Early Romanian Contributions to Algebra and Polynomials

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Bucharest – 27 august 2018

Outline

- Solomon Marcus and the History of Mathematics
- Davidoglu, Țițeica and polynomials
- Lalescu and Algebra

Introduction

Solomon Marcus was always interested by the historical roots of fundamental concepts and results in Mathematics.
His historical approaches covered not only the fields in which he has obtained outstanding results but a broad view on all branches of our science and related fields.
He was particularly interested by Romanian mathematics. He has pointed to professor Nicolae Radu the fact that the Romanian researches in algebra started three decades before the discoveries of Dan Barbilian (1895–1961).

Introduction (contd.)

Anton Davidoglu (1901)
Gheorghe Țițeica (1901)
Traian Lalescu (1907, 1908, 1926)

Solomon Marcus and the Mathematical Education.

Introduction (contd.)

Țicțeica and Davidoglu have obtained formulas for the computation of the number real roots of a univariate polynomial. Their results were related to a theorem of É. Picard.

Lalescu has obtained results concerning Galois theory, the algebraic study of polynomials and on binary quadratic forms.

Picard's theorem

Theorem (Picard, 1892)

The number of roots of polynomial f in the interval (a, b) is given by

$$-\frac{1}{\pi}\int_a^b \frac{\varepsilon(ff'' - f'^2)}{f^2\varepsilon^2 + f'^2}dx + \frac{1}{\pi}\arctan\frac{\varepsilon f'(b)}{f(b)} - \frac{1}{\pi}\arctan\frac{\varepsilon f'(a)}{f(a)},$$

where arctan *is between $-\pi/2$ and $\pi/2$ and ε is "small enough".*

Țițeica's theorem

Theorem (Țițeica)

Let f be a univariate polynomial with real coefficients. The number of its double roots in the interval (a, b) is equal to

$$\frac{1}{2\pi} \int_a^b \left(P(u,\varepsilon) - P(u,-\varepsilon)\right) du\,,$$

where $P(u,\varepsilon)$ is defined by

$$\frac{-f'(u-\varepsilon)\left(f'(u+\varepsilon)+f''(u+\varepsilon)\right)+f''(u-\varepsilon)\left(f(u+\varepsilon)+f'(u+\varepsilon)\right)}{\left(f(u+\varepsilon)+f'(u+\varepsilon)\right)^2+f'^2(u-\varepsilon)}$$

with "small enough" ε .

Țițeica's theorem (contd.)

This is equivalent to the following formula for the number of double roots:

$$\frac{1}{2\pi} \lim_{\varepsilon \longrightarrow 0} \int_a^b (P(u,\varepsilon) - P(u,-\varepsilon)) \, \mathrm{d}\, u$$

Davidoglu's theorem

Theorem (Davidoglu)

If f is a univariate polynomial with real coefficients, the number of its double roots in the interval (a, b) is

$$-\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_a^b Q(f, \varepsilon)\, dx\,,$$

where

$$Q(f, \varepsilon) = \frac{f'^2 - f'' + \varepsilon^2 (f' f''' - f''^2)}{f'^2 + (f + \varepsilon^2 f'')^2} - \frac{f'^2 - f'' - \varepsilon^2 (f' f''' - f''^2)}{f'^2 + (f - \varepsilon^2 f'')^2}\,.$$

Computational aspects

We have used the `gp pari` package. Considering $f(x) = x^2 + 2x + 1$, we have considered the expressions

$$gro(u, v) = \frac{N1(u, v)}{N2(u, v)},$$

where

$$\begin{aligned} N1(u, v) = \; & (4 * v * u^5 + 48 * v * u^4 \\ & +(24 * v^3 + 184 * v) * u^3 \\ & +(112 * v^3 + 320 * v) * u^2 \\ & +(4 * v^5 + 184 * v^3 + 260 * v) * u \\ & +(112 * v^3 + 80 * v)) \end{aligned}$$

Computational aspects (contd.)

$$
\begin{aligned}
N2(u,v) = \; & (u^8 + 12 * u^7 + (12 * v^2 + 64) * u^6 \\
& +(108 * v^2 + 196) * u^5 \\
& +(38 * v^4 + 376 * v^2 + 374) * u^4 \\
& +(228 * v^4 + 648 * v^2 + 452) * u^3 \\
& +(12 * v^6 + 496 * v^4 \\
& +580 * v^2 + 336) * u^2 \\
& +(36 * v^6 + 468 * v^4 + 252 * v^2 + 140) * u \\
& +(v^8 + 24 * v^6 + 166 * v^4 + 40 * v^2 + 25))
\end{aligned}
$$

References on Țițeica and Davidoglu

A. Davidoglu: Sur le nombre des racines communes à plusieurs équations simultanées, *Journal de Math. Pures et Appl.*, 5–24 (1892).

L. Kronecker: Über Systeme von Funktionen mehrerer Variabeln, *Berl. Monatsber.*, 159–193, 688–698 (1869).

É. Picard: Sur le nombre des racines communes à plusieurs équations, *J. Math. Pures et Appl.*, IV–ème série, **8**, 5–24 (1892).

G. Tzitzeica: Sur le nombre des racines communes à plusieurs équations, *C. R. Ac. Sci. Paris*, 918-920 (1901).

The algebraic work of Traian Lalescu

Traian Lalescu (1882 – 1929) is well known for his pioneering work on integral equations.

During his studies in Paris he was also interested by algebraic subjects and he published 4 papers on Galois theory and on the study of univariate polynomials. One of them is devoted to the trinomial equations and other two to the study of binary quadratic forms. The fourth is a memory on Galois theory.

Later, in 1926, he published a note on polynomial division.

Lalescu's publications on Algebra

- *Sur le groupe des équations trinômes*, Bull. Soc. Math. de France, **35**, 75–76 (1907).
- *Sur la représentation des nombres par les classes de formes à un déterminant donné*, Bull. Soc. Math. de France, **35**, 248–252 (1907).
- *Sur la composition des formes quadratiques*, Nouv. Ann. Math., (4), **7**, 145–150 (1907).
- *La théorie générale de Galois*, Ann. Fac. de Toulouse, **10**, 113–123 (1908).
- *Diviziunea polinoamelor* (Division of polynomials), Rev. Mat. Timișoara, **6**, 4–6 (1926).

Three of them are devoted to polynomials.

The trinomial equation

Theorem

If the degree of the polynomial $f(X) = X^n + kaX + a$ is prime, where $a \in \mathbb{C}$ and $k = n/(n-1)^{\frac{n-1}{2}}$, then its Galois group is symmetric.

Lalescu has proved that the monodromy group is symmetric, and from this it follows that the Galois group should be symmetric.

On a theorem of Hilbert

Lalescu also gave an elegant proof to a theorem of Hilbert that states that

Theorem

The Galois group of a trinomial is symmetric for an infinity of values of the parameter a.

Lalescu's contribution to Galois Theory

The interest of Lalescu in Galois Theory were probably stimulated by a memory of Vessiot, which proposes a Galois Theory for homogeneous linear differential equations. The differential equations were in the core of Lalescu's studies in Paris.

In fact, Lalescu explains in his memory that he has studied the method of Vessiot for using Galois Theory in the study of homogeneous linear differential equations. He also mentions in the introduction of his memory a paper of J. T. Sörderberg (1888):

Lalescu's contribution to Galois Theory(contd.

Theorem

If an algebraic equation has no multiple roots, there exists a unique group of substitutions having the following two properties:

1. *Any rational functions of the roots that takes rational values, is invariable with respect to the substitutions of the group.*
2. *Reciprocally, any rational function of the roots whose value is invariable with respect to the substitutions of the group, is expressed rationally by the known quantities.*

The statement of Lalescu

Theorem

All the elements in the field of degree N of an equation satisfies the total or partial resolvents of degree N or a divisor of N. The resolvents of degree N are total and normal.

The algebraic equation considered above is only one of the rings in the chain of the equations which are resolvents of its fields and among which there is established an intimate link, translated in the links of their groups: if there are known the roots of the total resolvent, the roots of all the other are deduced rationally and the Galois groups of the other equations are formed through permutations of their roots, if there are applied to their rational expressions the permutations of the Galois group of the known equation. In particular the groups of the resolvents are isomorphic.

The statement of Lalescu (contd.

The memory of Lalescu on Galois Theory was discussed by I. Schur in **JFM 39.0203.02** *(1908). Schur explains that Lalescu has considered a polynomial having only simple roots in a field K and its Galois group $\mathfrak{G}$ and proved the fundamental Theorem of Galois Theory avoiding the use of resolvents. The result of Galois states that every rational function of the roots with coefficients from K which is invariant to all permutations of $\mathfrak{G}$, is an element from field K. This fundamental theorem of Galois theory is generally proved using the Galois resolvent of the equation $f(x) = 0$. Another proof was given by Söderberg but Lalescu simplifies this proof and shows how, on the basis of the fundamental theorem, the rest of the main results of Galois theory can be developed without significantly preferring the Galois resolvent of the equation over the other resolvents.*

Binary quadratic forms

Traian Lalescu has also published two other papers in algebra, namely concerning the composition of binary quadratic forms and the representation of integers by such forms. This is still today a central problem in algebraic number theory and class field theory, see F. Lemmermayer.
Lalescu's problem was to establish which integers are represented by a given primitive binary quadratic form. Lemmermeyer considers that "answering this seemingly innocent question quickly leads us into areas that were (and still are) important for the development of algebraic number theory: reciprocity laws and class fields".
Lalescu's papers on quadratic forms were cited recently, more than one hundred years after their publication, see F. Lemmermeyer (2010) and F. Pintore (2015).

Division of polynomials

In the 1920' Lalescu studied computational aspects of the division of univariate polynomials.

He communicated these results at the meetings of the Romanian Society of Sciences, the Mathematics branch. He has also published a note in 1926. His manuscripts should contain other results on this topic.