CDMTCS Research Report Series

Large Cayley Graphs and Digraphs with Small Degree and Diameter

P. R. Hafner
Department of Mathematics
University of Auckland

CDMTCS-005
June 1995

Centre for Discrete Mathematics and
Theoretical Computer Science

LARGE CAYLEY GRAPHS AND DIGRAPHS WITH SMALL DEGREE AND DIAMETER

P. R. HAFNER

ABSTRACT. We review the status of the Degree/Diameter problem for both, graphs and digraphs and present new Cayley digraphs which yield improvements over some of the previously known largest vertex transitive digraphs of given degree and diameter.

1. INTRODUCTION

Interconnection networks (for example of computers, or of components on a microchip) can be modelled conveniently by graphs or digraphs depending on whether the communication between nodes is two-way or only one-way. In practice, such networks are subject to two fundamental restrictions: the number of connections that can be attached at any one node is limited, as is the number of intermediate nodes on the communications path between two nodes. We have arrived at the

Degree/Diameter Problem: find (di-)graphs of maximal order with given (in- and out-)degree Δ and diameter D.

In this paper we discuss this problem for undirected and directed graphs and present new Cayley digraphs which improve known results in the case of vertex transitive graphs.

2. NOTATION AND TERMINOLOGY

We will consider directed and undirected graphs Γ. The *distance* from a vertex x to a vertex y is the length of a shortest path from x to y. The set of all vertices of Γ whose distance from a vertex x equals i is denoted by $\Gamma_i(x)$. The *diameter* of the (di)graph Γ is the maximum of all distances between pairs of vertices of Γ. Graphs of degree Δ and diameter D are called (Δ, D) graphs (similarly for digraphs). A graph is said to be Δ-*regular* if all its vertices have degree Δ; a digraph is called Δ-regular if all its vertices have in- and outdegree Δ.

A (di)graph is called *vertex transitive* if its automorphism group is transitive on the set of vertices, a digraph is called *arc transitive* if its automorphism group is transitive on the set of arcs. In the context of networks, vertex transitive (di)graphs are advantageous because identical routing algorithms can be used at each vertex.

1991 *Mathematics Subject Classification.* 05C, 68M, 94C.

Examples of vertex transitive graphs and digraphs can be obtained by constructing the *Cayley (di)graph* of a group G relative to a generating set X. The elements of the group G form the set of vertices of the Cayley digraph $Cay(G,X)$; there is an arc from g to h if $h = gx$ for some $x \in X$. If the generating set X is closed under inversion we can consider the pairs of opposite arcs between adjacent vertices as undirected edges and refer to this as the Cayley graph of G relative to X. Algebraically, the diameter of $Cay(G,X)$ is the maximum number of terms required to write elements of G as words in the alphabet X.

As an aside we recall that the Petersen graph is vertex transitive but not a Cayley graph. Sabidussi [37] showed that all vertex transitive graphs can be obtained as *Cayley coset graphs* of groups relative to some set of generators. Similar results hold for digraphs [27].

We now define some elementary classes of groups which have been used in [22] to construct large graphs. First of all, there are metacyclic groups, i.e. semidirect products of cyclic groups: if the multiplicative order of the unit $a \in Z_n$ divides m, a semidirect product of Z_m with Z_n can be defined using the following multiplication:

$$[x,y][u,v] = [x + u \bmod m, ya^u + v \bmod n].$$

In Table 1 groups of this kind are identified by DH; our detailed listing in the appendix uses the symbol $m \times_a n$, e.g. $6 \times_3 28$.

Another useful type of groups are semidirect products of a cyclic group Z_m with a direct sum $Z_n \times Z_n$. An automorphism σ of $Z_n \times Z_n$ is determined by the images of the generators $\sigma([1,0]) = [x,y]$ and $\sigma([0,1]) = [z,t]$. If the order of σ divides m we can define a multiplication on $Z_m \times Z_n \times Z_n$ by:

$$[c,d,e][f,g,h] = [c + f \bmod m,\ [d,e] \begin{bmatrix} x & y \\ z & t \end{bmatrix}^f + [g,h] \bmod n].$$

In Table 1 we identify groups of this kind by DH^*; our detailed listing in the appendix uses the symbol $m \times_\sigma n^2$, e.g. $8 \times_\sigma 3^2$; the action of the cyclic group is not encoded into this symbol and is specified separately.

A further kind of group is indicated in Table 1 by DH^{**}. These are semidirect products of $G = m \times_a n$ with itself, where the action is by conjugation.

The *line digraph* $L(\Gamma)$ of a digraph Γ has the arcs of Γ as vertices; arcs of $L(\Gamma)$ correspond to walks of length 2 in Γ. If Γ is Δ-regular with n vertices then $L(\Gamma)$ is also Δ-regular and has Δn vertices [28, 23]. Clearly, iteration of this construction produces an infinite sequence of Δ-regular graphs.

3. The Undirected Case

The order n of a graph Γ with diameter D and maximum degree Δ satisfies the inequality

$$n \le 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1}. \tag{1}$$

Graphs for which equality holds in (1) are called *Moore Graphs.* After distinguishing a vertex x in a Moore graph, any vertex in $\Gamma_i(x), 0 < i < D$, is adjacent only to vertices in $\Gamma_{i-1}(x)$ and $\Gamma_{i+1}(x)$, while vertices in $\Gamma_D(x)$ are adjacent only to vertices in $\Gamma_{D-1}(x)$ and $\Gamma_D(x)$. In other words: after the edges between vertices in $\Gamma_D(x)$ are removed, the Moore graph Γ becomes a tree whose internal vertices have degree Δ. Inequality (1) with '$\leq$' replaced by '$\geq$' applies to graphs of order n with maximum degree Δ and odd girth $2D+1$, so that Moore graphs appear also as solutions of the extremal problem associated with given girth $2D+1$ and given maximum degree Δ.

There are only very few Moore graphs [31, 4, 16]:

Δ	D	n	Description
2	D	$2D+1$	$(2D+1)$-gon
3	2	10	Petersen
7	2	50	Hoffman-Singleton
57	2	3250	?

It is not known if there exists a graph with $\Delta = 57$, $D = 2$ and $n = 3250$. Aschbacher [1] showed that a (57, 2)-graph of order 3250 cannot be distance-transitive.

The only graph whose order differs from the Moore bound by 1 is the square [5, 25]. This implies that the entries $(3, 3)$, $(4, 2)$, and $(5, 2)$ in Table 1 are indeed optimal, a fact which goes back to Elspas [24]. More recently it has been shown that for $\Delta = 3$, $D \geq 4$ the Moore bound cannot be missed by 2 [32].

Table 1 is an update of [9] and contains the orders of the largest known (Δ, D) graphs with annotations indicating the nature of the graphs. Comparison of the Moore bound with the orders listed in the table shows that, mostly, the Moore bound is missed by a considerable margin.

Graphs appearing in Table 1

$2cy$	connections between two cycles [3]
Allwr	graphs found by Allwright [2]
Cam	Cayley graphs of linear groups [12]
CR^*	chordal rings found by Quisquater [36]
vC	compound graphs by von Conta [40]
DH, DH^*, DH^{**}	Cayley graphs of metacyclic and related groups [22]
Dinn	Cayley graphs found by Dinneen [21]
C_n	cycle on n vertices
GFS	graph by Gómez, Fiol and Serra [29]
H_q	incidence graph of a regular generalized hexagon [6]
HS	Hoffman-Singleton graph
K_n	complete graph
Lente	graph designed by Lente, Univ. Paris Sud, France
P	Petersen graph
P_q	incidence graph of a projective plane [30]
Q_q	incidence graph of a regular generalized quadrangle [6]
T	tournament

Δ \ D	2	3	4	5	6	7	8	9	10
3	P 10	$C_5 * F_4$ 20	vC 38	vC 70	GFS 130	CR^* 184	CR^* 320	$2cy$ 540	$2cy$ 938
4	$K_3 * C_5$ 15	$Allwr$ 41	$C_5 * C_{19}$ 95	H_3' 364	$H_3(K_3)$ 740	DH 1 155	DH^{**} 3 025	DH 7 550	DH 16 555
5	$K_3 * X_8$ 24	$Lente$ 70	$Q_4(K_3)$ 186	$H_3'd$ 532	$H_4(K_3)$ 2 754	DH 5 334	DH 15 532	DH 49 932	DH 145 584
6	$K_4 * X_8$ 32	$C_5 * C_{21}$ 105	DH^* 360	DH 1 230	$H_5(K_4)$ 7 860	DH 18 775	DH 69 540	DH 275 540	DH 945 574
7	HS 502	DH^* 144	DH^* 600	DH 2 756	$H_4(K_4) < H_5$ 10 566	DH 47 304	DH 214 500	DH 945 574	Cam 4 773 696
8	P_7' 57	DH 234	DH 1 012	DH^* 4 704	$H_7(K_6)$ 39 396	DH 127 134	DH 654 696	DH^{**} 2 408 704	Cam 7 738 848
9	$P_8'd$ 57	Q_8' 585	DH 1 430	DH 7 344	$H_8(K_6)$ 75 198	DH 264 024	DH^{**} 1 354 896	DH 4 980 696	Cam 19 845 936
10	P_9' 91	$Q_8'd$ 650	DH 2 200	DH^* 12 288	$H_9(K_6)$ 133 500	DH 554 580	DH^{**} 3 069 504	DH 9 003 000	$Q_7\Sigma_2H_7$ 47 059 200
11	$P_9'd$ 94	$Q_8'd$ 715	$Q_7(T_4)$ 3 200	DH 17 458	$H_7(T_4)$ 156 864	DH 945 574	Cam 4 773 696	Cam 25 048 800	$Q_7\Sigma_6H_8$ 179 755 200
12	P_{11}' 133	$Q_8'd$ 780	$Q_8' * X_8$ 4 680	DH 26 871	$H_{11}(K_6)$ 355 812	$Dinn$ 1 732 514	DH 10 007 820	DH 48 532 122	$Q_8\Sigma_6H_9$ 466 338 600
13	$P_{11}'d$ 136	$Q_8'd$ 845	$Q_9(T_4)$ 6 560	DH 37 056	$H_9(T_4)$ 531 440	Cam 2 723 040	DH 15 027 252	DH 72 598 920	$Q_9\Sigma_6H_9$ 762 616 400
14	P_{13}' 183	$Q_8'd$ 910	$Q_9(T_5)$ 8 200	DH 53 955	$H_{13}(K_7)$ 806 636	$K_1\Sigma_8H_{11}$ 6 200 460	$Dinn$ 29 992 052	$P_9\Sigma_7H_{11}$ 164 755 080	$Q_8\Sigma_6H_{11}$ 1 865 452 680
15	$P_{13}'d$ 186	$(\otimes Q_{2,4})'$ 1 215	$Q_{11}(T_4)$ 11 712	DH 69 972	$H_{11}(T_4)$ 1 417 248	DH 7 100 796	DH 38 471 006	$P_{11}\Sigma_1H_{11}$ 282 740 976	$Q_{11}\Sigma_6H_{11}$ 3 630 989 376

TABLE 1. Largest known undirected (Δ, D) graphs

Operations on graphs used in Table 1

$G * H$	twisted product of graphs [7]
Gd	duplication of some vertices of G [20]
B'	quotient of a bipartite graph B by a polarity [19]
$B(K)$	substitution of vertices of a bipartite graph B by complete graphs K [15]
$B(K) < B$	compound of $B(K)$ and a bipartite graph B and a tournament T [29]
$\otimes B$	the component with polarity of the cartesian product of a bipartite graph B with itself [18]
$G\Sigma_i H$	various compounding operations [29]

4. The Directed Case

The order n of a digraph Γ with diameter D and maximum degree Δ satisfies the inequality

$$n \leq 1 + \Delta + \Delta^2 + \cdots + \Delta^D. \tag{2}$$

As in the undirected case, the right hand side in inequality (2) is known as the *Moore bound* (because it derives from a similar tree model). The only cases when equality holds in (2) are $\Delta = 1$ or $D = 1$ [35, 11]. Table 2 contains the orders of the largest known (Δ, D) digraphs. Almost all entries correspond to *Kautz digraphs* $K(\Delta, D)$ whose order is $\Delta^D + \Delta^{D-1}$. The vertices of a Kautz digraph are words $x_1 x_2 \cdots x_D$ of length D with $x_i \neq x_{i+1}$ in an alphabet of $\Delta + 1$ letters; arcs go from $x_1 x_2 \cdots x_D$ to $x_2 \cdots x_D y$. These digraphs can be obtained from the complete digraph on $\Delta + 1$ vertices (no loops) by line digraph iteration. For $i \geq 4$ the $(2, i)$ digraphs in Table 3 are obtained by line digraph iteration from a $(2, 4)$ digraph on 25 vertices found by computer search [28]. No improvements have been made on this list since 1984.

$\Delta \setminus D$	2	3	4	5	6	7	8
2	6	12	25	50	100	200	400
3	12	36	108	324	972	2 916	8 748
4	20	80	320	1 280	5 120	20 480	81 920
5	30	150	750	3 750	18 750	93 750	468 750
6	42	252	1 512	9 072	54 432	326 592	1 959 552
7	56	392	2 744	19 208	134 456	941 192	6 588 344

Table 2. Orders of largest known (Δ, D) digraphs

Recent studies have also focussed on the degree/diameter problem for vertex symmetric digraphs [26, 21, 27, 14]. In Table 3 we collect the current state of this problem, highlighting our new results by bold numbers. Details of the new digraphs are given in the appendix.

Δ \ D	2	3	4	5	6	7	8	9	10
2	K 6	FM 10	FM 20	$\mathbf{Z}\times_\sigma\mathbf{Z}$ 27	$\mathbf{Z}\times_\sigma\mathbf{Z}^2$ **72**	LD **144**	FM 171	FM 336	FM 504
3	K 12	$\mathbf{Z}\times_\sigma\mathbf{Z}$ 27	FM 60	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **165**	$\mathbf{Z}\times_\sigma\mathbf{Z}$ 333	$2G^2$ 1 152	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **1 860**	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **4 446**	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **10 849**
4	K 20	Γ 60	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **168**	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **444**	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **1260**	$2G^2$ 7 200	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **12 090**	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **38 134**	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **132 012**
5	K 30	Γ 120	Γ 360	$2G^2$ 1 152	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **3 582**	$2G^2$ 28 800	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **54 505**	$2G^2$ 259 200	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **752 914**
6	K 42	Γ 210	Γ 840	Γ 2 520	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **7 776**	$2G^2$ 88 200	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **170 898**	$2G^2$ 1 411 200	$3G^3C$ 5 184 000
7	K 56	Γ 336	Γ 1680	Γ 6 720	Γ 20 160	$2G^2$ 225 792	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **521 906**	$2G^2$ 5 644 800	$3G^3C$ 5 184 000
8	K 72	Γ 504	Γ 3 024	Γ 15 120	Γ 60 480	$2G^2$ 508 032	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **1 371 582**	$2G^2$ 18 289 152	$3G^3C$ 113 799 168
9	K 90	Γ 720	Γ 5 040	Γ 30 240	Γ 151 200	$2G^2$ 1 036 800	$\mathbf{Z}\times_\sigma\mathbf{Z}$ **2 965 270**	$2G^2$ 50 803 200	$3G^3C$ 384 072 192
10	K 110	Γ 990	Γ 7920	Γ 55 400	Γ 332 640	$2G^2$ 1 960 220	Γ 6 652 800	$2G^2$ 125 452 800	$3G^3C$ 1 119 744 000

TABLE 3. Largest known vertex symmetric (Δ, D) digraphs

The new results presented here have been produced by computer search. Typically, a group G is selected and a set of Δ generators chosen at random, followed by computation of the diameter of the resulting Cayley digraph. Vertex transitivity allows to restrict the computation to distances from the identity element. The output of such a run might look as follows ($\Delta = 4$, order 168):

```
dist  0:  new      1; total      1
dist  1:  new      4; total      5
dist  2:  new     16; total     21
dist  3:  new     55; total     76
dist  4:  new     86; total    162
dist  5:  new      6; total    168
```

Note that in the first few steps the number of elements produced is 1, Δ, Δ^2, but then the geometric progression stops in this example. The small number of elements with distance 5 might create the hope that a 'better' choice of generators would lead to a digraph of diameter 4 on 168 vertices.

The annotations in Table 3 have the following meaning:

Graphs appearing in Table 3

FM	digraph found by computer search by Faber and Moore [26]
Γ	digraph on permutations $\Gamma_\Delta(D)$ [26]
K	Kautz digraph [33, 34]
LD	line digraph of an arc symmetric digraph
nG^n	digraph composition [14]
nG^nC	generalized digraph composition [14]
$\mathbf{Z}\times_\sigma\mathbf{Z}$	digraphs built from semi-direct products of cyclic groups ([21] and present paper)
$\mathbf{Z}\times_\sigma\mathbf{Z}^2$	arc symmetric Cayley graph described in present paper

In addition to the new entries labelled $\mathbf{Z}\times_\sigma\mathbf{Z}$ there are two new ones of degree 2: a Cayley digraph of order 72, degree 2 and diameter 6 is obtained as Cayley graph of the 2-generator group

$$G = \langle x, y \mid xyxy = yxyx = xy^6x^{-2}y^{-3} = yx^6y^{-2}x^{-3} = 1\rangle.$$

It is not hard to see that this graph is arc-transitive, and therefore its line digraph is vertex transitive, providing the entry $(2, 7)$ of order 144. The group was found as a semidirect product of Z_8 acting on $Z_3 \times Z_3$.

5. Remarks

1. Semidirect products of cyclic groups are abundant and therefore a good hunting ground for Cayley graphs. We note that these graphs are also competitive with regard to *average distance* and can improve the results in [38].

2. Most computations were done by programs written in C; results for smaller orders were verified using the computer algebra package CAYLEY [13]. This package was also used to produce some auxiliary files.

3. In the drive for improved results, the following 'trick' was occasionally successful. When a group turned out to be a good candidate for a pair (Δ, D) by producing a good number of 'near misses' (as exemplified in section 4) the generators involved in these cases were collected and later the sampling of generators restricted to the pool of collected elements. No statistical analysis of this phenomenon is available, but the successes came as a surprise.

6. Acknowledgments

The author is grateful to the University of Auckland Research Committee for a grant supporting research reported in this paper. Thanks are also due to F. Comellas, M. A. Fiol, and M. J. Dinneen for sharing their tables of large digraphs.

Appendix A. Groups and Generators for New Cayley Digraphs

For an explanation of the entries in the column headed 'Group' refer to section 2.

(Δ, D)	Order	Group	Generators	Order of Generator
(2, 6)	72	$8 \times_\sigma 3^2$	[1 1 2]	8
		[1 0] → [1 1]	[3 2 2]	8
		[0 1] → [1 0]		
(3, 5)	165	$5 \times_4 33$	[4 17]	15
			[4 4]	15
			[2 21]	5
(3, 8)	1 860	$12 \times_{88} 155$	[8 108]	15
			[1 93]	12
			[11 68]	12
(3, 9)	4 446	$18 \times_4 247$	[12 50]	39
			[7 125]	18
			[10 231]	9
(3, 10)	10 849	$19 \times_{407} 571$	[2 19]	19
			[7 480]	19
			[15 502]	19
(4, 4)	168	$6 \times_3 28$	[4 3]	12
			[0 12]	7
			[1 22]	6
			[1 3]	6
(4, 5)	444	$12 \times_8 37$	[5 33]	12
			[11 25]	12
			[10 17]	6
			[3 18]	4
(4, 6)	1 260	$12 \times_2 105$	[9 87]	28
			[4 89]	15
			[7 8]	12
			[10 45]	6
(4, 8)	12 090	$30 \times_4 403$	[5 165]	186
			[12 285]	65
			[1 92]	30
			[22 39]	15
(4, 9)	38 134	$46 \times_{180} 829$	[15 507]	46
			[18 276]	23
			[6 637]	23
			[22 542]	23
(4, 10)	132 012	$36 \times_{1593} 3667$	[23 1710]	36
			[26 3100]	18
			[14 707]	18
			[15 2346]	12
(5, 6)	3 582	$18 \times_{37} 199$	[5 13]	18
			[16 53]	9
			[8 123]	9
			[14 34]	9
			[15 110]	6

(Δ, D)	Order	Group	Generators	Order of Generator
(5, 8)	54 505	$55 \times_{512} 991$	[17 201]	55
			[43 430]	55
			[49 898]	55
			[27 951]	55
			[9 528]	55
(5, 10)	752 914	$194 \times_{2069} 3881$	[183 1044]	194
			[30 1822]	97
			[184 1253]	97
			[188 1265]	97
			[160 2480]	97
(6, 6)	7 776	$24 \times_{\sigma} 18^2$	[7 2 8]	24
		[1 0] → [0 1]	[23 8 6]	24
		[0 1] → [7 16]	[13 7 5]	24
			[5 4 5]	24
			[2 14 5]	12
			[6 17 5]	4
(6, 8)	170 898	$78 \times_{1236} 2191$	[48 79]	91
			[66 76]	91
			[17 998]	78
			[7 872]	78
			[40 1389]	39
			[34 1491]	39
(7, 8)	521 906	$154 \times_{700} 3389$	[139 798]	154
			[71 2968]	154
			[50 3016]	77
			[48 1301]	77
			[33 2433]	14
			[70 2042]	11
			[56 1956]	11
(8, 8)	1 371 582	$414 \times_{408} 3313$	[305 241]	414
			[95 1838]	414
			[193 353]	414
			[287 650]	414
			[224 138]	207
			[102 3186]	69
			[330 2254]	69
			[153 1940]	46
(9, 8)	2 965 270	$770 \times_{32} 3851$	[419 1605]	770
			[431 3461]	770
			[194 3715]	385
			[514 1381]	385
			[334 943]	385
			[296 304]	385
			[755 2906]	154
			[623 3338]	110
			[60 733]	77

References

1. M. Aschbacher, "The nonexistence of rank three permutation groups of degree 3250 and subdegree 57," *J. Algebra*, **19** (1971), 538–540.
2. J. Allwright, "New (Δ, D) graphs discovered by heuristic search," *Discrete Appl. Math.*, **37/38** (1992), 3–8.
3. R. Bar-Yehuda and T. Etzion, "Connections between two cycles – a new design of dense processor interconnection networks," *Discrete Appl. Math.*, **37/38** (1992), 29–43.
4. E. Bannai and T. Ito, "On finite Moore graphs," *J. Fac. Sci. Univ. Tokyo*, **20** (1973), 191–208.
5. E. Bannai and T. Ito, "Regular graphs with excess one," *Discrete Math.*, **37** (1981), 147–158.
6. C. T. Benson, "Minimal regular graphs of girth eight and twelve," *Canad. J. Math.*, **18** (1966), 1091–1094.
7. J.-C. Bermond, C. Delorme and G. Farhi, "Large graphs with given degree and diameter. II," *J. Combin. Theory Ser. B*, **36** (1984), 32–48.
8. J.-C. Bermond, C. Delorme and J.-J. Quisquater, "Strategies for interconnection networks: some methods from graph theory," *Journal of Parallel and Distributed Computing*, **3** (1986), 433–449.
9. J.-C. Bermond, C. Delorme and J.-J. Quisquater, "Table of large (Δ, D)-graphs," *Discrete Appl. Math.*, **37/38** (1992), 575–577.
10. J. Bond, C. Delorme and W.F. de La Vega, "Large Cayley graphs with small degree and diameter," *Rapport de Recherche no. 392*, LRI, Orsay, 1987.
11. W. G. Bridges and S. Toueg, "On the impossibility of directed Moore graphs," *J. Combin. Theory*, **29** (1980), 339–341.
12. L. Campbell, "Dense group networks," *Discrete Appl. Math.*, **37/38** (1992), 65–71.
13. J. Cannon and W. Bosma, *CAYLEY. Quick Reference Guide*, Sydney, October 1991.
14. F. Comellas and M. A. Fiol, "Vertex symmetric digraphs with small diameter," *to appear.*
15. F. Comellas and J. Gómez, "New large graphs with given degree and diameter," *7th Internat. Conference on Graph Theory*, Kalamazoo, June 1992.
16. R. Damerell, "On Moore graphs," *Proc. Cambridge Phil. Soc.*, **74** (1973), 227–236.
17. C. Delorme, "Grands graphes de degré et diamètre donnés," *European J. Combin.*, **6** (1985), 291–302.
18. C. Delorme, "Large bipartite graphs with given degree and diameter," *J. Graph Theory*, **9** (1985), 325–334.
19. C. Delorme, "Examples of products giving large graphs with given degree and diameter," *Discrete Appl. Math.*, **37/38** (1992), 157–167.
20. C. Delorme and G. Farhi, "Large graphs with given degree and diameter. I," *IEEE Trans. Computers*, **C-33** (1984), 857–860.
21. M. J. Dinneen, *Algebraic Methods for Efficient Network Constructions*, Master's Thesis, Department of Computer Science, University of Victoria, Victoria, B.C., Canada, 1991.
22. M. J. Dinneen and P. R. Hafner, "New results for the degree/diameter problem", *Report Series No. 265*, Department of Mathematics and Statistics, University of Auckland, August 1992.
23. D.-Z Du, Y.-D. Lyuu and D. F. Hsu, " Line digraph iteration and the spread concept—with application to graph theory, fault tolerance, and routing," *Graph-theoretic concepts in Computer Science* (G. Schmidt, R. Berhammer, eds.), LNCS 570, Springer Verlag (1992).
24. B. Elspas, "Topological constraints on interconnection-limited logic," *Proc. 5th Ann. Symp. Switching Circuit Theory and Logic Design* (1964),133–147.
25. P. Erdős, S. Fajtlowicz and A. J. Hoffman, "Maximum degree in graphs of diameter 2," *Networks*, **10** (1980), 87–90.

26. V. Faber and J. W. Moore, "High-degree low-diameter interconnection networks with vertex symmetry: the directed case," *Technical Report LA-UR-88-1051*, Los Alamos National Laboratory, Los Alamos, New Mexico, 1988.
27. V. Faber, J. W. Moore and W. Y. C. Chen, "Cycle prefix digraphs for interconnection networks," *submitted*.
28. M. A. Fiol, L. A. Yebra and I. Alegre de Miquel, "Line digraph iteration and the (d, k) digraph problem," *IEEE Trans. Computers*, **C-33** (1984), 400–403.
29. J. Gómez, M. A. Fiol and O. Serra, "On large (Δ, D)-graphs," *Discrete Math.*, to appear.
30. W. H. Haemers, "Eigenvalue techniques in design and graph theory," Mathematical Centre Tracts 121, Mathematisch Centrum, Amsterdam, 1980.
31. A. J. Hoffman and R. R. Singleton, "On Moore graphs with diameters 2 and 3," *IBM J. Res. Develop.*, **64** (1960), 15–21.
32. L. K. Jørgensen, "Diameters of cubic graphs", *Discrete Appl. Math.*, **37/38** (1992), 347–351.
33. W. H. Kautz, "Bounds on directed (d, k) graphs," *Theory of cellular logic networks and machines,* AFCRL-68-0668, SRI Project 7258, Final Report, pp. 20–28 (1968).
34. W. H. Kautz, "Design of optimal interconnection networks for multiprocessors, *Architecture and design of digital computers,* Nato Advanced Summer Institute, 249–272 (1969).
35. J. Plesník and Š. Znám, "Strongly geodetic directed graphs," *Acta Fac. Rer. Nat. Univ. Comen., Math.*, **29** (1974), 29–34.
36. J.-J. Quisquater, "Structures d'interconnection: constructions et applications," Thèse d'état, LRI, Orsay Cedex (1987).
37. G. Sabidussi, "Vertex transitive graphs," *Monatsh. Math.*, **68** (1969), 426–438.
38. S. T. Dchibell and R. M. Stafford, "Processor interconnection networks from Cayley graphs," *Discrete Appl. Math.*, **40** (1992), 333-357.
39. R. Storwick, "Improved construction techniques for (d, k) graphs," *IEEE Trans. Computers*, **C-19** (1970), 1214–1216.
40. C. von Conta, "TORUS and other networks as communication networks with up to some hundred points," *IEEE Trans. Computers*, **C-32** (1983), 657–666.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF AUCKLAND, PRIVATE BAG 92019, AUCKLAND, NEW ZEALAND

E-mail address: hafner@mat.aukuni.ac.nz