Pergamon

Stud. Hist. Phil. Mod. Phys., Vol. 32, No. 1, pp. 101–111, 2001

Printed in Great Britain
1355-2198/01 $ - see front matter

ESSAY REVIEW

Facets of Quantum logic

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K. Svozil, *Quantum Logic* (Springer Series in Discrete Mathematics and Computer Science) (Singapore: Springer, 1998), vi + 214 pp., ISBN 981-4021-07-5.

1. Four Attitudes Towards Quantum Logic

Since its first appearance in the works of von Neumann (1932) and Birkhoff and von Neumann (1936) in the late twenties and mid-thirties, quantum logic has become a vast, mixed field lying at the crossroads of and drawing on the methods of physics, mathematics, logic and philosophy. The approaches to and the interpretations of quantum logic have become very diverse in the past sixty years with literally thousands of papers and dozens of monographs in the field (see Pavicic's (1992) bibliography on quantum logic). One of the recent works on quantum logic is Svozil's book. In this essay I review Svozil's book by relating its content to the following four groups of major themes that comprise what came to be called quantum logic:

The attitude of the algebraist: non-distributive (in particular: orthomodular) lattices form a fascinating class of lattices, with a number of technically non-trivial problems. For the algebraist 'quantum logic' is just an exotic name for an ordinary, well-behaved and well-defined mathematical structure, which is part of a well-established branch of mathematics (lattice theory).

The viewpoint of the measure theorist/probabilist: a non-distributive lattice with additive real-valued maps on it is a natural generalisation of classical

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PII: S1355-2198(00)00013-7

measure theory. For the quantum *probabilist* in particular, quantum logic with the quantum states defined on it is just a non-classical (i.e. non-commutative) probability theory, and the challenge is to develop it to a full-fledged probability theory, complete with non-commutative integration and other non-commutative versions of the classical concepts.

The approach of the logician: in algebraic logic one transforms logical notions and problems into algebraic ones, whereby the investigation of logic and its properties gets subsumed under the authority of algebra. In this approach quantum logic appears as a logic which is determined semantically by a particular algebraic structure.

The perspective of the philosopher: the philosopher would like to understand how the above mentioned different aspects of quantum logic hang together, and whether any of the above attitudes helps in clarifying interpretational issues concerning quantum mechanics.

These four attitudes define four interdependent fields of research and monographs dealing with quantum logic are typically mixtures with different weights of these four 'pure' attitudes. Svozil's work is rather a superposition than a mixture of the above mentioned attitudes since the algebraic, measure theoretic, logical and philosophical topics are treated in it simultaneously.

I will discuss these attitudes, in order, in the following sections. Suffice it to say, by way of introduction, that the book consists of ten chapters (plus a mathematical Appendix). There does not seem to be a discernible logical structure behind the partition represented by the chapters; in fact, I find the logic of the presentation sometimes confusing: occasionally, and without a real didactic or conceptual advantage, the same topic disappears and re-surfaces several times in different chapters. For instance the issue of embeddability of an orthomodular lattice into a Boolean algebra, and the Kochen–Specker theorem in particular, are taken up and dropped in Section 6.2, and in Chapters 7, 8 and 9; also, the description of the algebraic properties of the Hilbert space lattice are scattered over different chapters in the book (in Chapter 2, Chapter 4 and Appendix A); hence the reader does not get a crisp, comprehensive picture of the elementary properties of a Hilbert space lattice by reading only Chapter 4 (entitled: 'Hilbert Lattices').

Svozil has chosen a semi-formal way of presenting the material: the mathematical definitions, statements and claims are not spelled out in the book in a technically and notationally formal manner and proofs are typically not given. This method has both advantages and drawbacks. The advantage is that the reader is not forced to digest a lot of technical notation, the disadvantage is that it is difficult to avoid ambiguities; indeed there remain a few in the text (see the examples in Section 2 below). On the other hand, it is a very attractive feature of the book that it presents many examples of lattices highlighting specific features of non-distributivity by using many excellently drawn diagrams (both Hasse and Greechie) of a number of lattices.

2. Algebras, Lattice Theory and Quantum Logic

The core observation which the standard concept of quantum logic is based on is the fact that the set $\mathscr{P}(\mathscr{H})$ of all closed linear subspaces of a (complex, finite or infinite dimensional) Hilbert space $\mathscr{H}$ is an (atomic, atomistic, irreducible) complete, orthomodular lattice with respect to the set theoretical inclusion as partial ordering $\subseteq = \leqslant$, set theoretical intersection as greatest lower bound $\cap = \wedge$, closure of the linear sum as least upper bound ('union') $A \vee B$ and the orthogonal $A^{\perp}$ complement as orthocomplementation $A \mapsto A^{\perp}$. Orthomodularity of the lattice means that the following equation holds:

$$\text{if } A \leqslant B \text{ and } A^{\perp} \leqslant C, \text{then } A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C). \tag{1}$$

Orthomodularity is a weakening of the following *distributivity* law (which is *not* valid in $\mathscr{P}(\mathscr{H})$):

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C) \quad \text{for all } A, B, C. \tag{2}$$

But the orthomodularity property is not the minimal weakening of distributivity: the following *modularity* property

$$\text{if } A \leqslant B, \text{then } A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C) \tag{3}$$

is stronger than orthomodularity. It is not difficult to prove (see p. 44 for a partial proof) the highly significant fact that $\mathscr{P}(\mathscr{H})$ is modular if and only if $\mathscr{H}$ is *finite-dimensional* (we write $(\mathscr{H}_n)$ for the Hilbert space of dimension n).

Hilbert lattices are not the only examples of orthomodular lattices: a particularly rich source of such lattices are the projection lattices $\mathscr{P}(\mathscr{N})$ of von Neumann algebras $\mathscr{N}$. Hilbert lattices are in fact special types of von Neumann lattices: $\mathscr{P}(\mathscr{H}) = \mathscr{P}(\mathscr{N})$ with $\mathscr{N}$ being the von Neumann algebra $\mathscr{B}(\mathscr{H})$ of *all* bounded operators on $\mathscr{H}$.

Non-Boolean lattices also emerge as finite *automaton logics or partition logics*, which is the topic of Chapter 10 ('Quasi-classical Analogies') of Svozil's book (see also Svozil (1998)). Mathematically considered, automaton logics are just pastings of Boolean algebras $\mathscr{B}_E$ generated by partitions E of a set $\mathscr{S}$ (a partition E of $\mathscr{S}$ is a family $\{m_j\}$ of sets $m_j \subseteq \mathscr{S}$ such that $m_i \cap m_j = 0\,(i \neq j)$ and $\cup_i m_i = \mathscr{S}$); given a family of partitions **B**, 'the pasting of the Boolean algebras $\mathscr{B}_E, E \in \mathbf{B}$ on the atomic level is called a partition logic, denoted by $(\mathscr{S}, \mathbf{B})$' (pp. 152–153 and Appendix A.4.2). Automaton logics emerge by taking $\mathscr{S}$ to be the *state space* of an *automaton*, which is a black-box machine that for any input i delivers an output o that can be given either by a computable function λ on $\mathscr{S}$, $\lambda(i) = o$, (a *Moore* automaton, (p. 148)) or by a computable function that also depends on the internal state i of the machine: $\lambda(s, i) = o$. The latter type of automaton is called a *Mealy* automaton (p. 148). The partition of $\mathscr{S}$ is then given by the inverse map λ^{-1}, using partitions of the output and input-output sets. Svozil formulates several claims concerning automaton logics, among them the following ones:

1. '[…] every partition logic corresponds to an automaton logic and *vice versa* (p. 153).
2. '[…] the set of two-valued probability measures on any automaton logics is separating. That is,
3. automaton logics *can* be embedded into classical Boolean algebras, whereas certain quantum logics cannot' (p. 157). In particular 'Kochen–Specker configurations cannot be realized' as automaton logics (p. 157).
4. '[…] all finite subalgebras of finite dimensional Hilbert logics can be obtained by automaton partition logics' (p. 167). ('A subalgebra of an orthocomplemented lattice is a subset which is closed under the operations $\perp$, $\wedge$ and $\vee$ and which contains 0 and 1', i.e. a subalgebra of $\mathscr{P}(\mathscr{H}_n)$ is a sub-orthocomplemented lattice of $\mathscr{P}(\mathscr{H}_n)$ (p. 187)).

The above claims entail the interesting and surprising consequence (not spelled out in Svozil's work) that only an *infinite* sub-orthocomplemented lattice of $\mathscr{P}(\mathscr{H}_n)$ can contain a Kochen–Specker type *finite* partial Boolean subalgebra of $\mathscr{P}(\mathscr{H}_n)$, where a partial algebra is called *Kochen–Specker type* if it cannot be embedded into a Boolean algebra by a partial Boolean algebra homomorphism (see below).

Chapter 10 is the most original in the book; it contains the author's own contribution to the field. This chapter gives many examples of automaton logics and illustrates them by diagrams. A wide range of other deep issues—in connection with both automaton logics and with quantum mechanics in general—also pop up in this chapter's text, especially in Section 10.2.6 (such as logical, computational and physical reversibility and irreversibility, no-cloning theorems, quantum computing and modelling the measurement process). The treatment of these latter issues is rather sketchy, however.

About one fifth of Svozil's book is devoted to reviewing results related to the existence of certain embeddings of non-distributive lattices into Boolean lattices and Boolean algebras. Since the null space of a lattice homomorphism h from $\mathscr{P}(\mathscr{H})$ into a Boolean algebra is a non-trival prime ideal in $\mathscr{P}(\mathscr{H})$ and since there exist no such ideals in $\mathscr{P}(\mathscr{H})$, a Hilbert lattice cannot be mapped into a Boolean algebra by a lattice homomorphism h, whether injective (= embedding) or not. This leads to the question of what weakening of the lattice homomorphism property of h permits the embedding by h of an orthomodular lattice $\mathscr{L}$ into a Boolean algebra. In his chapter 'What Price Value Definiteness?' Svozil reviews the possible weakenings. A classic result of Kochen and Specker (1967) shows that relaxing the lattice homomorphism property of h by requiring it to be only a partial Boolean algebra homomorphism is not enough: there exists no partial algebra homomorphism h (injective or not) from the partial Boolean algebra $\mathscr{P}(\mathbb{R}^3)$ into a Boolean algebra, hence there is also no partial Boolean algebra homomorphism from $\mathscr{P}(\mathscr{H})$ into a Boolean algebra (where h is defined to be a partial Boolean algebra homomorphism from $\mathscr{P}(\mathscr{H})$ into a Boolean algebra if the restriction of h to any Boolean subalgebra of $\mathscr{P}(\mathscr{H})$ is a Boolean algebra homomorphism). Consequently, there exist no partial Boolean algebra embeddings or *weak embeddings* (= restrictions of which to Boolean subalgebras of $\mathscr{P}(\mathscr{H})$ are injective) of $\mathscr{P}(\mathscr{H})$ into any Boolean algebra. The Kochen–Specker result is described in great detail in Svozil's Chapter 'Contextuality', and besides describing the original Kochen–Specker construction the author also reviews similar arguments given by Bell, Peres and Mermin and Greenberger–Horne–Zeilinger–Mermin. It is very useful to have these arguments available in a collected form.

A very recent result obtained by Meyer (1999) in this direction shows that the Kochen–Specker theorem is not stable under a natural topological weakening of the assumptions: the original Kochen–Specker proof is based on showing that the points on the unit sphere S^2 cannot be coloured by two colours in such a manner that one in every three points that define an orthogonal set of unit vectors is coloured differently from the other two. Meyer shows that points on S^2 having *rational* coordinates *can* be coloured in the required manner. This result has recently been used to create non-contextual hidden variable models (Kent, 1999; Clifton and Kent, 1999).

However, embeddings of non-Boolean lattices to Boolean ones that do not preserve either $\wedge$ or $\vee$ but preserve orthogonality are known to exist. Svozil mentions these (p. 130) but only in four lines, so that for the details the reader is referred to the original papers. Another type of result mentioned in the book is when h '[...] preserves the order relation. However, it neither preserves the binary operations *and* and *or* nor the complement' (p. 130; 'Malhas embedding'). In connection with Malhas embeddings Svozil refers to Malhas' original papers and gives an example of a Malhas embedding of the six element 'chinese lantern' lattice on pp. 130–132. In the example the $\wedge$ operation is preserved under the embedding (the orthocomplement and $\vee$ are not).

A related embeddability result that Svozil mentions is the existence of *Kalmbach embeddings* h_K of partially ordered sets $\mathscr{P}$ (posets) into orthomodular lattices $\mathscr{L}_{\mathscr{P}}$: by definition such a h_K preserves the ordering and the partial lattice structure of $\mathscr{P}$ (if any). Furthermore, $\mathscr{L}_{\mathscr{P}}$ is embeddable into a Boolean algebra by an injective map h preserving orthocomplementation and the lattice operations between orthogonal elements. However, as Svozil emphasises, the combined map $h \circ h_K$ (Svozil calls it the *combined* Kalmbach embedding) is *not* a lattice homomorphism (p. 127), in conformity with the basic no-go results. Again, it would be nice to have the proof of these statements in the book, but the reader interested in the details has no choice but to turn to the papers of Kalmbach. Further sufficient conditions excluding embeddability of an orthomodular lattice into Boolean algebras, formulated in terms of properties of the set of probability measures on the lattice, are also discussed in the book: see the next section.

3. Quantum Logic and Non-Commutative Measure Theory

Replacing a Boolean (σ) algebra $\mathscr{S}$ with an orthomodular (σ) lattice $\mathscr{L}$ and a classical measure μ on $\mathscr{S}$ with an additive (σ-additive in case of a σ-lattice) map $\phi:\mathscr{L}\to\mathbb{R}_+$ (called a 'state' if $\phi(I)=1$), one obtains a non-commutative measure space $(\mathscr{L},\phi)$. Note that it is not at all trivial that, given an $\mathscr{L}$ there exists an additive h on $\mathscr{L}$. In fact, examples of simple, finite element orthomodular lattices have been given by Greechie and by Ptak and Pullmannova that do not admit such an additive measure on them; Svozil's nicely drawn Greeche diagrams of these lattices can be found on p. 65.

The Hilbert lattice $\mathscr{P}(\mathscr{H})$ does, however, admit a large number of states, in the case of $\dim(\mathscr{H})\geqslant 3$ this being one consequence of Gleason's theorem, which Svozil recalls on p. 60: given a state ϕ there exists a density operator ρ such that $\phi(X)=Tr(\rho X)$, and conversely, every ϕ given by $\phi(X)=Tr(\rho X)$ with a density matrix ρ is a state.

The significance of Gleason's theorem is not only that it shows that there exist a lot of non-commutative probability measures. Equally important is the fact that the theorem shows at the same time that the non-commutative probability measures can be extended from the lattice of projections to a linear state on the set of all bounded observables on the Hilbert space—exactly as a classical measure can be extended from the Boolean algebra of measurable sets to the set of bounded measurable functions. This is because, given a density operator ρ, the expression $\phi(Q)=\mathrm{Tr}(Q\rho)$ makes sense for any bounded operator Q on $\mathscr{H}$. The procedure of extending a classical measure from the Boolean algebra of measurable sets to the set of bounded functions is known as integration theory, so Gleason's theorem is a theorem in non-commutative integration.

Gleason's theorem remains valid for von Neumann lattices: if the von Neumann algebra $\mathscr{N}$ does not have the complete matrix algebra M_2 on the two-dimensional Hilbert space ($\mathscr{H}_2$) as a component in its direct sum decomposition (i.e. '$\mathscr{N}$ does not have I_2 as a direct summand') then every additive map on the lattice $\mathscr{P}(\mathscr{N})$ can be extended to a normal state on the algebra $\mathscr{N}$. So von Neumann lattices and algebras with normal states on them are the natural non-commutative generalisations of classical probability theory in measure theoretic form, and non-commutative analogues of concepts in classical measure theory can and have been worked out in von Neumann algebra theory (L^p spaces, non-commutative conditional expectation etc.).

One area of investigation concerns properties of the states, and the features of the set $\varepsilon(\mathscr{L})$ of all states, on an orthomodular lattice $\mathscr{L}$. The state ϕ is called a Jauch–Piron state if $\phi(A)=1$ and $\phi(B)=1$ implies $\phi(A\wedge B)=1$ (p. 77). A lattice is called a Jauch–Piron lattice if every state on it is a Jauch–Piron state. All normal states on a von Neumann lattice are Jauch–Piron, and a complete classification of Jauch–Piron lattices is known in the von Neumann algebra category: if $\mathscr{N}$ does not contain an I_2 direct summand, then $\mathscr{P}(\mathscr{N})$ is Jauch–Piron if and only if it is the direct sum of a commutative algebra and finitely many finite-dimensional factors (Hamhalter, 1993). A set $\varepsilon'\subseteq\varepsilon$ of states is called *full* (p. 86) if for any two non-orthogonal A, B there exists a $\phi \in \varepsilon'$ such that $\phi(A) = \phi\ (B) = 1$; *separating* (p. 88) if for any two elements $A \neq B$ there is a $\phi \in \varepsilon'$ such that $\phi(A) \neq \phi\ (B)$ and *unital* (p. 112) if for every $A \neq 0$ there is a $\phi \in \varepsilon'$ such that $\phi(A) = 1$.

Svozil formulates several claims concerning these measure theoretic notions, but again, since the claims are not spelled out in a mathematically explicit manner and are not accompanied with proofs, it is not always clear what the precise content of the claim is or how it could be true. For instance Svozil states that if ε' is full, then it is separating and if ε' is separating then it is unital (p. 112), which is fairly clear. Less obvious is in precisely what sense of 'embedding' it is true that both non-separability and non-unitality (of the whole state space $\varepsilon(\mathscr{L})$?) is sufficient for non-embeddability of $\mathscr{L}$ into a Boolean algebra (p. 84); and so the precise content of the conjecture '[...] nonunitality may be the weakest measure theoretic criterion for nonembeddability [...]' (p. 112) remains vague.

It also is unclear in what sense '[t]he notion of unitality introduced here is a special case of Kochen and Specker's notion of *weak embeddability* [...]', since Svozil does not specify for which set of states on $\mathscr{L}$ is (non-)unitality relevant for the weak (non-)embeddability of $\mathscr{L}$. Certainly not the whole set of states, since the set of all states on $\mathscr{P}(\mathscr{H})$ is clearly unital but $\mathscr{P}(\mathscr{H})$ is *not* weakly embeddable into a Boolean algebra by the Kochen–Specker theorem. Maybe Svozil's last claim is to be understood as 'if the set of *dispersion-free* states on $\mathscr{L}$ is unital then $\mathscr{L}$ is weakly embeddable into a Boolean algebra (but not conversely)', a claim which might be true; however, so interpreted, unitality does not seem to be a genuine special case of weak embeddability but appears equivalent to it by another result of Kochen and Specker: a partial Boolean algebra $\mathscr{A}$ is weakly embeddable into a Boolean algebra if and only if for every $0 \neq A \in \mathscr{A}$ there exists a partial Boolean algebra homomorphism h from $\mathscr{A}$ into the two element Boolean algebra $\mathscr{B}_2$ such that $h(A) = I$ (Kochen and Specker, 1967)—and a partial Boolean algebra homomorphism into $\mathscr{B}_2$ is just a two-valued state.

4. Algebraic Semantics and Quantum Logic

Interpreting statements of the form *sent* (Q, d) = 'Observable Q has its value in set d (with probability 1)' as sentence letters in the sense of formal logic, one can define by induction in the standard way the set of formulas $\mathscr{F}$ using $\sim$ (*not*) and & (*and*) as primitive connectives. In algebraic semantics the semantic notions for $\mathscr{F}$ (such as truth, falsity, logical truth etc.) are given in terms of a map $v: \mathscr{F} \to \mathscr{L}$, where $\mathscr{L}$ is a certain algebraic structure. In the case of quantum logic $\mathscr{L}$ is assumed to be at least an ortholattice, and the map v is required to be a 'homomorphism' in the sense of satisfying $v(\sim\alpha) = v(\alpha)^{\perp}$ and $v(\alpha \& \beta) = v(\alpha) \wedge v(\beta)$. The map v is called a *valuation* (not to be confused with the notion of valuation as a Boolean algebra homomorphism from a lattice into the two element Boolean algebra). The pair $(\mathscr{L}, v)$ is called a *realisation* of $\mathscr{F}$. The formula α is called *true* in realisation $(\mathscr{L}, v)$ (or $(\mathscr{L}, v)$ is called a *model* of α) if $v(\alpha) = 1$, and α is called a *tautology* if it is true in *every* realisation (model). A formula α is a *consequence* of a set T of formulas in realisation $(\mathscr{L}, v)$ if for any $x \in \mathscr{L}$ we have: if for every $\beta \in T$ it holds that $x \leqslant v(\beta)$ then it also holds that $x \leqslant v(\alpha)$. A formula α is a *logical consequence* of T if it is a consequence of T in every realisation.

An alternative way of defining models for $\mathscr{F}$ is in terms of a *Kripkean semantics* $\mathscr{K} = \langle \mathscr{J}, R, \Pi, \rho \rangle$: here one associates with every formula α the set $\rho(\alpha) \in \Pi \subseteq \mathscr{J}$ *of possible worlds* in which α is true, where R is the *accessibility relation* on the set $\mathscr{J}$ of possible worlds. The map ρ is again required to satisfy $\rho(\sim\alpha) = \rho(\alpha)^{\perp}$ and $\rho(\alpha \& \beta) = \rho(\alpha) \cap \rho(\beta)$, where $X^{\perp}$ is now the set of possible worlds that are inaccessible from every world in X. The set $\Pi = \{\rho(\alpha) : \alpha$ a formula$\}$ is called the set of *propositions*; it turns out that Π is an ortholattice if the accessibility relation R is reflexive and symmetric, in which case $(\mathscr{J}, R)$ is called an *orthoframe*. Truth, logical truth, consequence and logical consequence are defined in $\mathscr{K}$ in analogy with the algebraic definitions (by replacing v by ρ). One method of obtaining a $\mathscr{K}$ is to take a (pre-)Hilbert space $\mathscr{H}$ as $\mathscr{J}$, to define $R = R_{\langle,\rangle}$ by $R_{\langle,\rangle}(\psi, \xi)$ iff $\langle \psi, \xi \rangle \neq 0$, and to take Π as the set of subsets $X \subseteq \mathscr{H}$ such that $(X^{\perp})^{\perp} = X$; we call $(\mathscr{H}, R_{\langle,\rangle})$ a *Hilbertian orthoframe*.

Semantic ideas are treated by Svozil briefly on pages 9–12 and 42–45. His treatment is in less general terms than the ones used above; in particular, algebraic and Kripkean semantics are not distinguished in the book and Svozil's description of the mentioned semantic notions is intertwined in his book's text with the description of the lattice properties of Hilbert lattices. Presenting the semantic notions in the book in this convoluted way is somewhat unfortunate because the algebraic properties of Hilbert lattices and the logical ideas related to algebraic semantics are conceptually separate issues and keeping them— and algebraic and Kripkean semantics— apart enables one to raise certain logical and metalogical issues concerning quantum logic, issues which otherwise are difficult to explicate.

One such topic is whether the algebraic semantics and the Kripkean semantics determine the same logic, where by 'determining the same logic' is meant that for any formula α and any set T of formulas α is a logical consequence of T in the sense of algebraic semantics iff it is a logical consequence of T in the sense of Kripkean semantics. As it turns out, the algebraic semantics and the Kripkean semantics determine the same logic.

However, there is a significant difference between the orthomodular and ortholattice cases: the ortholattice property of Π can be characterised in terms of properties of R (Π is an ortholattice if the accessibility relation is reflexive and symmetric) but *orthomodularity* of Π *cannot* be defined in terms of *elementary* properties of the accessibility relation, in the sense that there exists no first-order language containing the name of the accessibility relation $\hat{R}$ and containing a sentence S such that for every orthoframe $(\mathscr{J}, R)$ we have: S is a theorem in $(\mathscr{J}, R)$ iff Π is orthomodular: this is Goldblatt's theorem, the 'intractability of orthomodularity'. This can be seen by referring to the Hilbertian orthoframe:

a Π determined by a Hilbertian orthoframe $(\mathscr{H}, R_{\langle,\rangle})$ is orthomodular iff $\mathscr{H}$ is a Hilbert space (i.e. metrically complete), and, in conformity with metric completeness not being an elementary property, orthomodularity is also a non-elementary property. Partly due to the fact that Svozil deliberately restricts his investigations in the book to finite quantum logic (finite-dimensional Hilbert spaces and finite lattices), a treatment of such metalogical properties of quantum logic is not aimed at in his work (see Chiara and Giuntini (2000) for an extensive discussion of metalogical properties of quantum logic).

5. Philosophical and Historical Comments

The ideal relation of logic and probability theory is exemplified by the classical case: a Boolean algebra $\mathscr{S}$ represents *both* the set of events for classical probability theory $(\mathscr{S}, \mu)$ *and* the propositional logic, the latter being related to the events in the most natural way; namely an event can be identified with the proposition stating that the event happens, and this identification preserves the Boolean algebra structures of the logic and of the events. This classical harmony is made complete by the fact that the probability measure μ has the following 'strong subadditivity' property:

$$\mu(A) + \mu(B) = \mu(A \vee B) + \mu(A \wedge B). \tag{4}$$

The significance of (4) is that it is necessary if the probabilities $\mu(X)$ are to be interpretable as relative frequencies, an interpretation which, the standard conceptual difficulties relating to the frequency view notwithstanding, seems to be the only serious candidate for an interpretation in the context of physics.

In the Birkhoff–von Neumann paper (1936) the authors' original intention was not simply to create a non-classical, i.e. quantum logic, but to create it in such a manner that an interpretation of quantum logic can be given that mirrors the classical situation just described. Specifically, they wished to interpret quantum logic as an event structure for a non-commutative probability with probability understood as relative frequency. The trouble is that no quantum state ϕ on $\mathscr{P}(\mathscr{H})$ satisfies (4), hence no quantum probability can be interpreted as relative frequency— with the understanding that $A \wedge B$ represents the *joint occurrence* of A and B.

Svozil mentions this problem by discussing, following Szabó (1998), 'counter-intuitive probabilities' in two chapters (Chapter 6 'Probabilities' and Chapter 10, p. 75 and 181), i.e. probabilities that violate (4) in the strong form of $\phi(A) = 0.999999 = \mu(B)$ and $\phi(A \wedge B) = 0$. However Svozil's position concerning the problem posed by these counter-intuitive probabilities is not entirely clear in the book. Some passages in the book (e.g. p. 76) seem to indicate that he considers results of counterfactual arguments *irrelevant* on the grounds that they do not have operational consequences, and $\phi(A \wedge B) = 0$ in the above situation is indeed a 'counterfactual' probability in the sense that A and B are non-commuting, so that $\phi(A \wedge B)$ cannot be experimentally measured. But one just cannot declare $\phi(A \wedge B) = 0$ irrelevant if one wishes to interpret quantum logic as an event structure along the lines of the classical case. Quite to the contrary: the existence of nonsensical (in the sense of relative frequency) probabilities is highly relevant, and their significance is that (4) is generally violated. Hence one is forced to choose between

1. giving up the interpretation of quantum logic as an event structure;
2. giving up the relative frequency view of quantum probability;
3. giving up the interpretation of $\mathscr{P}(\mathscr{H})$ as quantum logic.

None of these options is particularly attractive. (1) and (2) mean abandoning the hope of creating a non-commutative version of the classical harmony, whereas option (3) means abandoning the Hilbert space formalism of quantum mechanics. Surprisingly, Birkhoff and von Neumann both saw this difficulty and had already opted for (iii) in their original 1936 paper. That is to say, they postulated quantum logic to be a *modular* lattice, which $\mathscr{P}(\mathscr{H})$ is not. The concept of quantum logic they preferred over $\mathscr{P}(\mathscr{H})$ was the modular von Neumann lattice $\mathscr{P}(\mathscr{N})$ of a type II_1 von Neumann algebra $\mathscr{N}$, a type of algebra that had been discovered by Murray and von Neumann just before the idea of quantum logic was formulated by Birkhoff and von Neumann. This von Neumann algebra is distinguished by the fact that there exists a probability measure τ on $\mathscr{N}$ whose restriction to $\mathscr{P}(\mathscr{N})$ satisfies (4), and this was the main reason why von Neumann preferred $\mathscr{P}(\mathscr{N})$ to $\mathscr{P}(\mathscr{H})$. It is worth adding that von Neumann was not satisfied even with this notion of quantum logic, and after 1936 he abandoned the relative frequency interpretation of probability in connection with quantum logic (see Rédei 1998, 2000) for a detailed analysis of von Neumann's views.)

Svozil's book can be recommended for two types of readers: the expert and the novice. Those who already know the topic well and do not need detailed proofs of the facts and statements presented will enjoy Svozil's picture of the world of non-Boolean lattices in finite-dimensional Hilbert space. The uninitiated who just wishes to get a first taste of quantum logic without being overburdened by technical subtletics will also benefit from reading this book.

Acknowledgements—This work has been supported by OTKA (contract numbers T 025841 and T 023447).

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