Phase Transition between Unidirectionality and Bidirectionality

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Introduction

Definition [Chaitin Ω Number, Chaitin 1975]

$$\Omega := \sum_{p \in \mathrm{Dom}\, U} 2^{-|p|}.$$

Here U is the optimal prefix-free machine. $\square$

Theorem [Calude & Nies 1997] $\Omega \equiv_{wtt} \mathrm{Dom}\, U$. $\square$

Definition [Generalization of Chaitin Ω Number, Tadaki 1999]

$$Z(T) := \sum_{p \in \mathrm{Dom}\, U} 2^{-\frac{|p|}{T}}$$

for any real $T > 0$. $\square$

In the case of $T = 1$, $Z(1) = \Omega$.

Theorem Suppose that T is a computable real with $0 < T \le 1$. Then $Z(T) \equiv_{wtt} \mathrm{Dom}\, U$. $\square$

Introduction

In this talk, we introduce an elaboration of the notion of weak truth-table reducibility, called reducibility in query size f, where we try to follow the fashion in which computational complexity theory is developed, while staying in computability theory.

Theorem [Calude & Nies 1997, posted again] $\Omega \equiv_{wtt} \mathrm{Dom}\, U$. $\square$

Using the notion of reducibility in query size f, this theorem is elaborated to show the unidirectionality between Ω and $\mathrm{Dom}\, U$.

Theorem [posted again] Suppose that T is a computable real with $0 < T \leq 1$. Then $Z(T) \equiv_{wtt} \mathrm{Dom}\, U$. $\square$

Using the notion of reducibility in query size f, this theorem is elaborated to show the bidirectionality between $Z(T)$ and $\mathrm{Dom}\, U$.

Thus, the notion of reducibility in query size f can reveal a critical difference of the behavior between $T = 1$ and $T < 1$, which cannot be captured by the notion of weak truth-table reducibility.

Physical Motivation: Statistical Mechanical Interpretation of AIT

[Calude & Stay, Information and Computation 204 (2006)] pointed out that $Z(T)$ is similar to a partition function in statistical mechanics.

- In statistical mechanics, the partition function Z is given as:

$$Z = \sum_{n} e^{-\frac{E_n}{kT}}.$$

Here, n denotes the quantum number of an energy eigenstate of a quantum system, E_n its energy, and T temperature.

- On the other hand, $Z(T)$ is given as:

$$Z(T) = \sum_{p \in \mathrm{Dom}\, U} 2^{-\frac{|p|}{T}} \qquad (T > 0).$$

Thus, Z coincides with $Z(T)$ by performing the following replacements:

An energy eigenstate n	$\Longrightarrow$	A program $p \in \mathrm{Dom}\, U$,
The energy E_n of n	$\Longrightarrow$	The length $\lvert p\rvert$ of p,
Boltzmann constant k	$\Longrightarrow$	$1/\ln 2$.

Physical Motivation: Statistical Mechanical Interpretation of AIT

In our former work, we developed the statistical mechanical interpretation of algorithmic information theory (AIT, for short) where we introduced the thermodynamic quantities into AIT by performing the following replacements for the corresponding thermodynamic quantities of a physical system at temperature T.

An energy eigenstate n	$\Longrightarrow$	A string p in $\mathrm{Dom}\,U$,
The energy E_n of n	$\Longrightarrow$	The length $\lvert p\rvert$ of p,
Boltzmann constant k	$\Longrightarrow$	$1/\ln 2$.

Partition function $$Z(T)=\sum_{n} e^{-\frac{E_n}{kT}} \quad\Longrightarrow\quad Z(T)=\sum_{p\in \mathrm{Dom}\,U} 2^{-\frac{|p|}{T}},$$

Free energy $$F(T)=-kT\ln Z(T) \quad\Longrightarrow\quad F(T)=-T\log_2 Z(T),$$

Energy $$E(T)=\frac{1}{Z(T)}\sum_{n} E_n e^{-\frac{E_n}{kT}} \quad\Longrightarrow\quad E(T)=\frac{1}{Z(T)}\sum_{p\in \mathrm{Dom}\,U} |p|\,2^{-\frac{|p|}{T}},$$

Entropy $$S(T)=\frac{E(T)-F(T)}{T} \quad\Longrightarrow\quad S(T)=\frac{E(T)-F(T)}{T}.$$

Physical Motivation: Statistical Mechanical Interpretation of AIT

Theorem [Tadaki 2008]

(i) If $0 < T < 1$ and T is computable, then each of $Z(T)$, $F(T)$, $E(T)$, and $S(T)$ converges to a real whose compression rate equals to T, i.e.,

$$\lim_{n\to\infty} \frac{H(Z(T)\restriction_n)}{n} = \lim_{n\to\infty} \frac{H(F(T)\restriction_n)}{n} = T,$$

$$\lim_{n\to\infty} \frac{H(E(T)\restriction_n)}{n} = \lim_{n\to\infty} \frac{H(S(T)\restriction_n)}{n} = T.$$

(ii) If $1 < T$, then $Z(T)$, $E(T)$, and $S(T)$ diverge to ∞, and $F(T)$ diverges to $-\infty$. □

Implication of (i): The compression rate of the values of all the thermodynamic quantities equals to the temperature T.

Thermodynamic Interpretation of (ii): "Phase Transition" occurs at temperature 1.

The purpose of this talk is to reveal a new aspect of the phase transition at temperature $T = 1$, based on the notion of *reducibility in query size f*.

Elaborating Weak Truth-Table Reducibility

Weak Truth-Table Reducibility

Definition [Weak Truth-Table Reduction of A to B]

Let $A, B \subset \mathbb{N}$. We say that *A is weak truth-table reducible to B*, denoted $A \leq_{wtt} B$, if there exist a total recursive function $f \colon \mathbb{N} \to \mathbb{N}$ and an oracle Turing machine M such that

(i) A is Turing reducible to B via M, and

(ii) on every input $n \in \mathbb{N}$, M only queries natural numbers at most $f(n)$. $\square$

Elaborating Weak Truth-Table Reducibility

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(i) A is Turing reducible to B via M, and

(ii) on every input $n \in \mathbb{N}$, M only queries natural numbers at most $f(n)$. $\square$

Note that, in the definition of weak truth-table reducibility (wtt-reducibility, for short), we only require the existence of the total recursive bound f on the use for the oracle B.

In this talk, we introduce an elaboration of the notion of wtt-reducibility, where the total recursive bound f on the use for the oracle B is explicitly specified.

In doing so, in particular we try to follow the fashion in which computational complexity theory is developed, while staying in computability theory.

Elaborating Weak Truth-Table Reducibility

Definition [Weak Truth-Table Reduction of A to B]

Let $A, B \subset \mathbb{N}$. We say that *A is weak truth-table reducible to B*, denoted $A \leq_{wtt} B$, if there exist a total recursive function $f\colon \mathbb{N} \to \mathbb{N}$ and an oracle Turing machine M such that

(i) A is Turing reducible to B via M, and

(ii) on every input $n \in \mathbb{N}$, M only queries natural numbers at most $f(n)$. $\square$

Recall that the notion of input size plays a crucial role in computational complexity theory since computational complexity such as time complexity and space complexity is measured based on it. Note that this is already true in AIT since the program-size complexity is measured based on input size.

Thus, in elaborating wtt-reducibility we consider a reduction between subsets of $\{0, 1\}^*$ and not a reduction between subsets of $\mathbb{N}$ as in the original wtt-reducibility.

Reducibility in Query Size f

Definition [Weak Truth-Table Reduction of A to B]

Let $A, B \subset \mathbb{N}$. We say that *A is weak truth-table reducible to B*, denoted $A \leq_{wtt} B$, if there exist a total recursive function $f\colon \mathbb{N} \to \mathbb{N}$ and an oracle Turing machine M such that

(i) A is Turing reducible to B via M, and

(ii) on every input $n \in \mathbb{N}$, M only queries natural numbers at most $f(n)$. □

The notion of wtt-reducibility is elaborated as follows:

Definition [Reduction of A to B in Query Size f]

Let $f\colon \mathbb{N} \to \mathbb{N}$, and let $A, B \subset \{0,1\}^*$. We say that *A is reducible to B in query size f* if there exists an oracle Turing machine M such that

(i) A is Turing reducible to B via M, and

(ii) on every input $x \in \{0,1\}^*$, M only queries strings of length at most $f(|x|)$. □

$\boxed{\text{Reducibility in Query Size } f}$

For any fixed sets A and B, the new definition allows us to consider the notion of asymptotic behavior for the function f which bounds the use of the reduction, i.e., which imposes the restriction on the use of the computational resource (i.e., the oracle B).

Thus, even in the context of computability theory, we can deal with the notion of asymptotic behavior in a manner like in computational complexity theory.

The notion of wtt-reducibility is elaborated as follows:

Definition [Reduction of A to B in Query Size f]
Let $f\colon \mathbb{N} \to \mathbb{N}$, and let $A, B \subset \{0,1\}^*$. We say that *A is reducible to B in query size f* if there exists an oracle Turing machine M such that

(i) A is Turing reducible to B via M, and

(ii) on every input $x \in \{0,1\}^*$, M only queries strings of length at most $f(|x|)$. $\square$

Reducibility in Query Size f

Note that in the elaboration we require the bound $f(|x|)$ to depend only on input size $|x|$ as in computational complexity theory, and not on input x itself as in the original wtt-reducibility.

We pursue a formal correspondence to computational complexity theory in this manner, while staying in computability theory.

We apply the elaboration to sets which appear in AIT and demonstrate the power of the elaboration.

The notion of wtt-reducibility is elaborated as follows:

Definition [Reduction of A to B in Query Size f]
Let $f: \mathbb{N} \to \mathbb{N}$, and let $A, B \subset \{0,1\}^*$. We say that *A is reducible to B in query size f* if there exists an oracle Turing machine M such that

(i) A is Turing reducible to B via M, and

(ii) on every input $x \in \{0,1\}^*$, M only queries strings of length at most $f(|x|)$. □

$\boxed{\text{Elementary Properties of Reducibility in Query Size } f}$

Observation

Let $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$, and let $A, B, C \subset \{0, 1\}^*$.

(i) If A is reducible to B in query size f and B is reducible to C in query size g, then A is reducible to C in query size $g \circ f$.

(ii) Suppose that $f(n) \leq g(n)$ for every $n \in \mathbb{N}$. If A is reducible to B in query size f then A is reducible to B in query size g.

(iii) Suppose that A is reducible to B in query size f. If A is not recursive then f is unbounded. $\square$

Observation

For every $A \subset \{0, 1\}^*$, A is reducible to A in query size n.
Here "n" denotes the identity function $I\colon \mathbb{N} \to \mathbb{N}$ with $I(n) = n$ and not a constant. $\square$

We follow the notation in computational complexity theory.

Review of Chaitin Ω Number

Review: Program-size Complexity

Definition $P \subset \{0,1\}^*$ is called *prefix-free* if no string in P is a prefix of another string in P.

Definition A partial recursive function $M \colon \{0,1\}^* \to \{0,1\}^*$ is called a *prefix-free machine* if Dom M is prefix-free, where Dom M is the domain of definition of M.

Definition For any prefix-free machine M and any $x \in \{0,1\}^*$,

$$H_M(x) := \min\left\{ |p| \;\middle|\; p \in \{0,1\}^* \;\&\; M(p) = x \right\}.$$

Definition A prefix-free machine U is called *optimal* if, for every prefix-free machine M, there exists $d \in \mathbb{N}$ such that, for every $x \in \{0,1\}^*$,

$$H_U(x) \leq H_M(x) + d.$$

Definition [Program-Size Complexity] We choose a particular optimal prefix-free machine U as a standard one. Then the *program-size complexity* (or *Kolmogorov complexity*) $H(x)$ of $x \in \{0,1\}^*$ is defined by

$$H(x) := H_U(x).$$

Review: Chaitin Ω Number

Definition [Randomness of Real]

A real α is called *random* if $n \leq H(\alpha\restriction_n) + O(1)$ for all $n \in \mathbb{N}^+$.
Here $\alpha\restriction_n$ denotes the first n bits of the base-two expansion of $\alpha - \lfloor \alpha \rfloor$.

The fractional part of α.

Definition [Chaitin Ω Number, Chaitin 1975]

$$\Omega := \sum_{p \in \mathrm{Dom}\, U} 2^{-|p|},$$

where U is the optimal prefix-free machine.

The first n bits of the base-two expansion of Ω solve the halting problem of U for inputs of length at most n. Namely, for every n, if $\Omega\restriction_n$ is given, then the list of all halting inputs for U of length at most n can be calculated.

Theorem [Chaitin 1975] Ω is random. □

Unidirectionality between Ω and $\operatorname{Dom} U$

Prefixes of Real

In what follows, we investigate the relative computational power between Ω and $\mathrm{Dom}\, U$, based on the notion of *reducibility in query size f*.

In the case of wtt-reducibility, we regard reals as subsets of $\mathbb{N}$ and then study the wtt-reducibility between them since the wtt-reducibility is originally defined for a pair of subsets of $\mathbb{N}$.

To be precise, in that case, each real α is identified with the subset of $\mathbb{N}$ whose characteristic sequence is the base-two expansion of α.

On the other hand, the notion of reduction in query size f is originally defined for a pair of subsets of $\{0,1\}^*$.

Thus, to investigate the relative computational power between a real and a subset of $\{0,1\}^*$, based on the notion of reducibility in query size f, we have to specify first how to identify a real with a subset of $\{0,1\}^*$.

We do this as follows.

Prefixes of Real

We identify a real with a subset of $\{0,1\}^*$ as follows:

Definition [Prefixes of Real]

For each real α, the subset $\mathrm{Pf}(\alpha)$ of $\{0,1\}^*$ is defined by

$$\mathrm{Pf}(\alpha) := \{\alpha\restriction_n \mid n \in \mathbb{N}\}.$$

Namely, $\mathrm{Pf}(\alpha)$ is the set all prefixes of the base-two expansion of $\alpha - \lfloor \alpha \rfloor$. $\square$

The notion of prefixes of real would seem natural especially for AIT, since the following holds.

Observation A real α is Chaitin random, i.e., $n \le H(\alpha\restriction_n) + O(1)$, if and only if there exists $d \in \mathbb{N}$ such that, for every $x \in \mathrm{Pf}(\alpha)$, $|x| \le H(x) + d$. $\square$

Recall that the first n bits of the base-two expansion of Ω solve the halting problem of U for inputs of length at most n.

This can be rephrased as follows.

Observation $\mathrm{Dom}\, U$ is reducible to $\mathrm{Pf}(\Omega)$ in query size n. $\square$

Main Results: Reduction of Ω to Dom U in Query Size f

Definition [Order Function] An *order function* is a non-decreasing and unbounded total recursive function $f\colon \mathbb{N} \to \mathbb{N}$. $\square$

Theorem [Main Result I]

For every order function f, the following two are equivalent:

(i) $\mathrm{Pf}(\Omega)$ is reducible to Dom U in query size $f(n) + O(1)$.

(ii) $\sum_{n=0}^{\infty} 2^{n-f(n)} < \infty$ (Kraft inequality). $\square$

The implication (ii) $\Rightarrow$ (i) results in:

Corollary $\mathrm{Pf}(\Omega)$ is reducible to Dom U in query size $n+(1+\epsilon)\log_2 n+O(1)$ for every real $\epsilon > 0$. $\square$

On the other hand, the implication (i) $\Rightarrow$ (ii) results in:

Corollary $\mathrm{Pf}(\Omega)$ is not reducible to Dom U in query size $n + \log_2 n + O(1)$. $\square$

Corollary $\mathrm{Pf}(\Omega)$ is not reducible to Dom U in query size $n + O(1)$. $\square$

One-Wayness

$$\Omega = 0.111111010010100000111110011101011000101\ldots\ldots$$

(with n and $f(n)$ marked above positions of Ω)

$\mathrm{Dom}\,U$	1
	0011
	001001
	000110010
length $\leq n$	$\underline{0001100110\,01110011}$
length $\leq f(n)$	$0001100110\,0111000100\,111100111$
	$0001100110\,0111000100\,1111001100\,01110$
	$\vdots$

$$\underbrace{n \qquad\qquad f(n)}_{f(n)-n} \quad \text{with} \quad \sum_{n=0}^{\infty} 2^{n-f(n)} < \infty$$

Main Results: Reduction of Dom U to Ω in Query Size f

$\boxed{\text{Theorem}}$ [Main Result II]

For every order function f, the following two are equivalent:

(i) Dom U is reducible to Pf(Ω) in query size $f(n)+O(1)$.

(ii) $n \leq f(n)+O(1)$. $\square$

The implication (ii) $\Rightarrow$ (i) results in:

$\boxed{\text{Corollary}}$ Dom U is reducible to Pf(Ω) in query size $n+O(1)$. $\square$

On the other hand, the implication (i) $\Rightarrow$ (ii) says that the upper bound "$n+O(1)$" of the query size in this corollary is, in essence, tight.

One-Wayness

$$\Omega = 0.111111010010100000\overset{n}{1}1111001110101\overset{f(n)}{1}000101\ldots\ldots$$

Dom U

1

0011

001001

000110010

length $\leq n$ 0001100110 01110011

length $\leq f(n)$ 0001100110 0111000100 111100111

0001100110 0111000100 1111001100 01110

$\vdots$

$$\underbrace{n \qquad\qquad f(n)}_{f(n)-n} \quad \text{with} \quad \sum_{n=0}^{\infty} 2^{n-f(n)} < \infty$$

Definitions of Unidirectionality and Bidirectionality

Definition [Unidirectionality and Bidirectionality] Let $A, B \subset \{0, 1\}^*$.

(i) We say that *the computation from A to B is unidirectional* if the following holds: For every order functions f and g, if B is reducible to A in query size f and A is reducible to B in query size g then the function $g(f(n)) - n$ of $n \in \mathbb{N}$ is not bounded to the above.

(ii) We say that *the computations between A and B are bidirectional* if the computation from A to B is not unidirectional and the computation from B to A is not unidirectional. $\square$

Theorem The computation from $\mathrm{Pf}(\Omega)$ to $\mathrm{Dom}\, U$ is unidirectional and also the computation from $\mathrm{Dom}\, U$ to $\mathrm{Pf}(\Omega)$ is unidirectional. $\square$

Meaning of Unidirectionality

Let f be an order function. The notion of the reduction of A to B in query size f is equivalent to that, for every $n \in \mathbb{N}$, if n and $B\restriction_{f(n)}$ are given, then $A\restriction_n$ can be calculated, where $A\restriction_n$ denotes $\{x \in A \mid |x| \leq n\}$.

Definition [Unidirectionality and Bidirectionality, posted again]

Let $A, B \subset \{0,1\}^*$.

(i) We say that *the computation from A to B is unidirectional* if the following holds: For every order functions f and g, if B is reducible to A in query size f and A is reducible to B in query size g then the function $g(f(n)) - n$ of $n \in \mathbb{N}$ is not bounded to the above.

(ii) We say that *the computations between A and B are bidirectional* if the computation from A to B is not unidirectional and the computation from B to A is not unidirectional. $\square$

The notion of unidirectionality of the computation from A to B in the above definition is, in essence, interpreted as follows: No matter how a order function f is chosen, if f satisfies that $B\restriction_n$ can be calculated from n and $A\restriction_{f(n)}$, then $A\restriction_{f(n)}$ cannot be calculated from n and $B\restriction_{n+O(1)}$.

One-Wayness

$$\overset{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \qquad\qquad\quad f(n)}{\Omega = 0.111111010010100000111110011101011000101\ldots\ldots}$$

$\mathrm{Dom}\, U$

1
0011
001001
000110010
length $\leq n$ $\quad \underline{000110011001110011}$
length $\leq f(n)$ $\quad \underline{000110011001110001001111001111}$
$000110011001110001001111001100\,01110$

$$\underbrace{n \qquad\qquad\qquad f(n)}_{f(n)-n}$$

⋮

$f(n)-n$ with $\displaystyle\sum_{n=0}^{\infty} 2^{n-f(n)} < \infty$

Bidirectionality between $Z(T)$ and $\mathrm{Dom}\,U$

Review: Partition Function $Z(T)$

Definition [Partition Function $Z(T)$ at Temperature T, Tadaki 1999]

$$Z(T) := \sum_{p \in \mathrm{Dom}\, U} 2^{-\frac{|p|}{T}}$$

for any real $T > 0$. $\square$

In the case of $T = 1$, $Z(1) = \Omega$.

Suppose that T is a computable real with $0 < T \leq 1$.
Then the first n bits of the base-two expansion of $Z(T)$ solve the halting problem of U for inputs of length at most Tn.

In other words,

Theorem $\mathrm{Dom}\, U$ is reducible to $\mathrm{Pf}(Z(T))$ in query size $\lceil n/T \rceil$. $\square$

Review: Partition Function $Z(T)$

Theorem

(i) If $0 < T < 1$ and T is computable, then

$$H(Z(T)\restriction_n) = Tn + O(1),$$

and therefore

$$\lim_{n\to\infty} \frac{H(Z(T)\restriction_n)}{n} = T,$$

i.e., the compression rate of $Z(T)$ equals to temperature T.

(ii) If $1 < T$, then $Z(T)$ diverges to ∞. $\square$

Recall that the partition function $Z(T)$ is one of the thermodynamic quantities of AIT. The above theorem shows that the partition function $Z(T)$ diverges when temperature T exceeds 1. Thus, from the point of view of the statistical mechanical interpretation of AIT, this means that phase transition occurs at temperature 1.

The purpose of this talk is to reveal a new aspect of the phase transition, based on the notion of reducibility in query size f

Implication of the Computability of $Z(T)$

Theorem [fixed point theorem on partial randomness, Tadaki 2008]

For every $T \in (0,1)$, if $Z(T)$ is a computable real, then

$$\lim_{n\to\infty} \frac{H(T{\restriction_n})}{n} = T,$$

i.e., the compression rate of T equals to T itself. $\square$

Intuitive Meaning; Metaphor

Consider a file of infinite size whose content is

"The compression rate of this file is $0.100111001\cdots\cdots$"

When this file is compressed, the compression rate of this file actually equals to $0.100111001\cdots\cdots$, as the content of this file says.

This situation forms a fixed point and is self-referential !

Main Results: Bidirectionality between $Z(T)$ and $\mathrm{Dom}\, U$

Theorem [Main Result III]

Suppose that T is a computable real with $0 < T < 1$. For every order function f, the following two are equivalent:

(i) $\mathrm{Pf}(Z(T))$ is reducible to $\mathrm{Dom}\, U$ in query size $f(n) + O(1)$.

(ii) $Tn \le f(n) + O(1)$. $\square$

Theorem [Main Result IV]

Suppose that T is a computable real with $0 < T \le 1$. For every order function f, the following two are equivalent:

(i) $\mathrm{Dom}\, U$ is reducible to $\mathrm{Pf}(Z(T))$ in query size $f(n) + O(1)$.

(ii) $n/T \le f(n) + O(1)$. $\square$

Note that the function Tn is the inverse of the function n/T. This implies the bidirectionality between $Z(T)$ and $\mathrm{Dom}\, U$.

Theorem Suppose that T is a computable real with $0 < T < 1$. Then the computations between $\mathrm{Pf}(Z(T))$ and $\mathrm{Dom}\, U$ are bidirectional. $\square$

Two-Wayness

$Tn \qquad n$

$$Z(T) = 0.1111110100100000100110011101011000101\ldots\ldots$$

Dom U 1

0011

001001

length $\leq Tn$ 000110010

length $\leq n$ 0001100110 01110011

0001100110 0111000100 111100111

0001100110 0111000100 1111001100 01110

$\vdots$

$Tn \qquad n$

$\vdots$

Key Theorem for the Proof of the Bidirectionality (Main Result III)

Proof of Bidirectionality: Randomness of R.E. Real

Definition [R.E. Real]

A real α is called *recursively enumerable* (*r.e.*, for short) if there exists a computable, increasing sequence of rationals converging to α. $\square$

Proof of Bidirectionality: Randomness of R.E. Real

Theorem A [characterizations of randomness for an r.e. real]
Let α be an r.e. real with $0 < \alpha < 1$. Then the following conditions are equivalent:
(i) The real α is Chaitin random, i.e., $n \le H(\alpha\restriction_n) + O(1)$.
(ii) The real α is Martin-Löf random.
(iii) The real α is Ω-like.
(iv) $H(\beta\restriction_n) \le H(\alpha\restriction_n) + O(1)$ for every r.e. real β.
(v) For every r.e. real $\gamma > 0$, there exist an r.e. real $\beta \ge 0$ and a rational $q > 0$ such that $\alpha = \beta + q\gamma$.
(vi) There exists an optimal computer V such that $\alpha = \Omega_V$.
(vii) There exists a universal probability m such that $\alpha = \sum_{s\in\{0,1\}^*} m(s)$.
(viii) Every computable, increasing sequence of rationals which converges to α is universal.
(ix) There exists a universal computable, increasing sequence of rationals which converges to α. $\square$

Shown by [Schnorr 1973], [Chaitin 1975], [Solovay 1975], [Calude, Hertling, Khoussainov and Wang 2001], [Kučera and Slaman 2001], and [Tadaki 2005].

Proof of Bidirectionality: Partial Randomness of R.E. Real

Theorem B [characterizations of partial randomness for an r.e. real, Tadaki 2008]

Let α be an r.e. real with $0 < \alpha < 1$. Suppose that T is computable with $0 < T \leq 1$. Then the following conditions are equivalent:

(i) The real α is Chaitin T-random, i.e., $Tn \leq H(\alpha\restriction_n) + O(1)$.

(ii) The real α is Martin-Löf T-random.

(iii) The real α is $\Omega(T)$-like.

(iv) $H(\beta\restriction_n) \leq H(\alpha\restriction_n) + O(1)$ for every r.e. T-convergent real β.

(v) For every r.e. T-convergent real $\gamma > 0$, there exist an r.e. real $\beta \geq 0$ and a rational $q > 0$ such that $\alpha = \beta + q\gamma$.

(vi) There exist an optimal computer V and an r.e. real $\beta \geq 0$ such that $\alpha = \beta + \Omega_V(T)$.

(vii) There exists a universal probability m such that $\alpha = \sum_{s\in\{0,1\}^*} m(s)^{\frac{1}{T}}$.

(viii) Every computable, increasing sequence of rationals which converges to α is T-universal.

(ix) There exists a T-universal computable, increasing sequence of rationals which converges to α. $\square$

In the case of $T = 1$, Theorem B can result in Theorem A.

Proof of Bidirectionality: Partial Randomness of R.E. Real

Definition [T-convergence, Tadaki 2008]

- An increasing sequence $\{a_n\}$ of rationals is called *T-convergent* if

$$\sum_{n=0}^{\infty}(a_{n+1}-a_n)^T<\infty.$$

- An r.e. real α is called *T-convergent* if there exists a T-convergent computable, increasing sequence of rationals converging to α. □

Theorem B [characterizations of partial randomness for an r.e. real, posted again] Let α be an r.e. real with $0<\alpha<1$. Suppose that T is computable with $0<T\leq 1$. Then the following conditions are equivalent:

(i) The real α is Chaitin T-random, i.e., $Tn\leq H(\alpha{\restriction}_n)+O(1)$.

(iv) $H(\beta{\restriction}_n)\leq H(\alpha{\restriction}_n)+O(1)$ for every r.e. T-convergent real β. □

Proof of Bidirectionality: Partial Randomness of R.E. Real

Theorem [Calude, Hay, and Stephan 2011]

Suppose that T is a computable real with $0 < T < 1$. Then there exist an r.e. real $\alpha \in (0,1)$ and $d \in \mathbb{N}$ such that, for all $n \in \mathbb{N}^+$,

$$|H(\alpha\restriction_n) - Tn| \le d.$$

$\square$

Theorem [Key Result for Bidirectionality]

Suppose that T is computable with $0 < T < 1$. Then there exist an r.e. real $\alpha \in (0,1)$ and $d \in \mathbb{N}$ such that, for all $n \in \mathbb{N}^+$,

$$|H(Z(T)\restriction_n) - Tn| \le d.$$

$\square$

Main Results: Bidirectionality between $Z(T)$ and Dom U

Theorem [Main Result III]

Suppose that T is a computable real with $0 < T < 1$. For every order function f, the following two are equivalent:

(i) Pf$(Z(T))$ is reducible to Dom U in query size $f(n) + O(1)$.

(ii) $Tn \leq f(n) + O(1)$. $\square$

Theorem [Main Result IV]

Suppose that T is a computable real with $0 < T \leq 1$. For every order function f, the following two are equivalent:

(i) Dom U is reducible to Pf$(Z(T))$ in query size $f(n) + O(1)$.

(ii) $n/T \leq f(n) + O(1)$. $\square$

Note that the function Tn is the inverse of the function n/T. This implies the bidirectionality between $Z(T)$ and Dom U.

Theorem Suppose that T is a computable real with $0 < T < 1$. Then the computations between Pf$(Z(T))$ and Dom U are bidirectional. $\square$

Two-Wayness

$$\begin{array}{c} \qquad\qquad\qquad\qquad Tn \qquad\quad n \\ Z(T) = 0.1111110100100000100110011101011000101\ldots\ldots \end{array}$$

Dom U	1
	0011
	001001
length $\le Tn$	000110010
length $\le n$	0001100110 01110011
	0001100110 011100100 111100111
	0001100110 011100100 1111001100 01110
	$Tn \qquad n$
	$\vdots$

Origin of Difference between $T = 1$ and $T < 1$

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In the case of $T = 1$, we use the following to establish the unidirectionality:

Theorem [Ample Excess Lemma, Miller & Yu 2008]

A real α is random if and only if $\sum_{n=1}^{\infty} 2^{n-H(\alpha\restriction_n)} < \infty$. $\square$

However, the "only if" part does not hold for the case of $T < 1$. Namely, $Tn \le H(\alpha\restriction_n) + O(1)$ does not imply that $\sum_{n=1}^{\infty} 2^{Tn-H(\alpha\restriction_n)} < \infty$ in the case of $0 < T < 1$ (Reimann & Stephan 2005).

In the case of $0 < T < 1$, we use the following to establish the bidirectionality:

Lemma [Reimann & Stephan 2005]

Suppose that $0 < T < 1$. Then there exists $c \in \mathbb{N}^+$ such that, for every $s \in \{0,1\}^*$, there exists $r \in \{0,1\}^*$ with $|r| = c$ for which

$$H(sr) \ge H(s) + T\,|r| \quad \text{i.e.} \quad H(s) - T\,|s| \le H(sr) - T\,|sr|\,. \qquad \square$$

However, this lemma does not hold for the case of $T = 1$.

I would like to thank Cris for his continuing support and encouragement over the years. Thanks to Cris, I could continue to do my research on algorithmic information theory.

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Congratulations on your 60th birthday !!