# Learning Families of Closed Sets in Matroids


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# Content

I. Matroids and closed sets
II. Learning theory
III. Learning of Noetherian matroids
IV. Partial learning
V. Learning of all recursive / r.e. sets
VI. Confident partial learning

# I. Matroids and Closed Sets

An $\mathbf{A}$-r.e. matroid $(\mathbb{N}, \mathbf{\Phi})$ satisfies below axioms for all sets $\mathbf{R}, \mathbf{S} \subseteq \mathbb{N}$ and all $\mathbf{a}, \mathbf{b} \in \mathbb{N}$:

- $\mathbf{S} \subseteq \mathbf{\Phi}(\mathbf{S})$;
- $\mathbf{\Phi}(\mathbf{\Phi}(\mathbf{S})) = \mathbf{\Phi}(\mathbf{S})$;
- $\mathbf{R} \subseteq \mathbf{S} \Rightarrow \mathbf{\Phi}(\mathbf{R}) \subseteq \mathbf{\Phi}(\mathbf{S})$;
- If $\mathbf{a} \in \mathbf{\Phi}(\mathbf{\Phi}(\mathbf{S}) \cup \{\mathbf{b}\}) - \mathbf{\Phi}(\mathbf{S})$ then $\mathbf{b} \in \mathbf{\Phi}(\mathbf{\Phi}(\mathbf{S}) \cup \{\mathbf{a}\})$;
- $\mathbf{\Phi}(\mathbf{S}) = \bigcup \{\mathbf{\Phi}(\mathbf{D}) : \mathbf{D}$ is finite and $\mathbf{D} \subseteq \mathbf{S}\}$;
- $\{(\mathbf{D}, \mathbf{x}) : \mathbf{x} \in \mathbf{\Phi}(\mathbf{D})\}$ is an $\mathbf{A}$-r.e. set.

A set is closed iff it is in the range of $\mathbf{\Phi}$; $\mathbf{C}_{\mathbf{\Phi}}^{\mathbf{A}}$ is the collection of all $\mathbf{A}$-r.e. closed sets.

# Background on Closure Operations

Matroids generalise the following two concepts:

- Vector spaces and linear closure.
- Equivalence relations and their closure.

Note every closure operation is a matroid:

- Topological closure does not satisfy the finiteness condition $\Phi(S) = \bigcup\{\Phi(D) : D$ is finite and $D \subseteq S\}$.
- Closure operations by forming subgroups or ideals in groups or rings, respectively, fail to satisfy the following axiom:

  If $a \in \Phi(\Phi(S) \cup \{b\}) - \Phi(S)$
  then $b \in \Phi(\Phi(S) \cup \{a\})$.

  To see this, consider $S$ to be the multiples of $9$ in the integers and $a = 3$ and $b = 1$.

# Matroids in Recursion Theory

If $\Phi(S) = S$ for all sets $S$ then $\Phi$ is the full matroid and $S \in C_{\Phi}^{A} \Leftrightarrow S$ is an $A$-r.e. set.

A matroid $(\Phi, \mathbb{N})$ is Noetherian iff there is no infinite ascending chain of sets in $C_{\Phi}^{A}$.

There is an $A$-recursive Noetherian matroid with $\Phi(S) = A$ for $S \subseteq A$, $\Phi(S) = \mathbb{N}$ for $S \not\subseteq A$ and $C_{\Phi}^{A} = \{A, \mathbb{N}\}$.

Let $\Phi(S) = S$ if $|S| < 60$ and $\Phi(S) = \mathbb{N}$ if $|S| \geq 60$. Then $(\Phi, S)$ is a recursive Noetherian matroid.

Let $B$ be a maximal set and $\approx$ be the equivalence relation given by $x \approx y \Leftrightarrow \{x, x+1, \ldots, y-1\} \subseteq B$. Let $\Phi(S)$ be the closure of $S$ under $\approx$. $(\Phi, \mathbb{N})$ is an r.e. matroid which is not Noetherian.

# II. Learning Theory

Learner reading data and outputting hypotheses.

| Data | Hypotheses |
|---|---|
| 2 | Set of even numbers; |
| 2,3 | Set of all numbers; |
| 2,3,5 | Set of prime numbers; |
| 2,3,5,13 | Set of prime numbers; |
| 2,3,5,13,1 | Set of Fibonacci numbers; |
| 2,3,5,13,1,8 | Set of Fibonacci numbers. |
| 2,3,5,13,1,8,21 | Set of Fibonacci numbers. |

Learner outputs a sequence of conjectures which eventually stabilizes on the correct one.

# General Setting

Class $\mathbf{C}$ of sets to be learnt; all sets are $\mathbf{A}$-r.e. subsets of $\mathbb{N}$.

Learner reads more and more data from an infinite sequence $\mathbf{a_0, a_1, \ldots}$ (called text) consisting of all members of some set $\mathbf{L} \in \mathbf{C}$.

Recursive learner (without access to $\mathbf{A}$) conjectures $\mathbf{A}$-r.e. index $\mathbf{e_n}$ for data $\mathbf{a_0\, a_1\, \ldots\, a_n}$.

Explanatory learning: Almost all $\mathbf{e_n}$ are the same $\mathbf{A}$-r.e. index $\mathbf{e}$ of $\mathbf{L}$.

Behaviourally correct learning: Almost all $\mathbf{e_n}$ are $\mathbf{A}$-r.e. indices of $\mathbf{L}$ (all $\mathbf{e_n}$ can be different).

Partial learning: Infinitely many $\mathbf{e_n}$ equal one correct $\mathbf{A}$-r.e. index $\mathbf{e}$ and no other index is output infinitely often.

# Learnable Classes

Explanatorily Learnable

The class of all finite sets is explanatorily learnable: learner conjectures at each time the range of the data seen so far.

The class of self-describing sets $\{\mathbf{L} : \mathbf{W}_{\min(\mathbf{L})} = \mathbf{L}\}$ is explanatorily learnable: learner conjectures minimum element seen so far as hypothesis.

The class of all subvector spaces of $\mathbb{Q}^{\mathbf{n}}$ is learnable (in appropriate coding).

Behaviourally correctly learnable

The class of all sets $\mathbf{A} \cup \mathbf{B}$ where $\mathbf{A}$ is a fixed r.e. non-recursive set and $\mathbf{B}$ is finite is behaviourally correctly learnable but not explanatorily learnable.

The class $\{\mathbf{L} : \exists \mathbf{e} < \min(\mathbf{L}) : \mathbf{W}_{\mathbf{e}} = \mathbf{L}\}$ is behaviourally correctly learnable but not explanatorily learnable.

# Unlearnable Classes

Theorem [Gold 1967]
A class containing an ascending chain $\mathbf{A_0}, \mathbf{A_1}, \ldots$ of sets and also their union $\mathbf{B}$ is not behaviourally correctly learnable.

Reason: Blum and Blum's Locking sequence argument. If $\mathbf{M}$ is a learner for $\mathbf{B}$, then there is a $\sigma \in \mathbf{L}^*$ such that the learner conjectures $\mathbf{B}$ on all inputs $\sigma\tau$ with $\tau \in \mathbf{B}^*$. Now there is an $\mathbf{A_n}$ with $\mathbf{range}(\sigma) \subseteq \mathbf{A_n}$ and $\mathbf{M}$ does not learn $\mathbf{A_n}$.

Theorem
The class of all graphs of recursive functions is not behaviourally correctly learnable.

Reason: Given a learner succeeding on a dense class of functions, one can make a recursive function which diagonalises exactly this learner.

# III. Noetherian Matroids

Theorem

For a recursive Noetherian matroid, one can make a recursive learner revising its hypotheses at most $\mathbf{c}$ times; the learner is consistent (each conjecture generates all the data seen so far) and conservative (each change of hypothesis is justified by an inconsistency of the previous conjecture with the current data).

Here $\mathbf{c}$ is the dimension of the matroid, that is, the minimum size of a set generating $\mathbb{N}$. In general, each closed set $\mathbf{R}$ has a dimension and if $\mathbf{R} \subset \mathbf{S}$ then the dimension of $\mathbf{R}$ is below the one of $\mathbf{S}$.

The learner updates its conjecture each time the dimension of the language generated by the data increases and conjectures then $\mathbf{\Phi(D)}$ for the data seen so far. In Noetherian r.e. matroids, one can compute the dimension of $\mathbf{\Phi(D)}$.

# Learning non-r.e. Matroids

**Theorem**

$C^A_\Phi$ can be learnt behaviourally correctly iff $(\Phi, \mathbb{N})$ is an $A$-r.e. Noetherian matroid.

**Algorithm**

If $D$ is the set of data seen so far then the learner conjectures an $A$-r.e. index for $\Phi(D)$.

**Necessity**

If an $A$-r.e. matroid $(\Phi, \mathbb{N})$ is not Noetherian then it contains a set which is not finitely generated. $\mathbb{N}$ is a superset of this set and also not finitely generated. Now the ascending chain $A_n = \Phi(\{0, 1, \ldots, n\})$ consists of $A$-r.e. sets and each of them is in $C^A_\Phi$. So also their union $\mathbb{N}$. This is impossible for a behaviourally correct learner.

# IV. Partial Learning

Theorem [Osherson, Stob and Weinstein 1986]
The class of all r.e. sets can be partially learnt.

Main idea: Taking a one-one numbering $\mathbf{W_{f(0)}, W_{f(1)}, \ldots}$, output $\mathbf{f(e)}$ at least $\mathbf{n}$ times iff there are at least $\mathbf{n}$ stages $\mathbf{s}$ such that each $\mathbf{x \leq n}$ is in $\mathbf{W_{f(e),s}}$ iff $\mathbf{x}$ appeared within the first $\mathbf{s}$ data-items observed.

Learner outputs no index infinitely often if language $\mathbf{L}$ to be learnt is not an r.e. set.

Reliable Partial Learning
A class $\mathbf{C^A}$ is reliably partially learnable iff
(a) on every text for some $\mathbf{L \in C^A}$, the learner outputs one index $\mathbf{e}$ infinitely often and this index satisfies $\mathbf{W_e^A = L}$ and
(b) on every text for some $\mathbf{L \notin C^A}$, the learner outputs no index infinitely often.

# Reliable Partial Learning

Theorem

A class $\mathbf{C}^{\mathbf{A}}$ is reliably partially learnable if there is a limit recursive set $\mathbf{E}$ such that every $\mathbf{B} \in \mathbf{C}^{\mathbf{A}}$ equals to some $\mathbf{W}_{\mathbf{e}}^{\mathbf{A}}$ with $\mathbf{e} \in \mathbf{E}$ and the set $\{(\mathbf{e}, \mathbf{x}) : \mathbf{e} \in \mathbf{E} \wedge \mathbf{x} \in \mathbf{W}_{\mathbf{e}}^{\mathbf{A}}\}$ is limit-recursive.

Algorithm

Let $\mathbf{f}$ be a padding function. Now conjecture an index $\mathbf{f}(\mathbf{e}, \mathbf{d})$ at least $\mathbf{n}$ times iff there is a time $\mathbf{t} > \mathbf{n}$ such that $\mathbf{e} \in \mathbf{E}_{\mathbf{t}}$, $\mathbf{d}$ times a pair $\mathbf{f}(\mathbf{e}', \mathbf{d}')$ with $\mathbf{e}' < \mathbf{e}$ has been conjectured so far and for all $\mathbf{x} \leq \mathbf{n}$, the $\mathbf{t}$-th approximation of $\mathbf{W}_{\mathbf{e}}^{\mathbf{A}}(\mathbf{x})$ is $\mathbf{1}$ iff $\mathbf{x}$ has been observed on the input so far.

Only correct indices in $\mathbf{E}$ have $\mathbf{f}(\mathbf{e}, *)$ output infinitely often; only the least such $\mathbf{e}$ has that $\mathbf{f}(\mathbf{e}, \mathbf{d})$ is output infinitely often for some $\mathbf{d}$.

# Applications

Theorem
The closed sets of a Noetherian $\mathbf{A}$-recursive matroid are reliably partially learnable iff $\mathbf{A} \leq_{\mathbf{T}} \mathbf{K}$.

Theorem
The class of all $\mathbf{A}$-r.e. sets is reliably partially learnable iff $\mathbf{A}$ is low $(\mathbf{A}' \equiv_{\mathbf{T}} \mathbf{K})$.

Theorem
The class of all $\mathbf{A}$-recursive sets is reliably partially learnable iff $\mathbf{A}$ is low$_2$ $(\mathbf{A}'' \equiv_{\mathbf{T}} \mathbf{K}')$ and $\mathbf{A} \leq_{\mathbf{T}} \mathbf{K}$.

For one direction, one just uses the the corresponding classes are uniformly $\mathbf{K}$-recursive. For the other direction, one has to make proofs that learning requires $\mathbf{A}$ to be of the corresponding form. Proofs require reliability of the learner.

# V. Learning all A-recursive Sets

Theorem

There is a learner $\mathbf{M}$ such that $\mathbf{M}$ learns, for uncountably many $\mathbf{A}$, the class of all $\mathbf{A}$-recursive sets partially.

Idea

There is a recursive tree $\mathbf{T}$ with only nonrecursive infinite branches such that each two infinite branches are Turing incomparable and all hyperimmune-free infinite branches have minimal Turing degree.

When $\mathbf{B}$ is a recursive set, $\mathbf{M}$ will partially identify an index $\mathbf{e}$ of $\mathbf{B}$ such that $\mathbf{W_e^A = B}$ for all oracles $\mathbf{A}$. If $\mathbf{B}$ is not recursive then $\mathbf{B \equiv_{tt} A}$ for a unique infinite branch $\mathbf{A}$ of $\mathbf{T}$; $\mathbf{M}$ will partially identify an index $\mathbf{e}$ which codes (in a padded way) a tt-reduction from $\mathbf{B}$ to $\mathbf{A}$ as well as a tt-reduction from $\mathbf{A}$ to $\mathbf{B}$ and satisfies that $\mathbf{W_e^A = B}$.

# Learning all A-r.e. Sets

Theorem

Assume that $\mathbf{A} \leq_{\mathbf{T}} \mathbf{K}'$. Then the class of all $\mathbf{A}$-r.e. sets is partially learnable iff $\mathbf{A}$ is low $(\mathbf{A}' \equiv_{\mathbf{T}} \mathbf{K})$.

Questions

1. Is there any further $\mathbf{A}$ such that the class of all $\mathbf{A}$-r.e. sets is partially learnable?
2. Is there a learner $\mathbf{M}$ such that $\mathbf{M}$ learns the class of all $\mathbf{A}$-r.e. sets for uncountably many $\mathbf{A}$?

# Some Remarks

One might ask why does the result that the classes of all $\mathbf{A}$-recursive sets can be learnt for uncountably many sets $\mathbf{A}$ carry over to the r.e. case?

Problem 1.: The proof method uses a recursive tree such that every nonrecursive set has only an index relative to one infinite branch; this branch was reconstructed from the set. When dealing with r.e. sets, these might be r.e. relative to various infinite branches, so that the branch is not uniquely determined by the set to be learnt.

Problem 2.: The proof uses that there is a tt-reduction from the set to be learnt to the infinite branch $\mathbf{A}$ relative to which the set is computed. In the r.e. case there might not even be a Turing reduction, this gives additional problems.

# Confident Partial Learning

Definition
A partial learner is confident iff it outputs on every text exactly one index, even if the text does not belong to any set to be learnt.

Theorem
The class of all cofinite sets does not have a confident partial learner.

Theorem [Ziyuan Gao, Master Thesis]
Every explanatorily learnable class is confidently partially learnable; confidently partially learnable classes are closed under union.

Hence there are classes which are confidently partially learnable but not behaviourally correctly learnable.

# Noetherian Matroids Revisited

Theorem

If $(\mathbb{N}, \Phi)$ is a Noetherian $A$-r.e. matroid with $A \leq_T K$ then $C_\Phi^A$ is confidently partially learnable.

Algorithm

Learner outputs an $A$-r.e. index for $\Phi(D)$ at least $n$ times iff there is a $t \geq n$ such that $|D| \leq c$ for the dimension $c$ of the matroid, all data in $D$ have been observed within $t$ steps and all data observed below $n$ are in $\Phi_t(D)$ and no length-lexicographically smaller set $D'$ has the same properties (at the current $t$).

Question

Let $(\mathbb{N}, \Phi)$ be a r.e. matroid whose r.e. closed sets are confidently partially learnable. Is $(\mathbb{N}, \Phi)$ Noetherian?

# Summary

I. Matroids and closed sets
Axiomatising closure in vector spaces and equivalence relations
II. Learning theory
Notions of explanatory, behaviourally correct and partial learning
III. Learning of Noetherian matroids
Good learning behaviour, but restricted models
IV. Partial learning of non-Noetherian matroids
Which **A**-r.e. matroids have recursive learners?
V. Learning all **A**-recursive / **A**-r.e. sets
Possible for uncountably many **A**?
VI. Confident partial learning
Can confident partial learning handle non-Noetherian matroids?

# Auckland

First journey to Auckland in 2000; came ill from the South Island.
Second journey to Auckland in 2004.
Subsequent visits in 2008 and 2010 and now 2012.
Cristian Calude returned visits and was in Singapore several times.

Fruitful collaboration on randomness:

- Representation of left-computable epsilon-random reals.
- Universal recursively enumerable sets of strings.

New-Zealand also home to many textbooks on randomness:

- Information and Randomness - An Algorithmic Perspective.

All the best for future fruitful work and Happy Birthday!