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FINE PROPERTIES OF DUALITY MAPPINGS

in

Mathematics Almost Everywhere: In Memory of Solomon Marcus,

A. Bellow, C. S. Calude and T. Zamfirescu Eds.,

World Scientific, Singapore, 2018, 93-146

MOTTO

Riemann has shown us that proofs are better achieved through ideas than through long calculations.

David Hilbert (1857)

A mathematical idea is "significant" if it can be connected, in a natural and illuminated way, with a large complex of other mathematical ideas.

G. H. Hardy, *A mathematician's apology*, Cambridge University press, 1967 (first edition, 1940).

THE CONCEPT OF DUALITY MAPPING

It was first introduced and studied by A. Beurling and A. E. Livingston, Ark. Math. 4 (1962), 405-411.

Major contributions: Browder, Laursen, Kato, Asplund, Dubinsky, Petryskyn, Ciarlet

Definition 1

Gauge function: $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ as $r \to \infty$.

Definition 2

Duality mapping corresponding to a gauge function φ:
X real Banach space,
$J_\varphi : X \to 2^{X^*}$, $J_\varphi 0_X = 0_{X^*}$,
$J_\varphi x = \varphi(\|x\|) \{x^* \in X^* | \, \|x^*\| = 1, \ \langle x^*, x \rangle = \|x\|\}$, if $x \neq 0_X$.

Hahn-Banach $\Longrightarrow$ $\operatorname{dom} J_\varphi = \{x \in X | J_\varphi x \neq \emptyset\} = X$.
Equivalently:
$J_\varphi x = \{u^* \in X^* | \, \langle u^*, x \rangle = \varphi(\|x\|) \|x\|, \ \|u^*\| = \varphi(\|x\|)\}$.

Theorem 1

a) $J_\varphi : X \to 2^{X^*}$ *is monotone in the Minty-Browder sense:*

$$\left.\begin{array}{r}\forall x, y \in X \\ \forall x^* \in J_\varphi x \\ \forall y^* \in J_\varphi y\end{array}\right\} \Rightarrow \begin{array}{l}\langle x^* - y^*, x - y\rangle \geq \\ \geq \left(\varphi\left(\|x\|\right) - \varphi\left(\|y\|\right)\right)\left(\|x\| - \|y\|\right) \geq 0.\end{array}$$

b) (Equivalent definition-Asplund) $J_\varphi u = \partial\Phi\left(u\right)$, *with* $\Phi\left(u\right) = \int_0^{\|u\|} \varphi\left(t\right) \mathrm{d}t$, $\forall u \in X$, *i.e.*

$$J_\varphi u = \left\{u^* \in X^* \middle| \left\langle u^*, u\right\rangle = \varphi\left(\|u\|\right)\|u\|,\ \|u^*\| = \varphi\left(\|u\|\right)\right\} =$$
$$= \left\{u^* \in X^* \middle| \Phi\left(v\right) - \Phi\left(u\right) \geq \left\langle u^*, v - u\right\rangle, \forall v \in X\right\} = \partial\Phi\left(u\right).$$

Remark

$u \in X \longmapsto \Phi\left(u\right) = \int_0^{\|u\|} \varphi\left(t\right) \mathrm{d}t \in \mathbb{R}$ is continuous and convex.

MONOTONE OPERATORS	$\supset$	SUBDIFF. of conv. cont. functionals	$\supset$	$\partial\Phi$, $\Phi(u) = \int_0^{\|u\|} \varphi(t)\,\mathrm{d}t$	$\ni J_\varphi$

CONNECTING DUALITY MAPPINGS TO SIGNIFICANT MATHEMATICAL IDEAS

(a) $J_\varphi \longleftrightarrow$ Hahn-Banach theorem

Theorem 2 (G. D., 2012)

X is separable $\Longrightarrow$
$\mathcal{A} = \{x \neq 0_X | \exists$ a unique $x^* \in X^*$ s.t. $\|x^*\| = 1,\ \langle x^*, x\rangle = \|x\|\}$
is norm dense in X.

Proof. Zarantonello, 1973:

$$X \text{ is separable, } T : D(T) \subset X \to 2^{X^*} \text{ monotone} \Longrightarrow$$
$$\Rightarrow Z(T) = \{x \in D(T)\,|\,Tx \text{ is not a sigleton}\} \text{ has empty interior:}$$
$$\overset{\circ}{Z(T)} = \emptyset.$$

Take: X separable, $D(T) = X$, $T = J_\varphi : X \to 2^{X^*}$ and

$$J_\varphi x = \begin{cases} 0_{X^*}, & x = 0_X, \\ \varphi(\|x\|)\{x^* \in X^* |\, \|x^*\| = 1,\ \langle x^*, x\rangle = \|x\|\}, & x \neq 0_X \end{cases}$$

T. Zarantonello $\Longrightarrow Z(J_\varphi)$ has no interior: $\overset{\circ}{Z(J_\varphi)} = \emptyset$.

(b) $J_\varphi \longleftrightarrow$ James' theorem on reflexivity

$J_\varphi \longleftrightarrow$ Laursen's theorem on reflexivity

(c) $J_\varphi \longleftrightarrow$ Bishop-Phelps theorem on subreflexivity

(d) $J_\varphi \longleftrightarrow$ Geometry of Banach spaces

(d$_1$) If X is smooth ($\Leftrightarrow$ the norm of X is Gateaux differentiable), then J_φ is single-valued, i.e.

$$J_\varphi : X \to X^*$$

and

$$J_\varphi x = \begin{cases} 0_{X^*}, & \text{if } x = 0_X, \\ \varphi(\|x\|)(\text{grad}\,\|\cdot\|)(x), & \text{if } x \neq 0_X. \end{cases}$$

(d$_1$) If X is strictly convex $\Leftrightarrow J = J_\varphi$, $\varphi(t) = t$, $t \geq 0$, is strictly monotone, i.e.

$$\left. \begin{array}{r} x, y \in X, \ x \neq y \\ x^* \in Jx, \ y^* \in Jy \end{array} \right\} \Rightarrow \langle x^* - y^*, x - y \rangle > 0.$$

(e) $J_\varphi \longleftrightarrow$ PDE

Example

$\Omega \subset \mathbb{R}^n$ bounded and smooth domain, $1 < p < \infty$,
$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega) : u|_{\partial\Omega} = 0\right\}$, $\|u\| = \||\nabla u|\|_{L^p(\Omega)}$,
$-\Delta_p : W_0^{1,p}(\Omega) \to \left(W_0^{1,p}(\Omega)\right)^* = W^{-1,p'}(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$,

$$-\Delta_p u = -\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right).$$

Theorem

$-\Delta_p = J_\varphi : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, $\varphi(t) = t^{p-1}$, $t \geq 0$.

Remark

$p = 2 \Rightarrow \Delta_p = \Delta$.

Thank you!