + |f(x) - \nu_{\mathbb{D}}(g(v_j)(m+3))| \\ \le\ & 2\cdot 2^{-(m+3)}\,. \end{aligned}$$

Hence, the numbers t and i, j show that $h_5(m,p)$ exists. We already remarked that at least one of the words $h_3(t,p)$ or $h_4(t,p)$ always is a prefix of a ρ_b-name of x. If $h_3(t,p)$ is such a prefix, then

$$|f(x) - \nu_{\mathbb{D}}(g(v_i)(m+3))| \leq 2^{-(m+3)} < 2^{-(m+1)},$$

if $h_4(t,p)$ is the prefix, then

$$\begin{array}{ll} & |f(x) - \nu_{\mathbb{D}}(g(v_i)(m+3))| \\ \leq & |f(x) - \nu_{\mathbb{D}}(g(v_j)(m+3))| + |\nu_{\mathbb{D}}(g(v_j)(m+3)) - \nu_{\mathbb{D}}(g(v_i)(m+3))| \\ < & 2^{-(m+3)} + 3 \cdot 2^{-(m+3)} = 2^{-(m+1)}. \end{array}$$

This ends the proof of $|f(x) - \nu_{\mathbb{D}}(h_5(m,p))| < 2^{-(m+1)}$. We have proved that every (ρ_b, ρ_C)-computable function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is also (ρ_C, ρ_C)-computable.

Now assume that $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ is a (ρ_b^n, ρ_C)-computable function where the input dimension $n \in \{2,3,4,\ldots\}$ is greater than 1 but finite. Let $p = \langle p_0, \ldots, p_{n-1}\rangle \in \operatorname{dom} f\rho_C^n$ be a ρ_C^n-name for a vector $(x_0, \ldots, x_{n-1})$ in the domain of f. Using p, we can for each $i \in \{0, \ldots, n-1\}$ proceed as above and compute $h_3(t,p_i)$ and $h_4(t,p_i)$ for any n_i. For t tending to infinity each sequence $(h_3(t,p_i))_t$ and $(h_4(t,p_i))_t$ converges to a ρ_b-name for x_i. And one of the words $h_3(t,p_i)$ and $h_4(t,p_i)$ is a prefix of a ρ_b-name for x_i. Hence, one of the 2^n combinations containing for each $i \in \{0,\ldots,n-1\}$ either $h_3(t,p_i)$ or $h_4(t,p_i)$ consists solely out of prefixes of ρ_b-names for x_i, $i = 0, \ldots, n-1$. So we proceed as in the one-dimensional case, but checking all 2^n combinations at the same time and comparing the results for all of them pairwise (in the part of computation which is done by h_5 above). We leave the precise definition of the corresponding function $h_5^{(n)}$ to the reader. Since all combinations converge to ρ_b^n-names for x, the algorithm $h_5^{(n)}(m,.)$ will always stop, and since one of the combinations consists solely out of prefixes of names for x_i, the algorithm will give an approximation to the desired result. Hence the assertion is also true for finite dimension $n > 2$.

Finally we consider a (ρ_b^ω, ρ_C)-computable function $f :\subseteq \mathbb{R}^\omega \to \mathbb{R}$. We wish to show that f is (ρ_C^ω, ρ_C)-computable. Again, we can proceed similarly as above, but we must take care of a growing number of components. We need some notation. For an infinite sequence p we define the sequences $p^{[i]}$ by $p = \langle p^{[0]}, p^{[1]}, \ldots\rangle$. For a finite word w we define $w^{[i]}$ to be the longest prefix of $(w\$^\omega)^{[i]}$ not containing the dummy symbol \$.

Assume that $F :\subseteq \Sigma_b^\omega \to \mathbb{F}$ is a function that (ρ_b^ω, ρ_C)-tracks f, and that $g :\subseteq \Sigma_b^* \to \mathbb{N}^*$ is a monotone, computable function that induces F. Let $v_0, v_1, v_2, \ldots$ be a computable sequence of all words in $\operatorname{dom} g$. For each $t \in \mathbb{N}$ let $d_t : \{0, \ldots, 2^{t+1}-1\} \to \{0,1\}^{t+1}$ be the lexicographic bijection and for $0 \leq j \leq t$ let $d_t(k,j)$ be the j-th digit in the vector $d_t(k)$. We define a computable function $h_5^{(\omega)} :\subseteq \mathbb{N} \times \mathbb{F} \to \mathbb{N}$ by

$$\begin{array}{lll} h_5^{(\omega)}(m,p) := \pi_1^3 \min\{\langle k,s,t\rangle \mid & (\exists\, i_0, \ldots, i_{2^{t+1}-1} \leq s) \\ & [(\forall\, 0 \leq k \leq 2^{t+1}-1)\ |g(v_{i_k})| \geq m+4 \\ & \quad \&(\forall\, 0 \leq j \leq t)\ v_{i_k}^{[j]} \text{ is prefix of } h_{3+d_t(k,j)}(t, p^{[j]})\,] \\ & \&(\forall\, 0 \leq k,l \leq 2^{t+1}-1) \\ & \quad |\nu_{\mathbb{D}}(g(v_{i_k})(m+3) - \nu_{\mathbb{D}}(g(v_{i_l})(m+3))| < 3 \cdot 2^{-(m+3)} \\ & \&\ g(v_{i_0})(m+3) = k\}\,. \end{array}$$

We claim that

$$|f(x) - \nu_{\mathbb{D}}(h_5^{(\omega)}(m,p))| < 2^{-(m+1)}$$

for all $m \in \mathbb{N}$ and $p \in \mathrm{dom}\,\rho_C^\omega$. It is clear that this claim proves the assertion (that f is (ρ_C^ω, ρ_C)-computable) since then we can proceed further as in the one-dimensional case. Once we know that $h_5^{(\omega)}(m,p)$ for $m \in \mathbb{N}$ and $p \in \mathrm{dom}\,\rho_C^\omega$ exists, our claim follows as in the one-dimensional case, since one of the vectors

$$(h_{3+d_t(k,0)}(t,p^{[0]}),\ldots,h_{3+d_t(k,t)}(t,p^{[t]}))$$

for $0 \le k \le 2^{t+1}-1$ contains only prefixes of ρ_b-names of the corresponding components $(x_0,\ldots,x_t)$ of $x = (x_0,x_1,x_2,\ldots) = \rho_C^\omega(p)$. So we assume that $m \in \mathbb{N}$ and $p \in \mathrm{dom}\,\rho_C^\omega$. We have to show only that $h_5^{(\omega)}(m,p)$ exists.

The set of ρ_b^ω-names of x may be infinite, but it is compact, since each real number x_j has only one or two ρ_b-names. Hence, there is a number h such that each ρ_b^ω-name of x has a prefix $v \in \mathrm{dom}\,g$ with $|v| \le h$ and $|g(v)| \ge m+4$. Let $s \in \mathbb{N}$ be a number such that all these prefixes v are contained in the set $\{v_0,v_1,\ldots,v_s\}$. On the other hand, if t is large enough, then for each $k \in \{0,1,\ldots,2^{t+1}-1\}$ the prefix of length h (and hence any prefix of length less or equal to h) of

$$\langle h_{3+d_t(k,0)}(t,p^{[0]})\$^\omega,\ldots,h_{3+d_t(k,t)}(t,p^{[t]})\$^\omega,\$^\omega,\$^\omega,\ldots\rangle$$

is a prefix of a ρ_b^ω-name of x (where \$ is just a dummy symbol). This is true since the first h digits of these sequences depend only on the first h digits of the words $h_3(t,p^{[j]})$ and $h_4(t,p^{[j]})$ for $0 \le j \le h-1$, and these words tend to ρ_b-names of x_j for t tending to infinity. Furthermore, a word $v \in \mathbb{N}^*$ (not containing the symbol \$!) of length less or equal to $h \le t$ is a prefix of

$$\langle h_{3+d_t(k,0)}(t,p^{[0]})\$^\omega,\ldots,h_{3+d_t(k,t)}(t,p^{[t]})\$^\omega,\$^\omega,\$^\omega,\ldots\rangle$$

if and only if $v^{[j]}$ is a prefix of $h_{3+d_t(k,j)}(t,p^{[j]})$, for $0 \le j \le t$. Finally, for any prefixes $v_{i_k}, v_{i_l} \in \mathrm{dom}\,g$ of ρ_b^ω-names of x with $|g(v_{i_k})| \ge m+4$ and $|g(v_{i_l})| \ge m+4$ one has

$$\begin{aligned} & |\nu_{\mathbb{D}}(g(v_{i_k})(m+3) - \nu_{\mathbb{D}}(g(v_{i_l})(m+3))| \\ \le\ & |\nu_{\mathbb{D}}(g(v_{i_k})(m+3) - f(x)| + |f(x) - \nu_{\mathbb{D}}(g(v_{i_l})(m+3))| \\ \le\ & 2 \cdot 2^{-(m+3)}\,. \end{aligned}$$

We conclude that $h_5^{(\omega)}(m,p)$ exists. This ends the proof. □

5 The Structure of the Computable Real Numbers

In the last section we have seen that the real number structure is r-effectively categorical. At the end of Section 2 we observed that an r-effectively categorical structure S possesses a unique countable substructure whose universe S_c consists of the computable elements and whose functions and relations are obtained by appropriately restricting the functions and relations of S. In this section we collect some results on the structure $\mathbb{R}_c$ of the computable real numbers, derived from the real number structure of Theorem 4.1 by following Definition 3.8. Since it is so important we describe it explicitly.

The set $\mathbb{R}_c = \mathbb{R}^{\rho_C-\mathrm{comp}}$ of computable real numbers can also be written as

$$\mathbb{R}_c = \{x \in \mathbb{R} \mid \begin{array}{l} \text{there is an } (\mathrm{id}_{\mathbb{N}}, \nu_{\mathbb{D}})\text{-computable function } g : \mathbb{N} \to \mathbb{D} \\ \text{with } |x - g(n)| \le 2^{-n} \text{ for all } n \in \mathbb{N}.\} \end{array}$$

The constants 0 and 1 are computable and by Lemma 3.7.3 the basic arithmetic operations transform computable real numbers and computable pairs of real numbers into computable real numbers. Also, the limit NormLim $((x_n)_n)$ of a normed Cauchy sequence $(x_n)_n$ which is additionally computable, i.e. lies in

$$\begin{aligned} \mathbb{R}_c^{\omega,\mathrm{comp}} &= (\mathbb{R}^\omega)^{\rho_C^\omega-\mathrm{comp}} \\ &= \{(x_n)_n \in \mathbb{R}^\omega \mid (x_n)_n \text{ has a computable } \rho_C^\omega\text{-name}\} \\ &= \{(x_n)_n \in \mathbb{R}^\omega \mid (\exists g \in \mathrm{R}^{(1)})(\forall i,j)\, |x_i - \nu_{\mathbb{D}}(g\langle i,j\rangle)| \le 2^{-j}\}, \end{aligned}$$

is a computable real number.

Thus, on the computable real numbers we have a structure:

$$(\mathbb{R}_c, 0, 1, +, -, *, 1/., \mathrm{RecNormLim}, <)$$

consisting of the constants 0 and 1, the field operations addition "+": $\mathbb{R}_c^2 \to \mathbb{R}_c$, additive inverse "−": $\mathbb{R}_c \to \mathbb{R}_c$, multiplication "∗": $\mathbb{R}_c^2 \to \mathbb{R}_c$, multiplicative inverse "1/.': $\mathbb{R}_c \setminus \{0\} \to \mathbb{R}_c$, the recursive normed limit operator RecNormLim defined by dom (RecNormLim) $:= \mathbb{R}_c^{\omega,\mathrm{comp}} \cap \mathbb{R}_c^{\omega,\mathrm{normed}}$ and $\mathrm{RecNormLim}((x_n)_n) = \mathrm{NormLim}\,((x_n)_n)$ for all $(x_n)_n$ in its domain, and the order relation "<" $= \{(x,y) \in \mathbb{R}_c^2 \mid x < y\}$.

Theorem 5.1 *The structure* $(\mathbb{R}_c, 0, 1, +, -, *, 1/., \mathrm{RecNormLim}, <)$ *is n-effectively categorical.*

Proof. This follows immediately from Moschovakis' Theorem 4 in [26]. One detail should be observed: Moschovakis calls a numbered field *recursive* if all the field operations are strongly computable with respect to the numbering, compare Remark 2.8.1. The total operations "+", "−", "∗" are computable if and only if they are strongly computable. But for the partial function "1/." the two notions are not identical. However, if also the order relation "<" is enumerable, then the set $\{x \in \mathbb{R} \mid x \neq 0\}$, which is the domain of "1/.", is enumerable as well. Hence, if "1/." is computable and "<" is enumerable, then "1/." is strongly computable.

One can also easily obtain a direct proof. Let $\nu_C := \rho_C \varphi$ denote the numbering of $\mathbb{R}_c$ derived from the Cauchy representation ρ_C, compare Definition 3.6. Since ρ_C makes the real number structure $\mathbb{R}$ effective by Theorem 4.1, this numbering ν_C makes the structure $\mathbb{R}_c$ effective by Theorem 3.9. For the effective categoricity one can easily translate the second part of the proof of Theorem 4.1 from representations to numberings (by making use of the smn Theorem). □

Note that for the uniqueness of the numbering we do not demand that the four basic arithmetic operations are strongly computable, compare Remark 2.8.1. But they turn out to be strongly computable under any standard numbering. This is not the case for the limit operator, which, by the following result, is not strongly computable with respect to a standard numbering and cannot even be extended to a strongly computable operator. This is one of the reasons why we chose to consider relative computability of functions without restrictions on their domain.

Theorem 5.2 *Let $\nu :\subseteq \mathbb{N} \to \mathbb{R}_c$ be a standard numbering of $\mathbb{R}_c$. Every ν^ω-enumerable subset $U \subseteq \mathbb{R}_c^{\omega,\mathrm{comp}}$ with $\mathbb{R}_c^{\omega,\mathrm{comp}} \cap \mathbb{R}_c^{\omega,\mathrm{normed}} \subseteq U$ contains non-converging sequences.*

Proof. Consider the space $\mathbb{R}_c^{\omega,\mathrm{comp}}$ with the metric induced by the norm $||\cdot||$ defined by $||(x_n)_n|| = \sum_{n=0}^{\infty} 2^{-n} \frac{|x_n|}{1+|x_n|}$ and with its numbering ν^ω. This space is a "recursive metric space satisfying Condition (A)" in Moschovakis' [27] terminology. That is, one can compute its metric, and the limit of a computable and computably convergent sequence of elements in $\mathbb{R}_c^{\omega,\mathrm{comp}}$ is again an element of $\mathbb{R}_c^{\omega,\mathrm{comp}}$, and a ν^ω-index can be obtained from an index for the sequence and its convergence rate. Hence, by Moschovakis' Theorem 2 [27], the complement of any ν^ω-enumerable subset $U \subseteq \mathbb{R}_c^{\omega,\mathrm{comp}}$ must contain all points in $\mathbb{R}_c^{\omega,\mathrm{comp}}$ that are limits of computable sequences of elements in the complement. Let us assume that $U \subseteq \mathbb{R}_c^{\omega,\mathrm{comp}}$ in ν^ω-enumerable and contains $\mathbb{R}_c^{\omega,\mathrm{comp}} \cap \mathbb{R}_c^{\omega,\mathrm{normed}}$. The sequence $(0,0,0,\ldots)$ is an element of $\mathbb{R}_c^{\omega,\mathrm{comp}} \cap \mathbb{R}_c^{\omega,\mathrm{normed}} \subseteq U$. It is also the limit of the computable sequence $(y^{(n)})_n$ where $y^{(n)} = (0, \frac{1}{2^n}, 0, \frac{1}{2^n}, 0, \frac{1}{2^n}, \ldots) \in \mathbb{R}_c^{\omega,\mathrm{comp}}$. We conclude that not all of the $y^{(n)}$ lie in the complement $\mathbb{R}_c^{\omega,\mathrm{comp}} \setminus U$ of U. □

The same remarks as those after Theorem 4.1 apply here as well. If one wishes to analyze computability on $\mathbb{R}_c$ with respect to a numbering of $\mathbb{R}_c$, then a standard numbering is the natural choice. Indeed, many of the results of constructive analysis, as it is developed for example in Kushner [22], can be interpreted as formulated with respect to a standard numbering of the computable real numbers.

It is interesting to note several classical negative results about the computability structure of $\mathbb{R}_c$. By ν we always denote an arbitrary standard numbering of $\mathbb{R}_c$.

1. The order relation is not ν^2-decidable, that is, the function $t : \mathbb{R}_c^2 \to \{0,1\}$ is not $(\nu^2, \mathrm{id}_{\mathbb{N}})$-computable. Even more: if a predicate $P \subseteq \mathbb{R}_c^n$ is ν^n-decidable, then it must be either the empty set or $\mathbb{R}_c^n$. This is due to Markov [25]. For a simpler proof see Moschovakis [27], Section 3.

2. Given a computably enumerable set E of natural numbers contained in the domain of ν one can determine a ν-name of a real number in $\mathbb{R}_c \setminus \nu(E)$. This was observed by Moschovakis [26] and can be proved by a simple diagonalization, see e.g. Weihrauch [34, Lemma 3.8.9]. Hence, there is no c.e. set $E \subseteq \mathrm{dom}\,\nu$ with $\nu(E) = \mathbb{R}_c$. Especially, the domain of ν itself cannot be a computably enumerable subset of $\mathbb{N}$. This is the reason why we did not impose any restriction on the domain of a numbering. This fact also implies that $\mathbb{R}_c$ is not a computable field in the sense of Fröhlich and Shepherdon [11] and does not have an effective numbering as considered by Stoltenberg-Hansen and Tucker [31].

3. The numbering ν cannot be injective. Otherwise one could ν^2-decide the order relation. Hence, it is impossible to identify the structure of the computable real numbers with a computable substructure of $\mathbb{N}$.

4. The first point can be strengthened: every (ν^k, ν)-computable function $f :\subseteq \mathbb{R}_c^k \to \mathbb{R}_c$ with a ν^k-enumerable domain is computable with respect to a standard representation of $\mathbb{R}$ by Tseytin's theorem [32] on effective continuity, see also Moschovakis [27] and Kushner [22]. For a further strengthening see Hertling [17]. But there

exist (ν, ν)-computable real functions which are not continuous (and hence not computable with respect to a standard representation of $\mathbb{R}$) as we shall see in the proof of Theorem 3.11 below.

5. The "computation" of a discontinuous function like the test t in the first remark can be replaced by the computation of a continuous correspondence which approximates the discontinuous function, compare Remark 2.2.3 and see Weihrauch [34], Brattka and Hertling [5], and Brattka [4].

Finally we prove Theorem 3.11.

Proof of Theorem 3.11. Let again $\nu_C = \rho_C\varphi$ denote the standard numbering of $\mathbb{R}_c$ derived from the Cauchy representation ρ_C. We define a structure $\mathcal{S}$ by taking the structure of the computable real numbers $\mathbb{R}_c$ of Theorem 5.1 and adding the function $f :\subseteq \mathbb{R}_c \to \mathbb{R}_c$ with

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if for each } i \text{ with } \nu_C(i) = x \text{ there exists a } j \leq i \text{ with } \nu_{\mathbb{D}}(\varphi_i(j)) > 2^{-j} \\ \uparrow & \text{otherwise .} \end{cases}$$

This is an adaption of an example of Kreisel, Lacombe, Shoenfield [20, page 293], which they credit to Myhill. In order to prove that the structure $\mathcal{S}$ is n-effectively categorical we have to show only that the function f is (ν_C, ν_C)-computable. But this is shown by the computable function $g :\subseteq \mathbb{N} \to \mathbb{N}$ with

$$g(i) := \begin{cases} n_0 & \text{if } (\forall j \leq i)\ \nu_{\mathbb{D}}(\varphi_i(j)) \leq 2^{-j} \\ n_1 & \text{if } (\exists j \leq i)\ \nu_{\mathbb{D}}(\varphi_i(j)) > 2^{-j} \\ \uparrow & \text{otherwise} \end{cases}$$

where n_0 is a ν_C-index for the real number 0, and n_1 is a ν_C-index for the real number 1.

Now we show that the structure $\mathcal{S}$ is not r-effective. The first step is to show that the function f above is discontinuous in the point 0. Fix a number n. The set $\mathbb{R}_c \setminus \{x \in \mathbb{R}_c \mid (\exists i \leq n+1)\ \nu_C(i) = x\}$ is a dense subset of $\mathbb{R}$. Fix a number x in this set with $2^{-(n+1)} < |x| < 2^{-n}$. Then every i with $\nu_C(i) = x$ is larger than or equal to $n+2$. And for each such i we have $\nu_{\mathbb{D}}(\varphi_i(n+2)) > 2^{-(n+2)}$. Hence $f(x) = 1$. This shows that f is discontinuous in 0.

Let us assume that there is representation $\rho :\subseteq \mathbb{F} \to \mathbb{R}_c$ which makes $\mathcal{S}$ r-effective. We choose ρ-names p_i of the numbers $2^{-(i+1)}$. Then $\langle p_0, p_1, p_2, \ldots\rangle$ is a ρ^ω-name for a computable, normed Cauchy sequence converging to 0. Let $F_{RNL} :\subseteq \mathbb{F} \to \mathbb{F}$ be a computable function which (ρ^ω, ρ)-tracks the operator RecNormLim. Then the sequence $p := F_{RNL}(\langle p_0, p_1, p_2, \ldots\rangle)$ is a ρ-name for 0. Since the function F_{RNL} is continuous, for any finite prefix v of p there is a finite prefix w of $\langle p_0, p_1, p_2, \ldots\rangle$ with $F_{RNL}(w\mathbb{F}) \subseteq v\mathbb{F}$. We can fix any real number x with $|x| \leq 2^{-(|w|+1)}$ and replace all names p_j with $j > |w|+1$ by a ρ-name q for x. The resulting sequence $\langle p_0, p_1, p_2, \ldots, p_{|w|}, q, q, \ldots\rangle$ still lies in $w\mathbb{F}$ but is a ρ^ω-name for a computable, normed Cauchy sequence with limit x. Hence there is also a ρ-name for x in $v\mathbb{F}$. That means $\rho(v\mathbb{F})$ contains an open neighborhood of 0, for any prefix v of p.

On the other hand we assume that the function f is (ρ, ρ)-computable. Using the fact that the order relation "$<$" is ρ^2-enumerable one can easily show that f is also $(\rho, \mathrm{id}_{\mathbb{N}})$-computable. Let $F :\subseteq \mathbb{F} \to \mathbb{N}$ be a computable function which $(\rho, \mathrm{id}_{\mathbb{N}})$-tracks f. Since

$f(x) = 0$ we have $F(p) = 0$, and hence there is a finite prefix v of p such that $F(v\mathbb{F}) = \{0\}$. We just saw that $\rho(v\mathbb{F})$ contains a neighborhood of 0. We conclude that f is continuous in the point 0 in contradiction to the fact proved above that f is discontinuous in 0. □

6 Other Structures and Open Problems

We give one more general example of an r-effectively categorical structure which suggests that operators between different structures might be of use and conclude with open questions.

Consider a separable metric space M with a total numbering $\alpha : \mathbb{N} \to M$ of a dense subset and a metric $d : M^2 \to \mathbb{R}$. Define the normed limit operator NormLim as in the case of real numbers: $\mathrm{dom}\,(\mathrm{NormLim}\,) = \{(x_n)_n \in M^\omega \mid (\forall i, j)\ d(x_i, x_j) \leq 2^{-\min\{i,j\}}$ and $\lim_{n\to\infty} x_n$ exists in $M\}$, $\mathrm{NormLim}\,((x_n)_n) := \lim_{n\to\infty} x_n$ for all $(x_n)_n \in \mathrm{dom}\,(\mathrm{NormLim}\,)$. Finally define $P_i := \{(x, y) \in M^2 \mid d(x, y) < \nu_{\mathbb{D}}(i)\}$, for all i.

Theorem 6.1 *The structure*

$$(M, \alpha_0, \alpha_1, \alpha_2, \ldots, \mathrm{NormLim}\,, P_0, P_1, P_2, \ldots)$$

is r-effectively categorical if the set $B = \{(x, y, q) \in \mathrm{range}\,\alpha \times \mathrm{range}\,\alpha \times \mathbb{D} \mid d(x, y) < q\}$ *is* $(\alpha, \alpha, \nu_{\mathbb{D}})$*-enumerable.*

Proof. The definition of the normed Cauchy representation can be copied from the real number case: we define a representation $\delta_C :\subseteq \mathbb{F} \to M$ by

$$\mathrm{dom}\,\delta_C := \{p \in \mathbb{F} \mid (\forall m, n)\ |\alpha p(m) - \alpha p(n)| \leq 2^{-\min\{m,n\}} \text{ and } \lim_{n\to\infty} \alpha p(n) \text{ exists in } M\}$$

$$\delta_C(p) := \lim_{n\to\infty} \alpha p(n) \qquad \text{for all } p \in \mathrm{dom}\,\delta_C\,.$$

First we show that the representation δ_C makes the structure effective if the set B is $(\alpha, \alpha, \nu_{\mathbb{D}})$-enumerable. The function $i \mapsto (i, i, i, \ldots)$ proves $\alpha \leq \delta_C$. That NormLim is $(\delta_C^\omega, \delta_C)$-computable is as in the real number case proved by the function $F : \mathbb{F} \to \mathbb{F}$ with $F(\langle p_0, p_1, p_2, \ldots\rangle)(i) = p_{i+2}(i + 2)$. Let $A \subseteq \mathbb{N}$ be a c.e. set with $A \cap \mathrm{dom}\,(\alpha, \alpha, \nu_{\mathbb{D}}) = (\alpha, \alpha, \nu_{\mathbb{D}})^{-1}(B)$. The set

$$\begin{aligned} C := \{(n, w) \in \mathbb{N} \times \mathbb{N}^* \mid\ & |w| \text{ is even, } |w| \geq 2, \text{ and} \\ & d(\alpha(w(|w| - 2)), \alpha(w(|w| - 1))) < \nu_{\mathbb{D}}(n) - 2^{2-|w|/2}\} \end{aligned}$$

is c.e. and the sets $C_n := \{w \in \mathbb{N}^* \mid (n, w) \in C\}$ satisfy $C_n\mathbb{F} \cap \mathrm{dom}\,\delta_C^2 = (\delta_C^2)^{-1}(P_n)$ for all n. Hence δ_C makes the structure effective.

For the r-effective categoricity we assume that a representation $\gamma :\subseteq \mathbb{F} \to M$ makes the structure $(M, \alpha_0, \alpha_1, \alpha_2, \ldots, P_0, P_1, P_2, \ldots)$ effective and a representation $\delta :\subseteq \mathbb{F} \to M$ makes the structure $(M, \alpha_0, \alpha_1, \alpha_2, \ldots, \mathrm{NormLim}\,)$ effective. We show that this implies $\gamma \leq \delta$. We only sketch the proof, which is very similar to the second part of the proof of Theorem 4.1. Let p be a γ-name for a point in M. Using the algorithm which proves $\alpha \leq \gamma$ we can construct γ-names for the points α_i. Using the algorithm which γ^2-tracks the sets P_i we can estimate their distance from $\gamma(p)$ and in this way construct a sequence of points α_{n_i} with $d(\gamma(p), \alpha_{n_i}) < 2^{-(i+1)}$. Simultaneously we can compute δ-names for these points and hence a δ^ω-name for the whole sequence. Finally application of the algorithm which (δ^ω, δ)-tracks NormLim gives us the desired δ-name for $\gamma(p)$. This ends the proof. $\square$

It is clear that the metric space $\mathbb{R}$ with the numbering $\alpha = \nu_{\mathbb{D}}$ or with the numbering $\alpha = \nu_{\mathbb{Q}}$ of a dense subset satisfies the assumption of the theorem, and that the resulting class of representations is exactly the class of standard representations of the real numbers considered in Section 4.

It is interesting to note that already the condition that the distance should be approximable from below uniquely determines the equivalence class of the representation. For more results on metric spaces and an analysis of other representations and their relations the reader is referred to Weihrauch [35].

The formulation of the theorem above can simplified if one considers *composed* structures in which also operations defined on or mappings to other, already introduced (preferably effectively categorical) structures are allowed. For the components in other structures one could refer to standard numberings or representations of them. For example the sequence of relations $(P_i)_i$ above could be replaced by the distance function $d : M^2 \to \{x \in \mathbb{R} \mid x \geq 0\}$ and the condition that it should be $(\delta^2, \rho_<)$-computable (δ a representation of the metric space) where $\rho_< :\subseteq \mathbb{F} \to \{x \in \mathbb{R} \mid x \geq 0\}$ is the representation defined by $\rho_<(p) := \sup_{n\in\mathbb{N}} \nu_{\mathbb{D}}(p(n))$ if $\{\nu_{\mathbb{D}}(p(n)) \mid n \in \mathbb{N}\}$ is bounded and its supremum non-negative, and $\rho_<(p) =\uparrow$ otherwise, see Weihrauch and Kreitz [38]. This representation makes the following two r-effectively categorical structures effective:

1. $(\{x \in \mathbb{R} \mid x \geq 0\}, c, \mathrm{Sup}, Q)$,
2. $(\{x \in \mathbb{R} \mid x \geq 0\}, 0, x \mapsto x+1, x \mapsto 2 \cdot x, x \mapsto x/2, \mathrm{Sup}, \{x \in \mathbb{R} \mid x > 1\})$

where $c_{\langle i,k\rangle} = i/2^k$ for all $i, k \in \mathbb{N}$, where Sup is a (partial) infinitary function defined by $\mathrm{Sup}(x_0, x_1, x_2, \ldots) = \sup_{n\in\mathbb{N}} x_n$ (if this supremum is finite), and where $Q_{\langle i,k\rangle} = \{x \in \mathbb{R} \mid x > i/2^k\}$ for all $i, k \in \mathbb{N}$. In these structures the supremum of a bounded sequence is computable, compare Remark 4.2.

We conclude with some remarks. We have introduced the notions of effectivity and effective categoricity for structures with infinitary operations whose cardinality does not exceed the cardinality of the continuum. In this paper we concentrated on the real numbers. There are several areas where work needs to be done. For countable structures the relation between n-effectivity and r-effectivity needs to be analyzed more thoroughly. What can be transferred from the theory of effective algebras and from computable model theory and what role might uncountable effective or effectively categorical structures play there? Finally, analysis – and certainly also other fields — provides a rich source of natural uncountable structures which should be analyzed with respect to effectivity and effective categoricity. This last point seems to be of immediate practical interest.

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