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Paradise Lost, or Paradise Regained?

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Paradise Lost, or Paradise Regained?

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ABSTRACT. This paper outlines some of the history and philosophy of constructive mathematics, concentrating on the work of the late Errett Bishop and his followers.

HISTORY

No one shall be able to drive us from the paradise that Cantor created for us. (Hilbert, [31])

Lasciate ogni speranza, voi ch'entrate. (Dante, l'Inferno, Canto III)

For Hilbert, and all mathematicians who subscribe to his philosophy of *formalism*, in order to prove that

$$\exists x\, P(x)$$

it suffices to show that

$$\neg\forall x\, \neg P(x).$$

In other words, existence is equivalent to the impossibility of nonexistence.[1] As long as it can be proved, of a formal, axiomatic mathematical system, that it is impossible to derive both P and $\neg P$ from the axioms, then that system is acceptable to the formalist. What matters to him is consistency, rather than meaning; it is of no consequence that an existence proof might only establish the impossibility of the nonexistence of the object sought after, rather than provide the means of finding, or at least approximating, that object.

Even before Brouwer entered the debate (see below), Hilbert's view received some incisive criticism. Frege, for instance, asked: "Suppose we knew that the propositions

- A is an intelligent being,

[1] Such an interpretation of "existence" certainly raises philosophical questions. What does it mean to say that "nonexistence is impossible"? That the assumption of existence leads to a contradiction (such as $0 = 1$)? But then we appear to need a notion of "existence" before we can define "nonexistence", which, in turn, is used to define "existence"!

- A is omnipresent,
- A is omnipotent,

together with all their consequences, did not contradict each other; could we infer from this that there was an omnipresent, omnipotent, intelligent being?"

When, in 1889, Hilbert proved his basis theorem by contradiction, many mathematicians were disappointed because they expected a proof to tell them *how to find* the desired basis; hence Gordan's famous remark,

> *Das is nicht Mathematik. Das ist Theologie.*

The controversy continued, with contributions from Poincaré and several other highly reputed mathematicians. Then, in 1907, the Dutch mathematician L.E.J. Brouwer (1881-1966) burst upon the scene with the publication of his doctoral thesis [19], in which he put forward his mathematical–philosophical position known as *intuitionism.* Precursors of intuitionistic ideas can be traced back to the works of Gauss, Lebesgue, Borel, and Kronecker.[2]

According to Brouwer, mathematics is a free creation of the human mind; an object exists if and only if it can be mentally constructed. His programme, which can be regarded as the basis of constructive mathematics, included two main points:

- to build up mathematics constructively ("Mathematics can deal with no other matter than that which it has itself constructed"); and
- to point out the errors in all other conceptions of mathematics.

In his 1908 essay "The Unreliability of the Logical Principles" [20] Brouwer criticised the unrestricted use of the law of the excluded middle in logic. But, with the notable exception, for a period around 1920, of Hermann Weyl ([43], pp. 75-77), most mathematicians remained sceptical of, or hostile to, Brouwer's views, believing that too much mathematics had to be given up in order to accommodate them. Not only were the customary logic principles invalid in intuitionism, but also the set-theoretic approach to the definition of mathematical concepts had to be sacrificed. Even mathematicians sympathetic to a constructivist point of view believed that what one could do in a totally constructivist system was so limited that one would have only a truncated fraction of the mathematics needed by the sciences. For example, as late as 1952 we find Kleene writing

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[2]Though Kronecker never published a single paper on the foundations of mathematics, his views were well known and encapsulated in his dictum that

> *God made the natural numbers, everything else is the work of man,*

at a meeting in Berlin in 1886.

What kind of mathematics can be built within the intuitionistic restrictions? If the existing classical mathematics could be rebuilt within these restrictions, without too great increase in the labor required and too great sacrifice in the results achieved, the problem of the foundations of mathematics would appear to be solved. Intuitionistic mathematics employs concepts and makes distinctions not found in classical mathematics; and it is very attractive on its own account. As a substitute for classical mathematics it has turned out to be less powerful and in many ways more complicated to develop. ([32], p. 51)

It appeared that Hilbert was right when he said

No–one, though he speak with the tongues of angels, will keep people from using the law of excluded middle.

His disagreement with Brouwer became more and more acrimonious, culminating, in 1928, with the notorious incident in which Brouwer was sacked from the Editorial Board of *Mathematische Annalen* (see pp. 99-102 of [43]).

Hilbert's formalist programme received a severe blow with the publication of Gödel's Theorem in 1931 [25]. Despite Hilbert's continued opposition, though, constructive approaches to mathematics survived.[3] Brouwer continued to lecture and write on intuitionism [42]. His student Heyting continued in the master's footsteps, and eventually became the leader of a small, but active, group of intuitionists spread through a number of universities in the Netherlands. Among Heyting's achievements was the first formalisation of intuitionistic logic, abstracted from the practice of intuitionistic mathematics. On the other hand, a completely different approach to constructive mathematics—essentially recursive mathematics with intuitionistic logic—was initiated by A.A. Markov in the Soviet Union in 1948-49, and achieved a number of technical successes ([34], [33]).

Nevertheless, by the mid-1960s constructive mathematics was, when compared with its classical (traditional) counterpart, relatively stagnant. This situation changed virtually overnight (at least as far as the mathematical public was concerned), with the appearance, in 1967, of the seminal monograph "Foundations of Constructive Analysis" [2]. This book was the fruit of an astonishingly fertile period of 2-3 years in which Errett Bishop (1928-1983) developed, single–handedly and more

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[3] Philosophic intuitionism is as robust as it ever was. Indeed, according to Dummett,

Of the various attempts made [in the early twentieth century] *to create over–all philosophies of mathematics providing, simultaneously, solutions to all the fundamental philosophical problems concerning mathematics, only the intuitionist system originated by Brouwer survives today as a viable theory to which, as a whole, anyone now could declare himself an adherent.* ([23], p. 1) or less from scratch, a large part of modern analysis by rigorously constructive methods. Founding his mathematics on a primitive, unspecified notion of *algorithm* and on the Peano properties of the natural numbers, and keeping strictly to the interpretation of "existence" as "computability", Bishop was able to develop his analysis in the style of the classical analyst (but, of course, with a different logic), without a commitment to either Brouwer's quasi-metaphysical intuitionistic principles or the recursive function theoretic formalism of the Markov school.

Bishop's refusal to make such commitments, and in particular to pin down his notion of algorithm, led to criticism, particularly from philosophers of mathematics and from those committed to Church's thesis (that all computable partial functions from $\mathbf{N}$ to $\mathbf{N}$ are recursive). But his imprecision about the nature of algorithms is precisely what gives Bishop's work so many possible interpretations: his results and proofs are valid, *mutatis mutandis*, in classical mathematics, intuitionism, and all reasonable models of computable mathematics—such as, for example, recursive function theory [33] or Weihrauch's TTE ([46], [47]).

Recently, Richman has advocated the view, based on his experience as a practitioner of constructive mathematics for more than twenty–five years, that Bishop's mathematics is simply mathematics with intuitionistic logic (see [37], [38]). This viewpoint, which is prefigured in [6], allows the constructive mathematician to work with *any* mathematical objects, not just those that are, in some sense, constructive. On the other hand, as a philosophy of constructive epistemology, it does not preclude the ontological possibility that, as Brouwer maintained, mathematical objects are mental constructs. We adopt Richman's point of view in the remainder of this paper.

Logic and Practice

When using intuitionistic logic, we must take into account that it differs substantially from classical logic. For example,

- $P \vee Q$ holds if and only if we have either a proof of P or a proof of Q;
- $\exists x\, P(x)$ holds if and only if we have an algorithm for constructing x, and one for verifying that $P(x)$ holds;
- $\forall x \in A\, P(x)$ holds if and only if we have an algorithm which, applied to an object x and the data that witness that x belongs to the set A, demonstrates that $P(x)$ holds;
- $P \Rightarrow Q$ holds if and only if we have an algorithm which converts any proof of P into one of Q;
- if $P(n)$ is a decidable property of natural numbers n,

$$\forall n P(n) \vee \neg \forall P(n)$$

need not hold;

- the **law of excluded middle**, $P \vee \neg P$, does not hold.

There are certain classically trivial weak forms of the law of excluded middle that also cannot be derived within intuitionistic logic and are therefore excluded from constructive mathematics. Among these are the so-called (by Bishop) *omniscience principles*:[4]

- **The limited principle of omniscience (LPO)**: *If* (a_n) *is a binary sequence, then either* $a_n = 0$ *for all* n *or else there exists* n *with* $a_n = 1$.
- **The lesser limited principle of omniscience (LLPO)**: *If* (a_n) *is a binary sequence with at most one term equal to* 1, *then either* $a_{2n} = 0$ *for all* n *or else* $a_{2n+1} = 0$ *for all* n.

Any classical proposition that implies either of these omniscience principles is regarded as essentially nonconstructive. For example, each of the following is equivalent to LPO.

- The **law of trichotomy**: $\forall x \in R(x < 0 \vee x = 0 \vee x > 0)$.
- The **least-upper-bound principle**: each nonempty subset of R that is bounded above has a least upper bound.
- Every real number is either rational or irrational.

We show how the last of these implies LPO. Let $(a_n)_{n=0}^{\infty}$ be an increasing binary sequence, and define a real number by

$$x = \sum_{n=0}^{\infty} \frac{1 - a_n}{n!}.$$

Suppose that either x is rational or x is irrational. In the first case $|x - \mathrm{e}| > 0$, so there exists N such that

$$\sum_{n=0}^{N} \left(\frac{1}{n!} - \frac{1 - a_n}{n!} \right) > 0;$$

whence $a_n = 1$ for some $n \leq N$. In the case where x is irrational, we clearly have $a_n = 0$ for all n.

Each of the following is equivalent to LLPO.

[4] Both of these omniscience principles are false if interpreted recursively, even with classical logic (see Chapter 3 of [15]).

- $\forall x \in R(x \geq 0 \vee x \leq 0)$.
- If $x, y \in R$ and $xy = 0$, then $x = 0$ or $y = 0$.
- The **Intermediate Value Theorem**: if $f : [0,1] \to \mathbf{R}$ is a continuous function with $f(0) < 0 < f(1)$, then there exists $x \in (0,1)$ such that $f(x) = 0$.

Fortunately, there exist constructive principles and substitutes that enable us to obtain many positive results without using the ejected principles. Here are some examples.

- If $a < b$, then for each $x \in \mathbf{R}$ either $x > a$ or $x < b$.[5]
- The **constructive least–upper–bound principle:** if S is a nonempty[6] subset of $\mathbf{R}$ that it is bounded above, and if for all α, β with $\alpha < \beta$,

$$\forall x \in S(x < \beta) \vee \exists x \in S(x > \alpha),$$

  then the supremum of S exists.
- If $x > 0$ is contradictory, then $x \leq 0$.
- A version of the Intermediate Value Theorem: if $f : [0,1] \to \mathbf{R}$ is a continuous function with $f(0) < 0 < f(1)$, then for each $\varepsilon > 0$ there exists $x \in (0,1)$ such that $|f(x)| < \varepsilon$ ([5], Ch. 2, Thm (4.8)).
- A second version of the Intermediate Value Theorem:[7] if $f : [0,1] \to \mathbf{R}$ is a continuous function with $f(0) < 0 < f(1)$, and if f is **locally nonzero** in the sense that for each $x \in (0,1)$ and each $r > 0$ there exists y with $|x - y| < r$ and $f(y) \neq 0$, then there exists $x \in (0,1)$ such that $f(x) = 0$ ([5], p. 63, Problem 15).

A more controversial omniscience principle is **Markov's Principle (MP):**

If (a_n) is a binary sequence for which it is contradictory that all terms be 0, then there exists n such that $a_n = 1$.

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[5] This principle is often used to split a proof into cases. Typically, we have a split of the type "either $|x| > 0$ or $|x| < \varepsilon$", where the positive number ε is chosen carefully to enable the second case of the split to progress towards the desired conclusion.

[6] By **nonempty** we mean that there exists—in Bishop's terms, we can construct—an element of S.

[7] There are other versions: see page 63 of [5].

This principle represents an unbounded search, and certainly holds in the classical recursive model. It is accepted, at times reluctantly, by the practitioners of Markov's recursive constructive mathematics, but is rejected by the intuitionists. (Indeed, it is inconsistent with Brouwer's controversial theory of the creating subject—see [23], [30], or [1].) Followers of Bishop normally regard Markov's Principle as unusable, and reject as essentially nonconstructive any proposition that implies it.

An inevitable question in discussions about constructive mathematics is "What about the axiom of choice?". The following result, which appears in a disguised form as Problem 2 on page 58 of [2], was published by Goodman and Myhill in 1978 ([28]):

> *The axiom of choice entails the law of excluded middle.*

To prove this, let P be any constructively meaningful statement and define the set $A = \{s, t\}$ together with the equality relation[8] given by

$$s = t \text{ if and only if } P \text{ holds.}$$

Consider now the set $B = \{0, 1\}$ with the standard equality, and let

$$S = \{(s, 0), (t, 1)\} \subset A \times B,$$

with the equality relation derived from those on A and B:

$$(x, y) = (x_1, y_1) \text{ if and only if } x = x_1 \text{ in } A \text{ and } y = y_1 \text{ in } B.$$

Assume that there exists a function $f : A \to B$ such that $(x, f(x)) \in S$ for all $x \in A$. If $f(s) = 1$ or $f(t) = 0$, then $s = t$ and hence P holds; if $f(s) = 0$ and $f(t) = 1$, then $\neg(s = t)$ and therefore $\neg P$ holds.

Another classical result that does not hold in constructive mathematics is the following.

> *For each complex number z there exists $\theta \in [0, 2\pi)$ such that*[9] $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$*, and such that if $\theta \neq 0$, then $z \neq 0$.*

[8] In constructive mathematics a set is not completely described unless it is provided with an equivalence relation capturing the notion of equality of elements.

[9] When we write $x \neq y$ for two elements x, y of a metric space (X, ρ), we mean that $\rho(x, y) > 0$. The statement

$$\forall x \in \mathbf{R}\,(\neg\,(x = 0) \Rightarrow x \neq 0)$$

is equivalent to Markov's Principle.

To show that this proposition entails LPO,[10] consider an increasing binary sequence (a_n) with at most one term equal to 1. Define a sequence (z_n) of complex numbers such that

$$\begin{aligned} a_n = 0 \quad &\Rightarrow \quad z_n = 0, \\ a_n = 1 - a_{n-1} \quad &\Rightarrow \quad z_k = \mathrm{e}^{\mathrm{i}\pi/2}/n \text{ for all } k \geq n. \end{aligned}$$

It it easily seen that (z_n) is a Cauchy sequence and therefore converges to a limit z in $\mathbf{C}$. Assume that $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, 2\pi)$, and that if $\theta \neq 0$, then $z \neq 0$. Either $\theta < \frac{\pi}{2}$ or $\theta > 0$. In the first case we have $\forall n(a_n = 0)$: for if we suppose that there exists n such that $a_n = 1 - a_{n-1}$, then $z = \mathrm{e}^{\mathrm{i}\pi/2}/n$ and therefore $\theta = \frac{\pi}{2}$, a contradiction. In the case $\theta > 0$ we have $z \neq 0$, so there exists N such that $z_N \neq 0$; then $a_n = 1$ for some $n \leq N$.

Now, it might be thought that the failure of the modulus–argument decomposition of a complex number would mean that there was no constructive proof of the existence of square roots in $\mathbf{C}$. For, in order to find a square root of z, we normally would write $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$ and then compute $\pm\sqrt{|z|}\mathrm{e}^{\mathrm{i}\theta/2}$. Although this method of finding a square root certainly will not work unless we already can decide that $z = 0$ or $z \neq 0$—which, in general, we cannot—there is a constructive proof of the existence of square roots, one that uses the completeness of $\mathbf{C}$.[11]

To see this, consider any complex number z, and note that for each positive integer n we have either $|z| < 1/n$ or $|z| > 1/(n+1)$. Thus we can successively construct the terms of an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_n = 0 \quad &\Rightarrow \quad |z| < 1/n, \\ \lambda_n = 1 \quad &\Rightarrow \quad |z| > 1/(n+1). \end{aligned}$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $\zeta_n = 0$; if $\lambda_n = 1 - \lambda_{n-1}$, choose $\theta \in [0, 2\pi)$ such that $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$, and set $\zeta_k = \sqrt{|z|}\mathrm{e}^{\mathrm{i}\theta/2}$ for all $k \geq n$. Then (ζ_n) is a Cauchy sequence; in fact, $|\zeta_m - \zeta_n| \leq 1/(n-1)$ whenever $m \geq n \geq 2$. Also, $|\zeta_n^2 - z| < 1/n$ for each n. Since $\mathbf{C}$ is complete, (ζ_n) converges to a limit $\zeta \in \mathbf{C}$ that satisfies $\zeta^2 = z$.

This technique of "flagging" alternatives by the terms of a binary sequence (λ_n), constructing an appropriate Cauchy sequence, and using the limit of that sequence to circumvent our inability to make the sort of decision embodied in LPO, LLPO, or Markov's Principle, is quite common in constructive mathematics. For example, it is used in the proofs of the following results.

- A compactly generated Banach space is finite–dimensional ([15], Ch. 2, (6.1)).

[10] A variation of this argument shows that if, for each $z \in \mathbf{C}$, there exists $\theta \in [0, 2\pi)$ such that $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$, then LLPO holds.

[11] Contrary to a common misunderstanding, both $\mathbf{R}$ and $\mathbf{C}$ are complete in constructive mathematics.

- A linear mapping T from a normed space X onto a Banach space Y is **well behaved**, in the sense that if $x \neq y$ for all $y \in \ker(T)$, then $Tx \neq 0$ [12].

- If f is a nonnegative integrable function that is positive throughout a set of positive measure, then $\int f > 0$ ([5], Ch. 6, (4.13)).

- Let F be a finite–dimensional subspace F of a real normed space X, and x an element of X such that $\max\{\|x-y\|, \|x-y'\|\} > 0$ whenever y, y' are distinct elements of F. Then x has a (perforce unique) closest point in F ([5], Ch. 7, (2.12)).[12]

A special case of the first of these results is that if $\mathbf{R}a = \{ax : x \in \mathbf{R}\}$ is closed, then $a = 0$ or $a \neq 0$. (The converse is clearly true.) A proof using the foregoing flagging technique is given in [8], but Fred Richman has shown us the following simpler argument.

Assume that $\mathbf{R}a$ is complete. For each $\varepsilon > 0$ we have either $\sqrt{|a|} < \varepsilon$ or else $|a| > 0$; in the latter case,

$$\sqrt{|a|} = \pm \frac{a}{\sqrt{|a|}} \in \mathbf{R}a.$$

Since $\varepsilon > 0$ is arbitrary, $\sqrt{|a|} \in \overline{\mathbf{R}a} = \mathbf{R}a$ and there exists r such that $\sqrt{|a|} = ra$. Chose a positive integer $N > r$. Either $|a| > 0$ or $|a| < 1/N^2$. In the latter case, if $a \neq 0$, then

$$|r|\,|a| = |ra| = \sqrt{|a|} = \frac{|a|}{\sqrt{|a|}} > N\,|a|\,,$$

so $|r| > N$, a contradiction; whence $a = 0$.

One advantage of this proof over the one that uses the flagging technique is that it does not require any form of choice principle—not even countable choice. Although most constructive mathematicians have no qualms about the use of countable, or even dependent, choice in their proofs, there are some—notably Richman [39]—who prefer to avoid choice altogether whenever possible.

The reader must not get the impression that constructive mathematicians spend their time proving, by amusing arguments, results that are classically trivial. Most of our attention is devoted to classically nontrivial matters, but, as the arguments we have given demonstrate, there are significant constructive problems, often requiring considerable ingenuity for their solution, at levels nearer the surface than those normally mined by the classical mathematician.

So far, we have concentrated on analysis. Turning to algebra, we first note that whereas, in classical algebra, the splitting field associated with a given polynomial is unique up to isomorphism, in the constructive approach the uniqueness of splitting

[12] This is a constructive substitute for, and is classically equivalent to, the classical theorem that each point in a real normed space has a closest point in any given finite–dimensional subspace.

fields for polynomials over countable discrete fields is equivalent to **LLPO**. Nevertheless, such a splitting field does exist ([36], p. 152).

Classically, a ring is (left) Noetherian if each of its left ideals is finitely generated; constructively, even the field $\mathbf{Z}_2$ fails to satisfy this definition! Indeed, if we have a finite set of generators for an ideal of $\mathbf{Z}_2$ generated by a binary sequence, we can decide whether or not the sequence has a term equal to 1.

It might be suspected that, since we cannot prove constructively that $\mathbf{Z}_2$ is Noetherian, there will be no constructive version of the Hilbert Basis Theorem. This suspicion is doubtless reinforced by recollection of the furore that arose after Hilbert's original, highly nonconstructive proof of that theorem. But the real constructive problem lies with the definition of "Noetherian". Mines et al. [36] define a ring R to be **Noetherian** if for each ascending chain

$$J_1 \subset J_2 \subset J_3 \subset \cdots$$

of finitely generated left ideals in R there exists n such that $J_n = J_{n+1}$. This definition of "Noetherian" is classically equivalent to the standard classical one (see, for example, page 35 of [21]), is satisfied by the ring $\mathbf{Z}_2$, and leads to the following constructive version of the **Hilbert Basis Theorem**:

If R is a coherent Noetherian ring, then so is $R[x]$,

where, as usual, $R[x]$ is the ring of polynomials over R. (Regarding "coherence", suffice it to say that it is a property that automatically holds for a Noetherian ring in classical mathematics.)

The Hilbert Basis Theorem illustrates the point that many—if not most—classical results can be re-cast, sometimes with additional hypothesis that are trivially satisfied in classical mathematics, into forms that are constructively provable; moreover, those forms are often classically equivalent to the original classical theorem. For another instance of this, see the result, stated earlier, about best approximations in finite–dimensional spaces.

Conclusion

In this short article we have tried to sketch (that is certainly the appropriate word here) the history and philosophy of constructive mathematics, and to fill in a few details of modern, Bishop-style constructive mathematics. In emphasising the differences between constructive and classical mathematics, we have inevitably run the risk of creating an impression that the former is an essentially negative matter, at best concerned with filling in minor gaps that appear when the classical notion of existence is replaced by a computational one. We must emphasise that this impression, common in many mathematical quarters, is false. Not only have there been substantial positive developments in analysis and algebra, starting with Bishop's remarkable work in the 1960s and continuing through to the present, but also there have arisen several research groups investigating the theoretical and practical implications of constructive mathematics for logic and computer programming. The pioneering work in this area was Martin-Löf's theory of types, which provided a formal system for Bishop's mathematics and led to the extraction of programs from constructive proofs, the proof of a theorem being, essentially, a guarantee that the extracted program meets its specification. Good references for such "proofs as programs" approaches are [35], [22], and [26]; see also [14].

A perhaps surprising area where constructive mathematics has found applications is in topos theory (see [24], [27]). A topos is a special kind of category which can be used as a foundation for mathematics (the category of sets is just one example of a topos), and with which one can associate a logic. It turns out that the natural logic for a topos is intuitionistic, rather than classical, and that, roughly speaking, if a theorem is proved constructively, without the use of dependent choice, in the setting of one topos, then it can be reinterpreted as a theorem in completely different topoi. For example, by proving a certain one–complex–variable theorem in one topos, and then reinterpreting it in another topos, it is possible to obtain a theorem in several complex variables [40].

Returning to the expanse of constructive mathematics proper, we present a list of some of the areas of mathematics that have been developed constructively over the last thirty–five years. Where no specific reference for a topic has been given, the reader should consult [2] or [5].

Real and complex analysis: Abstract measure and probability theory; the Jordan curve theorem [4]; Riemann mapping theorem; Picard's theorem [11].

Functional analysis: the Hahn–Banach and separation theorems; the Krein–Milman theorem; duality in Banach spaces (in particular, the L_p spaces); the Stone–Weierstrass theorem.

Hilbert space: the functional calculus and spectral theory for selfadjoint operators [13].

Partial differential equations: the existence and continuity–in–parameters of weak solutions to the Dirichlet Problem in $\mathbf{R}^N$ ([44], [16], [18]).

Haar measure and Fourier transforms on locally compact groups.

Elements of Banach algebra theory.

Approximation theory: Chebyshev approximation [7]; best approximations on Jordan curves [17].

Mathematical economics: the existence and continuity of utility functions representing preference relations; the existence of a demand function [9].

Algebra: groups, rings, fields; Galois theory; valuation theory [36].

Perhaps it is best to let the last words flow from the pen of a true pioneer and master of constructive mathematics:

> *We are not contending that idealistic mathematics is worthless from the constructive point of view. This would be as silly as contending that unrigorous mathematics is worthless from the classical point of view. Every theorem proved with idealistic methods presents a challenge: to find a constructive version, and to give it a constructive proof.* (Errett Bishop: [2], p. x)

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