Exponential decay in Quantum mechanics

V. kruglov [1], K. Makarov [2], B. Pavlov [3,4], A. Yafyasov [4]

[1]Dept. of Physics at the University of Auckland, New Zealand,
[2]Department of Mathematics at the University of Missuri-Columbia, USA,
[3]New Zealand Institute of Advanced Study, Massey University, New Zealand,
[4]V.A. Fock Phys Inst. at the Dept. of Physics ,St. Petersburg University,Russia.

Computations, physics and beyond (WTCS)
The University of Auckland, New Zealand.
21- 24 February, 2012.

Exponential Decay: physical needs versus mathematical beauty.

V. Arnold, in his prominent interview (1993) with Sergey Kapitsa, commented on the controversial idea by Paul Adrien Moris Dirac (formulated, in particular in his lecture at Moscow University in 1955) that *Physical laws should have mathematical beauty* (see the above epigraph). Arnold's comment is even more straightforward than the original version by Dirac which was softer by nature. On top of Dirac's receipt about choosing direction of a new step: "it's future development should affect something, which was out of any doubts before, something which could not be revealed by the axiomatic formulation", see [**?**], Arnold comment contains an inspiring advice on how and where to find the new physics.

Exponential Decay: physical needs versus mathematical beauty.

Of course, both statements by Dirac/Arnold are literally wrong. Both are about the final formulation of the theory, when "research is already dead", but not the first revolutionary movement in the new direction. Max Planck's formulation of the leading idea of quantum physics - on the discrete nature of the light radiation from the cavity - had no connection with any beautiful mathematics at that time (in the beginning of the 20^{th} century). The initial mathematical formulation of the essence of Quantum Physics appeared almost 30 years later due to John von Neumann [**?**]. Since that moment a lot of new important details were added to it, but, really, all subject noticeably drifted towards mathematics.

In this paper, oriented on a wide audience of theoretical physicists and researchers in natural sciences we provide a review of a beautiful chapter of modern mathematics: the Harmonic Analysis of Operators in Hilbert Spaces, see [17], which arose in the first half of the previous century as a branch of Complex Analysis. However, it was never used by physicists as it probably deserves, according to our vision. Following the above receipt of Arnold, we choose the Decay problem in Quantum Physics, and consider the implications of Harmonic Analysis on it. We hope that our attempt may eventually bring this beautiful piece of mathematics into the arsenal of mathematical tools of natural sciences.

Exponential Decay: physical needs versus mathematical beauty.

The classical question on the validity of Quantum Mechanics for the description of Decay of the wave-packet rarely was on the front line of research in Physics. In fact, for almost 90 years after the revolutionary paper by Gamow [**?**], it has been considered more as an annoying nuisance. Nevertheless, many great minds contributed competing points of view on the subject. In 1930, Weisskopf and Wigner suggested a persuading concept (further referred to as the WW concept) of the exponential decay rate for a quantum system with discrete spectrum, see [1]. Their proposal was recognized by most experimentalists as a viable treatment of the subject.

Exponential Decay: physical needs versus mathematical beauty.

Unfortunately, 17 years later, Fock and Krylov spoiled the happy end, see [2], by showing “from the first principles”, that exponential decay cannot be explained based on the discrete spectrum hypothesis, leaving only one way out: considering quantum systems with continuous spectrum. This new concept (further denoted as KF-concept), proposed by Fock and Krylov, also sounds natural. Indeed, Fock was the first physicist to suggest, in his textbook on Quantum Mechanics [3], an accurate treatment of the continuous spectrum. This remained unchanged up to now in all modern texts on Mathematics and Mathematical Physics. Yet KF concept did not become a gravestone for the question on Decay.

Exponential Decay: physical needs versus mathematical beauty.

A paper by L. Khalfin, [8], communicated by Fock to the Russian Academy Doklady, contained an accurate calculation, again "from the first principles", on the decay of wave-packets for the simplest quantum problems, in particular for the 1D Schrödinger equation with compactly supported real potential. Represent the evolution of the wave-packet by the Riesz integral of the resolvent $R_\lambda \equiv (H - \lambda I)^{-1}$ of the corresponding Hamiltonian H,

$$e^{iHt} = -\frac{I}{2\pi i} \int_\Gamma R_\lambda \, e^{i\lambda t} \; d\lambda, \qquad (1)$$

on the contour Γ enclosing the spectrum σ of H. Using analyticity of the integrand on the two-sheets Riemann surface of the spectral parameter, one can deform $\Gamma \to \Gamma'$ to reveal i) components of th3e evolution operator.

Exponential Decay: physical needs versus mathematical beauty.

Unfortunately, the corresponding contribution to evolution (1) of the latter is estimated by the power function Const $t^{-\beta}$ of the time t, with β depending on the incident data.
This result actually revealed the error by Gamov and could possibly resolve the problem on the Decay, if the non-exponential component of the decay would be observed in an experiment. Surprisingly that was not the case up to now. Nevertheless, in the modern textbook on Quantum Mechanics [7], the result of L. Khalfin is quoted as an ultimate truth on Decay.

Exponential Decay: physical needs versus mathematical beauty.

In this paper we aim again on the quoted proposals by [1, 2], attempting to find a point of view that would permit for the ends to meet one another. Our program does not eliminate the naive theoretical analysis presented in [8], but reduces the discussion of validity of it to the problem of the *choice of measurement tools* that deliver the data from the quantum system to the observer. In this paper, we consider the case when the role of the “delivering tool” is played by the electromagnetic field, or, generally, by another zero-mass field . In the 1D example considered in this paper, the role of the delivering tool is played by a massless field governed by the wave equation, a 1D analog of photons.

Exponential Decay: physical needs versus mathematical beauty.

We will postpone for an upcoming publication a more realistic choice of the delivering tool as a classical electro-magnetic field in R_3 which also satisfies all natural assumptions we are basing on now.
Some of the basic mathematical tools that we use to interpret the exponential decay are already prepared by mathematicians. Similar situation was observed in Quantum Mechanics, where an exact understanding of selfadjointness (the physicists required for the Schrödinger theory by mid twenties) was already prepared by Hermann Weyl in 1916, see an adapted text in [10].

Exponential Decay: physical needs versus mathematical beauty.

In our treatment of the exponential decay, we use analysis of the acoustic scattering problem. Again, it was prepared by Peter Lax and Ralph Phillips in the 60-s of the last century (see [11]). Actually, Lax and Phillips succeeded to overcome, without even noticing it, the horror physicists survived when they discovered that the evolution of a quantum system with positive Hamiltonian L may be generated by another operator $\mathcal{L}$, which has both negative and positive branches of spectrum.

Exponential Decay: physical needs versus mathematical beauty.

This phenomenon, discovered by Dirac in the 30-s, was rigorously analyzed by Hegerfeld in his prominent theorem [16] only at the end of previous century. Hegerfeldt was able to show that the evolution of a quantum system with positive Hamiltonian always has “infinite tails”. For instance, the component of the 1D acoustic evolution in the positive frequency sector is represented by D’Alembertian waves that admit analytic continuation into the upper half-plane, and thus cannot vanish on a set with positive measure on the real axis. This was an essential step to legitimizing the non-semi-bounded generators of evolution. Another example of a non-semi-bounded generator is given by the supercharge in super-symmetric quantum mechanics.

Exponential Decay: physical needs versus mathematical beauty.

So, by the end of previous century, the trick suggested by Lax and Phillips would not look surprising any more. But in the mid-century, it was probably still too special and suspicious for physicists: Lax and Phillips represented the evolution of the Cauchy data $\mathbf{u} \equiv \left(u, c^{-1} u_t\right)$ of the wave equation

$$c^{-2}\, u_{tt} - \Delta u = 0$$

as a unitary transformation in the energy-normed space of the Cauchy data

$$\parallel \mathbf{u} \parallel^2 = \frac{1}{2} \int_{\Omega_{out}} \left[c^{-2} |u_t|^2 + |\nabla u|^2 \right] dx^3. \qquad (2)$$

Exponential Decay: physical needs versus mathematical beauty.

It was shown in [11] that the unitary evolution group $e^{i\mathcal{L}t} \equiv U_t \ : \mathbf{u}(0) \longrightarrow \mathbf{u}(t)$ is generated by a non-semi-bounded operator $\mathcal{L}$, an analog of the Dirac operator, that can be represents as an appropriate block operator matrix

$$\frac{1}{i}\frac{\partial \mathbf{u}}{c\partial t} = i\begin{pmatrix} 0 & -1 \\ -\Delta & 0 \end{pmatrix}\mathbf{u} \equiv \mathcal{L}\,\mathbf{u}, \quad \mathcal{L}^2 = -\Delta. \qquad (3)$$

Exponential Decay: physical needs versus mathematical beauty.

It turns out that (i) the generator $\mathcal{L}$ is self-adjoint in the energy-normed space $\mathcal{E}$ of all Cauchy data with finite energy, (ii) the spectrum of $\mathcal{L}$ in the energy-normed space of Cauchy data supported by the complement Ω_{out} of a compact domain $\Omega \subset R_3$ is absolutely continuous and it fills in the whole real axis (iii) the unitary group U_t has incoming and outgoing subspaces $\mathcal{D}_{in}, \mathcal{D}_{out}$ that are invariant with respect to the positive and negative semi-groups U_t and U_{-t}, $t \geq 0$, respectively. In fact, these subspaces consist of the data vanishing on the positive and negative light-cones respectively

$$U_t \mathcal{D}_{out} \subset \mathcal{D}_{out}, \quad t \geq 0, \qquad U_t \mathcal{D}_{in} \subset \mathcal{D}_{in}, \quad t \leq 0, \tag{4}$$

Incoming and outgoing waves on the complement $R_3 \backslash \Omega \equiv \Omega_{out}$ are mutually orthogonal with respect to the energy dot-product.

Exponential Decay: physical needs versus mathematical beauty.

Fortunately, by that time the question on description of invariant subspaces of an important isometry group in the Hilbert space was already solved by Arno Beurling [**?**] *with no connection to the above acoustic problem*. Beurling considered in 1947 the shift operator T (right translation) in the Hilbert space l_2 of all complex square-summable sequences
$\vec{x} = (x_0, x_1, x_1, x_2, x_3, \dots)$

$$(x_0, x_1, x_1, x_2, x_3, \dots) \xrightarrow{T} (0, x_0, x_1, x_1, x_2, x_3, \dots) \equiv t\vec{x}.$$

Exponential Decay: physical needs versus mathematical beauty.

One of Beurling's problems in [**?**] was the description of all invariant subspaces $\mathcal{D}$ of T : $T\mathcal{D} \subset T$. Obviously, the space of all sequences, $\sum_s |x_s|^2$, with (several) zeros on the first positions, like $(0, x_1, x_1, x_2, x_3, \dots)$, is invariant with respect to T. What are the others invariant subspaces? It is not that easy to answer the question using the language of the l_2 space. But if we change the language by translating the question into the one of Complex Analysis and substituting the sequences by the functions analytic in the unit disc $\mathrm{B} = \{\zeta : |\zeta| < 1\}$:

$$\vec{X}(x_0, x_1, x_1, x_2, x_3, \dots) \xrightarrow{T} x_0 + \zeta x_1 + \zeta^2 x_2 + \zeta^3 x_3 + \cdots \equiv x(\zeta) \equiv T\vec{x},$$

the problem of the description of invariant subspaces becomes almost trivial, *and this is the beauty*.

Exponential Decay: physical needs versus mathematical beauty.

Translating the question on invariant subspacec into the language of the3 Complex Analysis, we get a marvelous chance to view the problem from a completely new point, substituting T by the multiplication operator : $T\vec{x} \longrightarrow \zeta x(\zeta)$. Indeed, this transformation is a unitary mapping of l_2 onto the class of all analytic functions on the unit disc, with square integrable boundary data on the circle $\Gamma = \{\zeta : |\zeta| = 1\}$. This is the celebrated Hardy class H^2_+: a subspace of $L_2(\Gamma)$ consisting of all functions which admit an analytic continuation onto the unit disc equipped with the norm

$$\frac{1}{2\pi}\int_\Gamma |x(e^{i\theta})|^2 d\theta = |\vec{x}|^2_{l_2}.$$

Exponential Decay: physical needs versus mathematical beauty.

The subspace of all sequences $(0, x_1, x_1, x_2, x_3, \dots)$, with zero on the first position, is transformed into the class ζH^2_+ of all analytic functions in the unit disc vanishing at the center of the disc. It is clear now that all subspaces of functions vanishing at an inner point a are invariant with respect to T and are represented as $\frac{a-\zeta}{1-\bar{a}\zeta} H^2_+$. Of course, all subspaces of the analytic functions in the unit disk generated by finite or infinite Blaschke products $\Pi_{\vec{a}}(\zeta) \equiv \Pi_s \frac{a_s-\zeta}{1-\bar{a}_s\zeta} \frac{a_s}{|a_s|}$, with convergent series $\sum_s (1-|a_s^2|) < \infty$, are invariant subspaces $\mathcal{D}_{out} = \Pi_{\vec{a}} H^2_+$ of the shift operator T : $T\Pi_{\vec{a}} H^2_+ = \zeta \Pi_{\vec{a}} H^2_+ \subset \Pi_{\vec{a}} H^2_+$.

Exponential Decay: physical needs versus mathematical beauty.

Some uniform limits of the Blaschke products give rise to so called singular inner functions $\Theta_\mu(\zeta)$ on the unit disc. They are represented via positive singular measures μ supported by the unit circle as $\Theta_\mu(\zeta) = \exp \int_{|\eta|=1} \frac{\zeta+\eta}{\zeta-\eta} d\mu(\eta)$. The functions Θ_μ also produce invariant subspaces $\Theta_\mu H^2_+$ of the shift, [21, 17]. The full answer to the question about the structure of the *outgoing* invariant subspaces of the shift , $\zeta\mathcal{D}_{out} \subset \mathcal{D}_{out} \subset H^2_+$, is given by the formula

$$\mathcal{D}_{out} = \Theta_\mu \; \Pi \; H^2_+.$$

Similarly, the problem of the description of the invariant subspaces of the left shift U_t, $t < 0$, in the space of all sequences $x = (\dots, -3, -2, -1)$ can be considered with the use of the Hardy class H^2_- of analytic functions on the complement to the unit disc.

Exponential Decay: physical needs versus mathematical beauty.

These subspaces can be constructed from the singular inner factor and the Blaschke product Θ, Π. It is a remarkable fact, that the positive semi-group $\{\zeta^l\}$, $l = 0, 1, 2, 3, \ldots$, of the unitary group ζ^l on $L_2(\Gamma)$, restricted to the *co-invariant subspace* $H^2_+ \ominus \mathcal{D}_{out} \equiv \mathcal{K} = H^2_+ \ominus \Pi H^2_+ \equiv K$ *proves to be a contracting semi-group*

$$P_{\mathcal{K}} \zeta^l \Big|_{\mathcal{K}} \equiv Z^l, \; l = 0, 1, 2, 3, \ldots \; ,$$

with the generator Z. Indeed, since $\zeta P_{H^2_+} \in P_{H^2_+} \perp K$, for $l = 2$, we have:

$$Z^2 = P_K \zeta^2 P_K = P_K \zeta [P_{H^2_+} + P_K + P_{H^2_-}] \zeta P_K = P_K \zeta [P_{H^2_+} + P_K] \zeta P_K = P_K \zeta$$

Exponential Decay: physical needs versus mathematical beauty.

Moreover, the eigenvalues of the generator Z coincide with the zeros a_s of the Blaschke product $\Pi_{\vec{a}}$ and the corresponding eigenfunctions are $\psi_s[\zeta] = \frac{\Pi_{\vec{a}}(\zeta)}{a_s - \zeta}$. In addition, the bi-orthogonal system of eigenvectors of the adjoint operator Z^+ is constituted by the reproducing kernels $\phi_s(\zeta) = \frac{1}{1 - \bar{a}_s \zeta}$, so that the spectral decomposition of Z, with simple discrete spectrum, is given by the interpolation series

$$f = \sum_s \frac{\Pi_{\vec{a}}(\zeta)}{a_s - \zeta} \frac{f(a_s)}{\frac{d\Pi_{\vec{a}}}{d\zeta}(a_s)}, \quad f \in K$$

Similar explicit formulae are also true for the continuous shift of the real axis $f(x) \to f(x - t) \equiv U_t f$.

Exponential Decay: physical needs versus mathematical beauty.

The role of the incoming and outgoing subspaces $\mathcal{D}_{in,out}$ for the continuous shift group in the spectral (Fourier) representation $U_t \equiv e^{ipt}$ is played by the Hardy classes of square-integrable functions $H^2_\pm \subset L_2(R)$ that admit an analytical continuation to the upper and lower half-planes, respectively. In particular, the subspaces ΠH^2_+ generated by the Blaschke products in the upper half-plane are invariant with respect to the (continuous) shift in the Fourier representation. In general, the invariant subspaces of the positive semi-group $U_t, t \geq 0$ are parameterized by the inner functions $\Theta\Pi$ in the upper half-plane as $\Theta\Pi H^2_+ \equiv \mathcal{D}_{out}$, and, for the negative semi-group, the corresponding representation is of the form $\bar{\Theta}\bar{\Pi}H^2_- = \mathcal{D}_{in}$.

Exponential Decay: physical needs versus mathematical beauty.

The restriction of the positive semi-group of the continuous shift onto the orthogonal complement of $L_2(R) \ominus [\mathcal{D}_{in} \oplus \mathcal{D}_{out}] \equiv \mathcal{K}$ of the “incoming” and “outgoing” subspaces $\mathcal{D}_{in,out}$ in $L_2(R) \equiv \mathcal{E}$, with K the corresponding co-invariant subspace, is a strongly-continuous *Lax-Phillips semi-group* $P_{\mathcal{K}} U_t \big|_{\mathcal{K}} =: e^{i\mathcal{B}t}$, $t > 0$, of contractions generated by a dissipative operator $\mathcal{B}$. The spectral properties of the generator $\mathcal{B}$ are completely determined by the scattering matrix $S \equiv \Theta\Pi$ associated with the unitary group U_t and the corresponding unperturbed group U_t^0 which is a colligation of the components of the evolution on the reduced space $\mathcal{E}_0 =: \mathcal{D}_{in} \oplus \mathcal{D}_{out}$, see [20]. Again, similarly to the above discrete case, the spectral analysis of the Lax-Phillips semi-group can be done in an explicit form in terms of the corresponding inner function $\Theta\Pi$, the scattering matrix.

Exponential Decay: physical needs versus mathematical beauty.

Here is another source of beauty: the duality between the geometrical problem on invariant subspaces and relevant spectral questions for contracting and dissipative operators and classical questions on interpolation and approximation from the theory of Analytic functions. Unfortunately, the simple calculations above never appeared in elementary courses of Complex Analysis for physicists or engineers.

Exponential Decay: physical needs versus mathematical beauty.

The question on *exponential decay* for the acoustic problem on the complement of the scatterer Ω in a large ball B_R served as a central motivation for [11]. This problem is reduced to the study of spectral properties of the generator B of the Lax-Phillips semi-group: if all eigenvalues of the generator B are situated strictly in the upper spectral half-plane $\Im\lambda > \beta > 0$, then the Lax-Phillips semi-group admits an exponential estimation

$$\| e^{i\mathcal{B}t}\mathbf{u}_0 \| \leq Ce^{-\beta' t} \| \mathbf{u}_0 \|, \quad t \geq 0,$$

for any $\beta' < \beta$, with an appropriate absolute constant C, depending on β'.

Exponential Decay: physical needs versus mathematical beauty.

Highly nontrivial analysis was developed to prove the bound $\Im\lambda > \beta > 0$, $\lambda \in \sigma_{\mathcal{B}}$, for compact obstacles Ω that satisfy the exterior cone condition. Generally, the whole machinery, developed in [11] to reach the quoted exponential estimate for acoustic scattering, is based on Harmonic Analysis of matrix-valued analytic functions $u \in L_2(E)$. It was motivated by the problem of description of all invariant subspaces of the standard shift groups $u(p) \to e^{ipt}u(p) \equiv u(p,t)$ in the space $L_2(E)$ of vector-valued, square-integrable functions $u(p) \in E$ on the real axis $-\infty < p < \infty$. In fact, the above evolution group U_t is unitarily equivalent to the shift group, and the incoming subspaces of the evolution group U_t are equivalent to subspaces of the Hardy class $H^2_-(E) \subset L_2(E)$ of all square integrable functions admitting an analytic continuation into the lower half-plane $\Im p < 0$, see [21].

Exponential Decay: physical needs versus mathematical beauty.

The outgoing subspaces of the evolution group are unitarily equivalent either to the Hardy class $H^2_+(E) \subset L_2(E)$, or to subspaces ΘH^2_+ of the Hardy class defined by the *inner factors* Θ, which are unitary on the real axis and admit an analytic continuation into the upper half-plane $\Im p > 0$. In the case when Π is a Blaschke product

$$\Pi(p) = \prod_l \left[\frac{p - p_l}{p - \bar{p}_l} \theta_l P_l + P_l^{\perp} \right],$$

with appropriate phase factors θ_l and projections $P_l, \; P_l^{\perp} = I - P_l$, the quantities $\bar{p}_l$ coincide with the eigenvalues of the adjoint generator $\mathcal{B}^+$, and the eigenfunctions of the adjoint generator, in the "incoming" spectral representation of the original unitary group U_t in the energy-normed space $\mathcal{E}$, coincide with the reproducing kernels $\varphi_l = \frac{e_l}{p - \bar{p}_l}$.

Exponential Decay: physical needs versus mathematical beauty.

The bi-orthogonal system of eigenfunctions of the original operator $\mathcal{B}$ is formed as $\psi_l = \frac{\Theta(p)}{p-p_l} e_l^+$, with $e^+ \in \ker \Theta(p_l)$, see [17, 19, 26]. In the general case, these facts are derived from an extended theory of the "functional model" (see, for instance, [17, 19, 26]), which covers the Lax-Phillips generators with absolutely continuous spectrum. The modern theory of the Functional Model allows one to reduce all the questions of the spectral theory of the Lax-Phillips semi-group to the relevant questions of the theory of analytic functions and/or Harmonic Analysis.

Exponential Decay: physical needs versus mathematical beauty.

The crucial role of the theory of analytic functions for the theory of nonselfadjoint operator was predicted by M. G. Krein in his talk at the Moscow International Congress of Mathematicians in 1966, (see, [22, 23]). The problem on exponential Decay should be connected, from the point of view of mathematicians, with the list of problems on spectral analysis of dissipative or contracting operators. In the simplest case of a one-dimensional acoustic problem that we discuss in section 3, most of the above facts of spectral analysis of the Lax-Phillips semi-group are established via straightforward calculations.

Exponential Decay: physical needs versus mathematical beauty.

It must be noted that the first attempt to bridge the general theory of nonselfadjoint (in particular, dissipative) operators with relevant physics was undertaken by M. S. Livshits [18]. He was motivated by the observation that the problem of analysis of nonself-adjoint details of complex physical systems appears each time we attempt to substitute a whole complex system by a simpler surrogate system with similar properties. In [18], M. Livshits suggested a simplified model of a waveguide attached to a resonator, produced by substitution of a nonself-adjoint detail of the original system by a "triangular model", which, at the time was the only available general model of a dissipative operator. Based on Livshits' discovery, a new, more convenient "functional model" was suggested by B. Sz.-Nagy and C. Foias (see [17]). But the role of the scattering matrix as a basic parameter of the functional model was not yet recognized at that stage.

Exponential Decay: physical needs versus mathematical beauty.

Few years later, a seminal paper [20] provided an important connection between the Lax–Phillips scattering theory and the Sz.-Nagy–Foias Functional Model, see [11, 17]. One of he most important achievements of the theory was to give the spectral meaning to resonances, which never happened in the pure quantum mechanical treatment of the problem of the exponential decay.

All these important events succeeded just inside Mathematics. Physicists did not see, until now, any connection between an elegant analysis used by the community of analysts in their study of the acoustic problem or the corresponding abstract shift groups. One of the reasons for that is that the unitary group generated by the semi-bounded Schrödinger operator does not have orthogonal incoming and outgoing subspaces, as it follows from the Hegerfeldt Theorem [16].

Exponential Decay: physical needs versus mathematical beauty.

Nevertheless, an elegant analysis provided by the Lax–Phillips approach served as a motivation for the further research in a close area followed by publishing numerous physical papers. In particular, in [13, 14], the standard Hilbert space L_2 of square-integrable functions was supplied with additional structures transforming it into a space similar to the one used in [11]. In [12], a model Hamiltonian is constructed and an artificial analytic scattering matrix is suggested.

Exponential Decay: physical needs versus mathematical beauty.

In the case studied by Horwitz and Piron, the most important property of the model system in the Lax-Phillips approach, the orthogonality of the incoming and outgoing subspaces, was just formally derived from the analyticity of the constructed model scattering matrix. In recent papers, H. Baumgärtel with coauthors attempted to match the quantum mechanical condition of positivity of the generator of the evolution with spectral interpretation of the resonances to give the spectral meaning to the corresponding "Gamov vectors" (see [**?**, **?**]). Unfortunately, on this way all essential advantages of the Nagy-Foias functional model such as explicit expressions for the eigenvectors of the Lax-Phillips semigroup, the Gamov vectors, completeness of the corresponding bi-orthogonal system, and the relevant spectral decomposition were lost, because of the absence of natural, physically motivated, orthogonal pair of incoming and outgoing subspaces.

Exponential Decay: physical needs versus mathematical beauty.

Besides, no physical consequences were derived in [**?**, **?**] from the proposed matching of Quantum Mechanics with the corresponding analog of the Lax-Phillips theory. This most likely suggests that the scheme proposed in the papers is sentenced, according to the Arnold algorithm, to remain, for another period inside mathematics until all these details are completed. Contrary to that, in our version of bridging standard Quantum Mechanics with the Lax-Phillips theory, instead of inventing an artificial construction added on top of the standard quantum space of all square-integrable functions in order to imitate the Lax-Phillips structure, we consider excitations of the zero mass field playing the role of a channel passing information to the outside observer on the inner quantum system. Although the evolution of the "inner" the Quantum System, for a finite time, can be represented in the Schrödinger form as e^{iLt} with a positive Hamiltonian L, the study of its asymptotics as $t \to \infty$

Exponential Decay: physical needs versus mathematical beauty.

The substitution of the Lorenz invariant picture by the Schrödinger picture of evolution can only be done under the "positivity of mass" condition (see next section). It is not trivial to match this requirement with the zero-mass condition for the Lax-Phillips scheme.
Thus, the central question in our treatment becomes the matching of the Lax-Phillips scattering scheme with Quantum Mechanics with the positive Hamiltonian, that is the question on the physical realization. And again, the answer to this question is not general and does not look obvious.

Scattering of photons by a superconductor : an interplay between the Schrödinger equation and the Klein-Gordon-Fock equation

Yet an interesting example of similar matching can be found in the scattering of photons by a superconductor. Indeed, due to the Meissner effect, magnetic field cannot penetrate the super-conducting medium. The theoretical treatment of the phenomenon by Ginzburg and Landau (see [4]) is based on acquiring a non-zero mass by photons in the process of spontaneous symmetry breaking, the loss of abelian gauge invariance of the Lagrangian of the electromagnetic field in the superconductor. Hence, both contradictory requirements of zero-mass in the outer space, and the non-zero mass in the inner space are satisfied. Thus, we may hope to “put both ends together" in the problem .

Consider a compact domain in R_3 filled with a superconductor. The Lagrangian of the electro-magnetic field in the outer space is represented in terms of the field A, the electromagnetic potential, as

$$\frac{1}{4}\int_{\hat{\Omega}_s} F^+ F, \text{ where } F = dA,$$

(see for instance [9]). Here dA is an exterior differential of the field A, and $F^+ F$ is a 3-form obtained as an exterior product of 2-form F and its (hermitian) complement. In the inner space, due to the interaction of the electromagnetic field with the boson field of Cooper pairs, the Lagrangian is modified, in the boundary area of the superconductor, by additional massive terms containing the product of the electromagnetic field and

Scattering of photons by a superconductor : an interplay between the Schrödinger equation and the Klein-Gordon-Fock equation.

The depth of penetration of the magnetic field into the superconductor is estimated by the size δ of the Cooper pair, which is normally relatively large, greater than 10^{-7} cm. If the energy of photons does not exceed the Bardeen-Cooper-Schrieffer gap (the BCS - gap), the field of Cooper pairs can be eliminated and the scattering of photons by the superconductor can be treated in the one-body photon's sector, similar to the scattering problem in the classical Quantum Mechanics. In the one-body photon's sector, the scattering problem in vacuum $\hat{\Omega}_s$ can be reduced to the wave equation (the Klein-Gordon-Fock equation with zero mass). Similarly, the problem in Ω_δ is also reduced to the Klein-Gordon-Fock equation with non-zero mass.

The corresponding scattered waves satisfy smooth matching conditions on the common boundary of $\hat{\Omega}_s$ and Ω_δ. If the domain Ω_s is filled with a superconductor, then the electromagnetic potential should vanish on the common boundary $\partial\Omega_s \cap \partial\Omega_\delta$. Thus, one can consider, as a representative model, the Klein-Gordon-Fock equation in $R_3 = \Omega_s \cup \Omega_\delta \cup \hat{\Omega}_s$ assuming that the compact domain $\Omega_s \cup \Omega_\delta$ is filled with the superconductor, and $\hat{\Omega}_s$ is the vacuum. The mass is zero on $\hat{\Omega}_s$, but is non-zero on the δ-thin shell Ω_δ, separating the inner and the outer spaces. While Ω_s is filled by the superconductor, the electromagnetic field does not penetrate Ω_s, so that we can apply a zero boundary condition on $\partial\Omega_s \cup \partial\Omega_\delta$.

Scattering of photons by a superconductor : an interplay between the Schrödinger equation and the Klein-Gordon-Fock equation

Then the spectrum of the Klein-Gordon-Fock operator in Ω_δ is discrete, and the one on the complement $\hat{\Omega}_s$ is continuous. Hence, the scattering in the small energy region, for energy not exceeding the creation threshold of the Cooper pair, has a resonance character. The scattering matrix of the problem is unitary and analytic with respect to the energy on the complement of the discrete set of resonances. For small values of the added energy $E' = E - mc^2, \; E' << mc^2$, the evolution on Ω_δ can be described in a Schrödinger form:

$$E = c\sqrt{m^2c^2 + p^2} \approx mc^2 + \frac{p^2}{2m}.$$

Scattering of photons by a superconductor : an interplay between the Schrödinger equation and the Klein-Gordon-Fock equation

Indeed, considering on Ω_δ the Klein-Gordon-Fock equation with non-zero mass

$$\frac{\hbar^2}{c^2}\frac{\partial^2\psi}{\partial t^2} = \left[\hbar^2\Delta - m^2c^2\right]\psi,$$

permits to split off the fast oscillations by the unitary transformation $\psi = e^{-imc^2\hbar^{-1}t}\phi$:

$$\frac{\partial\psi}{\partial t} = \left[\frac{\partial\phi}{\partial t}e^{-imc^2\hbar^{-1}t} - imc^2\hbar^{-1}\phi\, e^{-imc^2\hbar^{-1}t}\right] \approx - \frac{imc^2}{\hbar}\phi\, e^{-imc^2\hbar^{-1}t},$$

$$\frac{\partial^2\psi}{\partial^2 t} \approx -\left[\frac{2imc^2}{\hbar}\frac{\partial\phi}{\partial t} + \frac{m^2c^4}{\hbar^2}\phi\right] e^{-imc^2\hbar^{-1}t}, \qquad (5)$$

Scattering of photons by a superconductor : an interplay between the Schrödinger equation and the Klein-Gordon-Fock equation

A nice feature of this equation is the possibility to interpret $|\phi|^2$ as the probability density for the particle to bound at the location marked by space coordinates (x, t) of the wave function $\phi(x, t)$, with the total probability to find the particle in the space is conserved $\int |\phi(x, t)|^2 dx = const$. But the formal use of it in the large time scale would give a non-exponential decay of the wave packet of the magnetic field. Moreover, vice versa, a straightforward analysis based on the Lax-Phillips scattering arguments for the zero-mass field in $\hat{\Omega}_s$ and the non-zero mass in the Klein-Gordon-Fock equation on Ω_δ shows an exponential decay, and even reveals the spectral meaning of resonances.

Scattering of photons by a superconductor : an interplay between the Schrödinger equation and the Klein-Gordon-Fock equation

Another interesting example of the exponential decay can be connected with a similar problem for a thin compact super-conducting shell Ω_δ separating the inner *vacuum* domain Ω_s from the outer domain $\hat{\Omega}_s$. Considering the one-particle scattering problem with smooth matching conditions on the inner and the outer components of the boundary of the shell, we again obtain a Lax-Phillips Scattering System. Taking into account the non-zero mass of the field on the shell, we see that the low-energy resonances arise from the discrete spectrum of the Dirichlet problem for the Klein-Gordon-Fock equation on the shell.

Scattering of photons by a superconductor : an interplay between the Schrödinger equation and the Klein-Gordon-Fock equation

A relevant physical phenomenon was observed on a multi-layer shell constructed of carbon nano-structures (see, for instance, [**?**]). In that paper, the resonance pumping phenomenon was discovered. Our previous analysis of the super-conducting shells allow us to formulate a question on the superconduction nature of the carbon shell in the experiment, which would explain the nature of pumping based on the classical Lax-Phillips resonance scattering (see next section).

The fields with nonzero mass play an important role in the transition from the Klein-Gordon-Fock evolution to the Schrödinger evolution. One may guess that other possible experiments revealing an exponential decay in quantum physics can be considered with involvement of some scalar boson fields

This gives us a pretext to underline a unique role of measurements based on zero-mass fields in quantum physics. In combination with the symmetry breaking and mass creation, these measurements may help to explain the exponential decay and resonance pumping in these experiments.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

Our review of Lax-Phillips technique and basic results presented in section 1 shows just a top of an Iceberg, with the rest of estimations, complex and harmonic analysis remained undercover. In the final part of section 1 of this paper we provided only a sketch of results obtained by Lax-Phillips technique for general multi-dimensional decay problem.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

In particular, modern technique of Lax-Phillips Scattering Theory permits to realize the one-body program mentioned in previous section. But we aim now on a simple aim concentrating on a simplest 1D model, for which all analytical details of the Lax-Phillips resonance scattering theory can be derived explicitly with use of standard tools of spectral theory of ordinary differential operators. Correspondingly, we select a simplest 1D model of Decay in form of a Klein-Gordon-Fock equation with quantum well potential supported by $[-a, 0]$ and zero boundary condition at the end $x = -a$.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

The quantum well is attached to the positive half-axis.

$$c^{-2}u_{tt} - \frac{\partial^2 u}{\partial x^2} + V_H(x)u = 0, \ -a < x < \infty, \ u(-a) = 0. \qquad (7)$$

Instead of the shell Ω_δ supporting the non-zero mass we assume that the potential contains a repulsing singularity $H\delta(x), \ H > 0$ at the origin, $V_H(x) = V(x) + H\delta(x)$, with a smooth real component $V(x), \ -a < x < 0$.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

This δ- singularity emulates the condition of domination of the BKS gap by the energy of photons and plays a role of a high potential barrier which separates the inner part Ω_s of the outer componen , with the zero-mass field in the outer space $x > 0$. Changing the "hight" H of the barrier one can approach the limit $H = \infty$, which corresponds to the zero boundary condition $u(0) = 0$ decoupling the inner and the outer subsystems.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

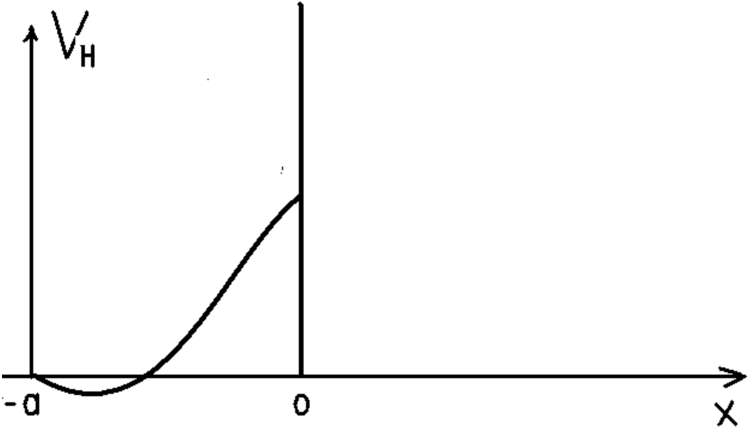

Figure: A simplest 1D model of a nuclear decay

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

The excitations inside the well $[-a, 0] \equiv \Omega_s$, are not observed independently, but only due to their connection to the photon's field. Following our proposal formulated in previous section, we introduce the slow varying component ψ of the wave-function $u = \psi(x)\, e^{-imc^2 t}$ and assume, that the variation of the kinetic energy associated with slow variables $\frac{d^2}{dt^2} c^{-2} \parallel \psi_t \parallel^2$ is relatively small and can be neglected so that we get the Schrödinger equation with $\omega = mc^2$:

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

$$-2i\omega\, c^{-2}\, \psi_t - \frac{\partial^2 \psi}{\partial x^2} + V_H(x)\psi - \omega^2\, c^{-2}\psi = 0,\ -a < x < \infty. \quad (8)$$

Eliminating the non-essential additive constant from the potential, we obtain the Schrödinger type equation

$$-2i\omega\, c^{-2}\, \Psi_t - \frac{\partial^2 \Psi}{\partial x^2} + V_H(x)\Psi = 0,\ -a < x < \infty, \quad (9)$$

which can be transformed to the standard form via introducing the effective mass of the excitation as $m_\omega = \omega\, \hbar\, c^{-2}$:

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

$$i\hbar\Psi_t + \frac{\hbar^2}{2m_\omega}\frac{\partial^2\Psi}{\partial x^2} - \frac{\hbar^2}{2m_\omega}V_H(x)\Psi = 0, \; -a < x < \infty. \qquad (10)$$

The above equation describes the evolution of the slow component of the excitation's in the quantum well, passed from the inner evolution inside to the evolution of 1D photons field outside - on the positive half-axis. Analysis of the wave-packets based on the Schrödinger equation (10), derived based on separation of the fast and slow variables, reveals a polynomial decay rate caused by the branching point at the origin $p = 0$ in the plane of the spectral parameter, see [8].

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

This theoretical proposal was never confirmed experimentally. We guess that the realistic decay rate can be theoretically extracted from the original equation (7) based on analysis of the corresponding Lax-Phillips dynamics, see below and more technical details in [11, 19].

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

Notice, first of all, that the basic Hilbert space associated with the Schrödinger equation (10) is the space of all square-integrable functions $L_2(-a, \infty)$, while the Hilbert space associated with (7) is an energy-normed space $\mathcal{E}$ of the Cauchy data $\mathbf{u} = (u, c^{-1} u_t) \equiv (u_0, u_1)$,

$$\parallel \mathbf{u} \parallel_{\mathcal{E}}^2 = \frac{1}{2} \int_{-a}^{\infty} \left[|u_x|^2 + V_H u\bar{u} + c^{-2} |u_t|^2 \right] dx. \qquad (11)$$

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

The basic equation (7) can be represented as a first order equation for the vector of Cauchy data, with a symmetric (selfadjoint) generator $\mathcal{L}$:

$$\frac{1}{i\,c}\,\frac{\partial \mathbf{u}}{dt} = i \begin{pmatrix} 0 & -1 \\ -\frac{d^2}{dx^2} + V_H & 0 \end{pmatrix} \mathbf{u}. \tag{12}$$

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

The evolution (12) of the Cauchy data is defined by the unitary group $\exp i\mathcal{L}t \equiv U_t$, which has an orthogonal pair of incoming and outgoing subspaces $\mathcal{D}_{in,out}$ consisting of Cauchy data $\{(u,\, u_x)\}\,,\, \{(u,\, -u_x)\}$ of the corresponding d'Alembertian waves $\mathbf{u}(x \pm ct)$ supported by the positive half-axis $0 < x < \infty$, see [11]. The orthogonal complement $\mathcal{K} \equiv \mathcal{E} \ominus [\mathcal{D}_{in} \oplus \mathcal{D}_{out}]$ - the corresponding co-invariant subspace - consists of the Cauchy data supported essentially by the quantum well $[-a, 0]$ and equal to $\mathbf{u} = (\mathrm{const}, 0)$ on the half-axis $(0, \infty)$.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

It is very easy to derive the semigroup property of the evolution reduced onto the co-invariant subspace-the Lax-Phillips semigroup:

$$\mathcal{P}_{\mathcal{K}} \; e^{i\mathcal{L}t}\Big|_{\mathcal{K}} \equiv e^{i\mathcal{B}t}, \;\; t > 0, \tag{13}$$

and calculate the corresponding generator as

$$\mathcal{B} = i \begin{pmatrix} 0 & -1 \\ -\frac{d^2}{dx^2} + V_H & 0 \end{pmatrix}$$

with the zero boundary condition at the end $x = -a$ and the impedance boundary condition at the origin $u_1 + \frac{du_0}{dx}\Big|_{x=0} = 0$.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

Similarly, the generator $-\mathcal{B}^{+}$ of the adjoint semigroup $e^{-i\mathcal{B}^{+}t}$ is defined by the same differential expression and the dual impedance boundary condition at the origin $u_1 - \frac{du_0}{dx}\Big|_{x=0} = 0$.

The generators $\mathcal{B}, -\mathcal{B}^{+}$ are dissipative operators, see [11], with discrete spectrum. It is important that the spectrum of $\mathcal{B}$ is defined by the zeros of the corresponding Lax-Phillips *scattering matrix* - the resonances.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

Indeed, the incoming and outgoing subspaces $\mathcal{D}_{in,out}$ of the Cauchy data are constituted by the Cauchy data of D'Alembertian waves $\Phi(x \pm ct)$, supported by the positive half-axis. Then the spectral images of them with use of the incoming scattered waves Ψ_{in} define the rescription B of $\mathcal{B}$ in the “incoming ” spectral representation of $\mathcal{L}$, attributing $\mathcal{D}_{in}$ to the Hardy class H^2_{min} of all square-integrable functions admitting an analytic continuation to the lower half-plane $\Im p < 0$ of the spectral parameter p.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

This spectral representation is defined by the incoming scattered waves of $\mathcal{L}$

$$^{-}{}_{in}(x,p) = \begin{pmatrix} \frac{1}{ip} \\ 1 \end{pmatrix} \psi_{in}(x,p), \qquad (14)$$

where $\psi_{in}(x,p)$ is the solution of the equation $-\frac{d^2\psi_{in}}{dx^2} + V_H(x)\psi_{in} = p^2\,\psi_{in}$, satisfying zero boundary condition at the end $x = -a$ and matching the scattering Ansatz

$$\psi_{in}(x,p) = e^{ipx} + S(p)e^{-ipx}, x > 0, \text{or} \;\; \psi_{out}(x,p) = \bar{\psi}_{in}(x,p)$$

at the origin to an appropriate solution $\varphi_{in,out}(x,p)$ of the original equation $-\frac{d^2\varphi}{dx^2} + V_H(x)\varphi = \lambda\varphi \equiv p^2\,\varphi$ on the well $(-a,0)$, satisfying the zero boundary condition at the end $x = -a : \varphi(-a,p) = 0$.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

The corresponding Weyl function $m_H(\lambda) = \varphi'(0, p)\ \varphi^{-1}(0, p) + H + m(\lambda) + H$ has a negative imaginary part in the upper half-plane $\Im\lambda > 0$. The stationary scattering matrix is found from the smooth matching condition at the origin as

$$S(p) = \frac{ip - m_H(\lambda)}{ip + m_H(\lambda)}, \lambda = p^2. \qquad (15)$$

It is analytic in the lower half-plane $\Im p < 0$, and has a sequence of zeros p_s, $\Im p_s < 0$, which is symmetric with respect to reflection $p_s = -\bar{p}_{-s}$.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

The scattered waves ψ obtained by matching ψ_{in}, ψ_{out} to $\phi_{in,out}$ form a complete orthogonal in $L_2(-a, \infty)$ systems of eigenfunctions of the spectral problem
$-\frac{d^2\psi}{dx^2} + V(x)\psi = p^2\,\psi,\;\; \psi(-a, p) = 0$ in $L_2(-a, \infty)$:

$$\delta(x - s) = \frac{1}{2\pi}\int_0^\infty \psi(x, p)\bar{\psi}(s, p)dp,$$

and the corresponding eigenfunctions
$\Psi(x, p) = \begin{pmatrix} \frac{1}{ip} \\ 1 \end{pmatrix} \psi(x, |p|),\; -\infty < p < \infty$ play a role of eigenfunctions of $\mathcal{L}$ of the generator of the evolution of Klein-Gordon-Fock equation, $\mathcal{L}\Psi_{in}(*, p) = p\Psi_{in}(*, p)$.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

The spectrum of $\mathcal{L}$ is $(-\infty, \infty)$. The incoming spectral representation $\mathbf{u} \xrightarrow{\mathcal{J}_{in}} \langle \Psi_{in}, \mathbf{u} \rangle_{\mathcal{E}} = V == \mathcal{J}_{in}\mathbf{u} =$

$$\frac{1}{2}\int_0^\infty \left[\bar{\Psi}'_{0,in}(x)u'_0(x) + V_H(x)\bar{\Psi}_{0,in}(x)u'_0(x) + \bar{\Psi}_{1,in}(p,x)u_1(x)\right] dx. \tag{16}$$

transforms the incoming subspace $\mathcal{D}_{in}$ into the Hardy class H^2_- of square-integrable functions on real axis and the outgoing subspace $\mathcal{D}_{out}$ into the invariant subspace $\bar{S}(p)H^2_+$ of the positive shift semigroup $f(p) \to e^{ipt} f(p),\ t > 0$.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

Thus the co-invariant subspace $\mathcal{K}$ is transformed into $H^2_+ \ominus \bar{S} H^2_+ \equiv K$, and the Lax -Phillips semigroup becomes $P_K e^{ipt}\Big|_K \equiv e^{iBt}$. The spectrum of the generator $B = \mathcal{J}_{in} \mathcal{B} \mathcal{J}_{in}^+$ in this representation coincides with the zeros $\bar{p}_s$ of $\bar{S}(\bar{p})$, and the eigenfunctions are just

$$\phi_s \equiv \bar{S}(\bar{p}) \sqrt{2|\Im p_s|} (p - \bar{p}_s)^{-1}. \qquad (17)$$

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

Together with the eigenfunctions

$$\phi^{+} \equiv \sqrt{2|\Im p_s|}(p - p_s)^{-1} \tag{18}$$

of the adjoint generator B^{+} they form a complete bi-orthogonal system in K,

$$B = \sum_s \phi_s\rangle \frac{1}{\langle \phi_s, \phi_s^{+} \rangle} \langle \phi_s^{+}, \quad e^{iBt} = \sum_s \phi_s\rangle \frac{e^{i\bar{p}_s t}}{\langle \phi_s, \phi_s^{+} \rangle} \langle \phi_s^{+}.$$

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

Here $\langle \phi_s, \phi_s^+ \rangle = \prod_{t \neq s} \frac{1-\bar{p}_s/\bar{p}_t}{1-p_s/p_t} \equiv \Pi_s$.
The system $\{\phi_s\}, \{\phi_s^+\}$ is similar to an ortho-normal basis if and only if the Carleson condition, see [19], is fulfilled:

$$\inf_t \prod_{s \neq t} \frac{|p_s - p_t|}{|\bar{p}_s - p_t|} > 0.$$

that is : under the Carleson condition there exist an orthogonal basis $\{\nu_s\}$ which is connected with the normalized families $\{\phi_s\}, \{\phi_s^+\}$ by an invertible transformation:

$$\phi_s = T\nu_s,\ \phi_s^+ = \left[T^{-1}\right]^+ \nu_s,\ \| T \|,\ \| \left[T^{-1}\right]^+ \| < \infty.$$

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

Unfortunately the Carleson condition is never fulfilled for potential of the type V_H. But it may be fulfilled for the corresponding polar problem with the potential substituted by density - a coefficient in front of the spectral parameter. Notice that the eigenvalues $\bar{p}_s, p_s$ of $\mathcal{B}, \ \mathcal{B}^+$ depend on the parameter H and approach the eigenvalues of the Schrödinger operator $L_H = -\frac{\partial^2 u}{\partial x^2} + V_H(x)$ in $L_2(-a, 0)$ with zero boundary conditions at the ends $-a, 0$.

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

The resonances $\bar{p}_s$ - the zeros of the Lax-Phillips Scattering matrix $S_{LP} = [S_H(p)]^{-1}$,

$$S(p) = \frac{ip + [m(\lambda) + H]}{ip - [m(\lambda) + H]}, \text{ with } \lambda = p^2$$

are found from the equation $ip + [m(\lambda) + H] = 0$. For large H the resonances are situated in the upper half-plane near the poles of of $m(\lambda)$ - the eigenvalues of the Dirichlet spectral problem on the interval $(-a, 0)$:

$$ip + H + \frac{q_s}{\lambda - \lambda_s^D} + b_s = 0.$$

An example: analysis of 1D model of Decay observed in 1D analog of an electro-magnetic experiment.

Denoting $\lambda_s^D = [p_s^D]^2$, we have for resonances p_s approaching p_s^D when $H \to +\infty$ the approximate expression

$$p_s \approx p_s^D + \frac{q_s(ip_s^D + H)}{2p_s^D(|p_s^D|^2 + (b_s + H)^2)} \approx p_s^D + \frac{q_s}{2p_s^D H} + \frac{iq_s}{2H^2}. \quad (19)$$

The eigenfunctions ϕ_s, ϕ_s^+ of the generators $\mathcal{B}$, $\mathcal{B}^+$ of the Lax-Phillips semigroup are calculated in spectral representation of the generator $\mathcal{L}$ of the evolution of the Klein-Gordon-Fock equation according to (17,18) in terms of the resonances.

Physics of the exponential decay via the Lax-Phillips scheme.

The spectral analysis of the Lax-Phillips semi-group, described in the brief review above, was based, on the one hand, on the presence of the continuous spectrum of the zero-mass Klein-Gordon-Fock evolution group generator $\mathcal{L}$, and, on the other hand, on the observation that the group possesses a pair of orthogonal incoming and outgoing subspaces. More specifically, the continuous spectrum of $\mathcal{L}$ fills in the whole real axis and the parts of the evolution in the incoming and outgoing subspaces are unitarily equivalent to the negative and positive semi-groups generated (in the p-representation) by the shift $f \rightarrow e^{ipt} f$ in the subspaces H^2_- and $S_{LP} H^2_+$, respectively.

Physics of the exponential decay via the Lax-Phillips scheme.

As a result, the remaining part of the corresponding positive evolution semi-group e^{iBt}, $t > 0$, reduced onto the co-invariant subspace $\mathcal{J}_{in} : \mathcal{K} \longrightarrow H^2_+ \ominus S_{LP} H^2_+ \equiv K$, is unitarily equivalent to the Lax-Phillips semi-group

$$P_{\mathcal{K}} U_t \big|_{\mathcal{K}} \xrightarrow{J^+_{in}} P_K e^{ikt} \big|_K, \quad t > 0.$$

One can see that these Lax-Phillips features were waived in the KF concept. Without them , the concept is not complete to guarantee the exponential decay. Adding these details to the KF proposal makes it sufficient not only to explain the exponential decay, but also to construct a solid bridge between the WW and KF schemes and even give a spectral meaning to resonances , which would be absolutely impossible in the pure Schrödinger approach. Indeed , firstly, the spectrum of the Lax-Phillips semi-group $P_K e^{ipt}\big|_K = e^{iBt},\ t > 0$, associated with a compact scatterer, is discrete, which meets the basic requirement of the WW approach.

Secondly, the corresponding eigenfunctions in the incoming spectral representation $\mathcal{J}_{in} : \mathcal{D}_{in} \xrightarrow{\mathcal{J}_{in}} H^2_-$ are calculated explicitly, as illustrated by (17,18) and, moreover, the corresponding eigenvalues of the dissipative generator coincide with the zeros $\bar{p}_s$ of the scattering matrix S_{LP}. In the case when the singular spectrum of the Lax-Phillips generator is absent and the discrete spectrum is simple, one can use a rational approximation to the scattering matrix given by a finite Blaschke product $S^N_{LP} = \Theta_0 \prod_{s=1}^N \frac{p-\bar{p}_s}{p-p_s}$, $\Im p_s < 0$, with Θ_0 a unitary constant. Based on this approximation we can obtain an approximate description of the exponential decay.

The LP bridge between the WW and KF concepts

In particular, the Lax-Phillips evolution of an initial state that coincides with the eigenvector ϕ_s can be described explicitly as

$$e^{i\mathcal{B}t}\phi_s = e^{i\bar{p}_s t}\ \phi_s.$$

Here the normalized eigenvectors ϕ_s are to be found as solutions of the impedance boundary problem for the Schrödinger equation, with a subsequent restriction on the coinvariant subspace, and the decrements $\Im\bar{p}_s$ can be obtained from the asymptotics (19). The resulting formula can be considered to be a unification of both the Fock–Krylov and the Weisskopf–Wigner approaches to resonances.

In previous example, see section 3 , we derived the formula (**??**) based on interaction of the inner quantum system (on a compact domain $[-a, 0]$) with the Klein-Gordon-Fock equation on the exterior domain defined by the appropriate matching at the common boundary $x = 0$. Using the Lax-Phillips approach we recovered the spectral meaning of resonances interpreting them as eigenvalues of the generator of the Lax-Phillips semi-group. In this particular case, the generator has a discrete spectrum located in the neighborhood of the spectrum of the unperturbed conservative system, the one which is defined by the same Schrödinger differential equation with zero boundary conditions at the end-points of the interval $[-a, 0]$.

This permits to observe the WW concept of the decay from the LP spectral point of view. In particular, in [1], an averaged decay is considered. Using the spectral representation for the Lax-Phillips semi-group, one can calculate the decrement by observing the decay on the initial stage for a relatively small t. Indeed, taking into account that $\langle \psi_r, \psi_r^+ \rangle = \Pi_r$ and that $\langle \psi_s, \psi_r \rangle = \frac{\sqrt{2\Im \bar{p}_s}\sqrt{2\Im \bar{p}_r}}{\Im \bar{p}_s + \Im \bar{p}_r}$, we get:

$$\left\| P_K U_t \big|_K u \right\|^2 = \sum_{s,r}^{N} e^{i[\bar{p}_s - ip_r]t} \langle \phi_s, \phi_r \rangle \frac{\langle \phi_s^+ , u \rangle \ \langle \phi_r^+ , u \rangle}{\langle \phi_s, \phi_s^+ \rangle \ \langle \phi_r, \phi_r^+ \rangle} \leq$$

The LP bridge between the WW and KF concepts

$$\sum_{s,r}^{N} e^{-[\Im\bar{p}_s+\Im\bar{p}_r]t} \left| \langle\phi_s,\phi_r\rangle \frac{\langle\phi_s^+,u\rangle\ \langle\phi_r^+,u\rangle}{\langle\phi_s,\phi_s^+\rangle\ \langle\phi_r,\phi_r^+\rangle} \right| = \sum_{s,r}^{N} e^{-[\Im\bar{p}_s+\Im\bar{p}_r]t}\ \frac{\sqrt{2\Im\bar{p}_s}\sqrt{}}{\Im\bar{p}_s +} \tag{20}$$

One can see from (20) that $\left\|P_K U_t\big|_K u\right\|^2 \leq C(u)e^{-\gamma t}$. The integral parameter γ can be estimated based on the asymptotics of (20) for small t.

Thus, we have

$$C(u)\gamma \approx t^{-1} \left[\| P_K U_t\big|_K u \|^2 - \| P_K u \|^2 \right] \leq 2 \sum_{s,r}^{N} \frac{\sqrt{\Im \bar{p}_s}\sqrt{\Im \bar{p}_r}\langle \phi_s^+, u\rangle \ \overline{\langle \varphi}}{\Pi_r \ \bar{\Pi}_s} \quad (21)$$

Note that the incoming spectral representation transforms $\mathcal{K}$ in to $K = H_+^2 \ominus S_{LP} H_+^2$. Then , for $u \in K$, we have $\langle \psi_s^+, u\rangle = \frac{1}{2\pi}\int_R \frac{u(p)dp}{p-\bar{p}_s} = iu(\bar{p}_s)$, with u calculated as $\mathcal{J}_{in}\mathbf{u}$ according to (16). The ultimate formula (21) bears some features of the exponential decay formulae derived according to the WW and KF concepts.

Indeed , the derivation of the exponential decay rate in the WW manner presented in [6], see the formula (80.13, chapter IX),gives the decay rate via the matrix elements of the perturbation in the interaction representation. If the perturbation is small, then the decay rate of the LP resonance state ϕ_s, see can be interpreted as the decay of the bound state state with the eigenvalue $\left(p_s^D\right)^2$ close to the resonance p_s, according to (19).

The spectral meaning of resonances.

Nevertheless, bridging together both of the contradictory concepts of the WW and KF is not the main achievement of the Lax-Phillips point of view. We suggest that the main achievement is the discovery of the *spectral meaning of resonances:* once we reduce the unitary evolution onto the co-invariant space $K = H^2_+ \ominus S_{LP} H^2_+$, the result is represented, via $\mathcal{J}_{in}$, by the Lax-Phillips semi-group

$$e^{iBt} u = \sum_s e^{-i\bar{p}_s t} \frac{\phi_s \rangle \langle \phi_s^+ , u \rangle}{\langle \phi_s , \phi_s^+ \rangle} . \qquad (22)$$

The spectral meaning of resonances.

Here the "Gamov vectors" ϕ_s, ϕ_s^+ have an unambiguous spectral meaning as the eigenvectors of the Lax-Phillips semigroup generator $\mathcal{B}$, and $\bar{p}_s$ are the corresponding eigenvalues. The spectrum of the generator is discrete, but the whole picture of the restricted evolution on the co-invariant subspace arose because of the specific features of the Lax-Phillips dynamics, first of all of those that are due to the presence of the constant multiplicity continuous spectrum on $R = (-\infty, \infty)$ for the shift group, exactly as it was expected in [2]. But the authors of [2] missed another essential point: the orthogonality in the energy-normed space of the incoming and outgoing invariant subspaces of the wave equation evolution.

The spectral meaning of resonances.

So , one can conclude that in the special case when the condition of orthogonality on the incoming and outgoing subspaces for the wave evolution is satisfied, the KF scheme of the exponential decay is confirmed mathematically. In that case , both the KF and WW schemes give expected results including that of the discreteness of the spectrum of resonances.

Quality of an oscillation system and the Resonance Pumping.

Note that the spectral decomposition for the Lax-Phillips semi-group ensures an exponentially decaying evolution for any single term of the spectral expansion of the semi-group, with the decrement $\Im p_s$. It is customary to interpret the slow decay of the terms of the spectral expansion as a “high quality” of the corresponding oscillatory system. There is, in principle, another method for the estimation of quality of the oscillatory system that is based on estimating the growth of the amplitudes of forced oscillations under periodic excitation. In radio-physics, these two estimations of "quality", based on the decay and on the "pumping", are considered to be alternative estimations of the quality, but the equivalence of them needs a justification using the spectral formulation of the Decay problem.

Quality of an oscillation system and the Resonance Pumping.

Indeed, let us consider the periodic excitation of the oscillatory system in the form

$$\frac{1}{i}\frac{du}{dt} = Bu + e^{i\omega t}\nu$$

with zero incident value. Using the spectral representation of the Lax-Phillips semi-group, one obtains that

$$u(t) = \sum_s i \int_0^t e^{i(\omega-\bar{p}_s)\tau} d\tau e^{i\bar{p}_s t} \frac{\phi_s\rangle\langle\phi_s^+,\nu\rangle}{\langle\phi_s,\phi_s^+\rangle} =$$

$$e^{i\omega t}\sum_s \frac{1-e^{i(\bar{p}_s-\omega)t}}{(\omega-\bar{p}_s)} \frac{\phi_s\rangle\langle\phi_s^+,\nu\rangle}{\langle\phi_s,\phi_s^+\rangle}.$$

Quality of an oscillation system and the Resonance Pumping

The phenomenon of resonance pumping is then observed when the frequency ω is close to one of the eigenvalues of the Lax–Phillips generator. For instance, if $\bar{p}_s - \omega = -i\Im p_s$, recall that $-\Im p_s > 0$, then the forced oscillation regime is

$$u(t) = \frac{e^{\Im p_1 t} - 1}{\Im p_1} \, e^{i\omega t} \frac{\phi_1\rangle\langle\phi_1^+, \nu\rangle}{\langle\phi_1, \phi_s^+\rangle} +$$

$$\sum_{s>1} \frac{1 - e^{i(\bar{p}_s - \omega)t}}{i(\omega - \bar{p}_s)} \frac{\phi_s\rangle\langle\phi_s^+, \nu\rangle}{\langle\phi_s, \phi_s^+\rangle}.$$

Therefore, the forced amplitude of the first term is linearly growing with time, until $t \approx (\Im p_1)^{-1}$, but eventually , at large time scale, it saturates at the value $-(\Im p_1)^{-1} \frac{\phi_1\rangle\langle\phi_1^+,\nu\rangle}{\langle\phi_1,\phi_1^+\rangle}$.

Complementarity of the Lax-Phillips Scattering Scheme and the Quantum Zeno Effect.

The celebrated Zeno Paradox, see [**?**], can also be treated from the viewpoint of the Lax–Phillips evolution. Indeed, consider the Lax-Phillips evolution defined by the unitary group $U_t = e^{i\mathcal{L}t}$ in an energy normed space $\mathcal{E}$ and suppose that the group possesses an orthogonal pair $\mathcal{D}_{in,out}$ of incoming and outgoing subspaces. The restriction $P_{\mathcal{K}} U_t P_{\mathcal{K}}$, $t > 0$, of the positive semi-group onto the co-invariant subspace $\mathcal{K} \equiv \mathcal{E} \ominus [\mathcal{D}_{in} \oplus \mathcal{D}_{out}]$ is the Lax-Phillips semi-group $P_{\mathcal{K}} U_t P_{\mathcal{K}} \equiv e^{i\mathcal{B}t}$.

It has a simple (with no self-adjoint/symmetric parts) dissipative generator $\mathcal{B}$ with discrete spectrum parameterized by the characteristic function S_{LP}, the Lax-Phillips scattering matrix, defined by a Blaschke product. Introducing the amplitude $\langle e^{i\mathcal{L}t}\phi, \phi\rangle_{\mathcal{E}} \equiv a_\phi(t)$ of the returning probability $p_t \equiv \bar{a}_\phi \, a_\phi$, for "smooth" elements $\phi \in \mathcal{K} \cap \mathcal{D}_{\mathcal{B}}$ such that

$\mathcal{B}\phi \in \mathcal{D}_{\mathcal{B}}$ we represent the amplitude as a $(t) = \langle e^{i\mathcal{B}t}\phi,\, \phi\rangle_{\mathcal{E}} =$

$$1 + it\langle \mathcal{B}\phi,\, \phi\rangle_{\mathcal{E}} - \frac{t^2}{2}\langle \mathcal{B}^2\phi,\, \phi\rangle_{\mathcal{E}} + \dots \ .$$

Complementarity of the Lax-Phillips Scattering Scheme and the Quantum Zeno Effect.

Then, Taylor's Theorem up to second order applied to the returning probability yields

$$p(t) = \bar{a}_\phi \ a_\phi = 1 - 2t\Im\langle\mathcal{B}\phi,\ \phi\rangle_\mathcal{E} - t^2\left[\Re\langle\mathcal{B}^2\phi,\ \phi\rangle_\mathcal{E} - |\langle\mathcal{B}\phi,\ \phi\rangle_\mathcal{E}|^2\right] + \cdots \ .$$

Complementarity of the Lax-Phillips Scattering Scheme and the Quantum Zeno Effect.

If $\Im\langle\mathcal{B}\phi,\,\phi\rangle_{\mathcal{E}} \neq 0$, then $1 - 2t\Im\langle\mathcal{B}\phi,\,\phi\rangle_{\mathcal{E}} \approx e^{-2t\Im\langle\mathcal{B}\phi,\,\phi\rangle_{\mathcal{E}}}$, and hence, despite a multiple control of the evolution we have $p(t) \approx [p(t/n)]^n$. This is the case of an exponential decay with the decrement $\Gamma = 2\Im\langle\mathcal{B}\phi,\,\phi\rangle_{\mathcal{E}}$. The alternative condition $\Im\langle\mathcal{B}\phi,\,\phi\rangle_{\mathcal{E}} = 0$ implies

$$p(t) = 1 - t^2\left[\Re\langle\mathcal{B}^2\phi,\,\phi\rangle_{\mathcal{E}} - \left|\langle\mathcal{B}\phi,\,\phi\rangle_{\mathcal{E}}\right|^2\right] + \cdots \approx 1 - At^2$$

which would give the following asymptotics for the probability under the evolution with the multiple control at the sequence of moments

Complementarity of the Lax-Phillips Scattering Scheme and the Quantum Zeno Effect.

$t_m = \frac{m}{n}t$, $m = 1, 2, \ldots$,

$$[p(t/n]^n) \approx [1 - A/n^2]^n \approx [e^{-A}]^{1/n} \approx 1 \text{ as } \quad t \to \infty.$$

This result corresponds to the quantum Zeno effect. The condition $\Im\langle \mathcal{B}\phi,\ \phi\rangle_{\mathcal{E}} = 0$ is not compatible with dissipativity of the simple (with no self-adjoint parts) generator $\mathcal{B}$ with Risz-basis property of eigenfunctions. Indeed the opposite condition $\langle \Im\mathcal{B}\phi, \phi\rangle > 0$ is obviously satisfied for all vectors from the domain of $\mathcal{B}$ in the coinvariant subspace, if the system of it's eigenvectors is a Riesz basis. Thus , we conclude that the Zeno effect is not compatible with the Lax–Phillips evolution for elements ϕ from the coinvariant subspace such that $\Im\langle \mathcal{B}\phi,\ ,\phi\rangle_{\mathcal{E}} > 0$.

Vice versa, the general Schrödinger type unitary evolution $U_t\phi = e^{iLt}\,\phi$ of a smooth state ϕ is compatible with the Zeno effect (whenever L is a self-adjoint generator in the Hilbert space E).
Indeed, the corresponding infinitesimal evolution for a smooth normalized state ϕ yields

$$p(t) = \langle e^{iLt}\,\phi, \phi\rangle_{\mathcal{E}} \langle\, \phi, e^{iLt}\,\phi\rangle_{\mathcal{E}} \approx 1 -$$

$$t^2\left[\langle L^2\phi, \phi\rangle_{\mathcal{E}} - (\langle L\phi, \phi\rangle_{\mathcal{E}})^2\right] + \dots \ .$$

Hence, in an experiment with the multiple control at the moments of time $t_m = \frac{m}{n}\, t$, $m = 1, 2, \dots$, we obtain:

$$[p(t/n)]^n \approx \left(1 - \frac{t^2}{n^2}\left[\langle L^2\phi, \phi\rangle_{\mathcal{E}} - (\langle L\phi, \phi\rangle_{\mathcal{E}})^2\right]\right)^n \approx$$

$$e^{-\left[\langle L^2\phi,\phi\rangle_{\mathcal{E}} - (\langle L\phi,\phi\rangle_{\mathcal{E}})^2\right]t^2 n^{-1}} \to 1, \text{ when } n \to \infty.$$

This corresponds to the standard Zeno effect in Quantum Mechanics, see [7]. It is worth mentioning that Quantum Mechanics is a description of dynamics and probability is not intrinsically involved in that. But probability arises as a detail of the measurement process: it is clearly seen from the preceding analysis that the interplay between the dynamics and the measurement process is different for the Schrödinger evolution [7] and for the Lax-Phillips one.

Conclusion

Our version of matching of a zero-mass field in the outer space with the Schrödinger evolution on the inner space of the quantum system allows one to derive the exponential decay based on the classical Lax-Phillips technique. Contrary to the constructions suggested in [13, 14] and and those in the recent papers [**?**, **?**], we use explicit functional model formulae for the eigenvalues and eigenvectors of the corresponding dissipative generator that gives rise to the reduced dynamics on the corresponding coinvariant subspace. For low energy, the dynamics on the inner space is matched with the corresponding Schrödinger dynamics that provides the standard probabilistic interpretation of the wave-function but would formally produce non-exponential terms in the large-time scale.

Conclusion

But the original dynamics, before being reduced to Schrödinger's scenario, exhibits an exponential decay for large time, with non-exponential terms absent. Our approach also reveals the spectral meaning of the resonances and the resonance states, and permits to bridge, on this base, the alternative concepts of resonances and the exponential decay proposed by Weisskopf–Wigner and Krylov–Fock. In turn, this proves that the lifetime of a resonance and the velocity of the resonance pumping are directly connected. We also establish duality between the exponential decay and the absence of the quantum Zeno effect on resonance initial data for the quantum system under a permanent control.

Bibliography I

- 1. V.E.Weiskopf and E.P. Wigner. Zeitshrift fü Physik, **63**, 54 (1930), **65**, 18 (1930)
- 2. V.A.Fock and V.A.Krylov , Journal of Experimental and Theoretical Physics (USSR), **17**, 93 (1947)
- 3. V.A. Fock *Selected works. Quantum mechanics and quantum field theory* Edited by L. D. Faddeev, L. A. Khalfin and I. V. Komarov. Chapman & Hall/CRC, Boca Raton, FL, (2004) xii+567 pp.
- 4. V. Ginzburg, L., Landau .... Zhurnal Eksp. Yheoret. Physics, **29**, 1950, p. 1064.
- 5. E. Stueckelberg ...... Helvetica Physica Acta **11** (1938) p. 299.

Bibliography II

- 6. A.S. Davydov *Quantum Mechanics*, Chapter IX, section 80.
- 7. J. Sakurai *Modern Quantum Mechanics. Revised Edition* Addison-Wesesley PC (1994), 500 pp.
- 8 L. Khalfin *On the theory of decay of a quasy-stationary state* In: Soviet Phys. Doklady V.2 (1958) p 340.
- 9. L.B. Okun *Leptons and quarks* Amsterdam,North Holland (1981) 351 p.
- 10. E.C.Titchmarsh *Eigenfunction expansion associated with second -order differential equations* Part 1, Clarendon Press, Oxford (1962) vi+203 pp
- 11. P. Lax, R. Phillips *Scattering theory* Academic Press,New York (1967)

- 12. Y. Strauss. L.P.Horwitz, E.Eisenberg *Representation of quantum mechanical resonances in the lax-Phillips Hilbert space* In: Journal of Mathematical Physics **41**, 12 (Quantum Physics, Particles and Fields) 8050 (2000)
- 13. C. Flesia and C. Piron, Helv. Phys. Acta **57** (1984) p 697
- 14. L. P. Horwitz and C. Piron, Helv. Phys. Acta **66** (1993) p. 694
- 15. E. Eisenberg and L. P. Horwitz, Advances in Chemical Physics, ed. I. Prigogine and S. Rice , Wiley, New York, Vol. XCIX (1997) p. 245.
- 16. G.C. Hegerfeldt *Causality, particle localization and positivity of the energy.* In: Irreversibility and causality: semigroups and rigged Hilbert spaces. Lecture Notes in Physics, v. 504 (1998) pp 238-245.

- 17. B.Sz.-Nagy, C., Foias,*Harmonic analysis of operators on Hilbert space*. Translated from the French and revised North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest 1970 xiii+389 pp.
- 18. M.S.Livshits *method of nonselfadjoint operators in the theory of waveguides* In: Radio Engineering and Electronic Physics. Pulb. by American Institute of Electrical Engineers, **1** (1962) pp 260-275
- 19. B. Pavlov *Spectral Analysis of a Dissipative Singular Schrödinger Operator in terms of a Functional Model* in the book: Partial Differential Equations,ed.M.Shubin in series Encyclopedia of Mathematical Sciences, Springer, 65 (1995) pp 87-153.

- 20. V.M.Adamjan;D.Z Arov. *On scattering operators and contraction semigroups in Hilbert space* (Russian) Dokl. Akad. Nauk SSSR 165 (1965) pp 9–12.
- 21. P. Koosis. *Introduction to Hp spaces*. Second edition. With two appendices by V. P. Havin. Cambridge Tracts in Mathematics, 115. Cambridge University Press, Cambridge 1998 xiv+289 pp.
- 22. M.G. Krein *Selected works. II : Banach spaces and operator theory* In Russian. Natsional'naya Akademiya Nauk Ukrainy, Institut Matematiki, Kiev, (1996) 348 pp.

Bibliography VI

- 23. M.G. Krein *Selected works. III . Topics in differential and integral equations and operator theory*. Edited by I. Gohberg. Translated from the Russian by A. Iacob. Operator Theory: Advances and Applications, 7. Birkhäuser Verlag, Basel, 1983. ix+302 pp.
- 24. N.K. Nikol'skii *Treatise on the shift operator. Spectral function theory* With an appendix by S. V. Hrucev [S. V. Khrushchëv] and V. V. Peller. Translated from the Russian by Jaak Peetre. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin (1986) 273 pp.
- 24. B. Pavlov *The theory of extensions and explicitly-soluble models* Russian Math. Surveys 42:6 (1987)pp 127-168.

- 25. N.K. Nikol'skii, S.V. Khrushchëv. *A functional model and some problems of the spectral theory of functions* (Russian). Trudy Mat. Inst. Steklov. 176 (1987) pp 97–210, 327. Translated fom Russian in Proc. Steklov Inst. Math., no. 3, 101–214. (Mathematical physics and complex analysis.) (1988)
- 26. B. Pavlov, V. Kruglov *Operator Extension technique for resonance scattering of neutrons by nuclei* In: Hadronic Journal 28(2005) pp 259-268.