Invariance and Universality of Complexity

H. Jürgensen
The University of Western Ontario
London, Canada

Descriptional Complexity:

- Conceptual Framework:
  (a) Set $\mathfrak{O}$ of *objects*,
  (b) set $\mathfrak{S}$ of *specification methods*,
  (c) set $\mathfrak{D}$ of potential *object descriptions*,
  (d) a *size measure* μ of object descriptions,
  (e) a *combined size measure* ν of specification methods and object descriptions.

- Goals:

  (1) For $\mathfrak{o} \in \mathfrak{O}$ and $\mathfrak{s} \in \mathfrak{S}$, determine a $\mathfrak{d} \in \mathfrak{D}$ such that $\mathfrak{s}(\mathfrak{d}) = \mathfrak{o}$ and $\mu(\mathfrak{d})$ is minimal (if it exists). Let $\mathfrak{o}^*$ be a $\mathfrak{d}$ with these properties. Then $\mu(\mathfrak{o}^*)$ is the *complexity of* $\mathfrak{o}$ *with respect to* $\mathfrak{s}$.

  (2) For $\mathfrak{o} \in \mathfrak{O}$, determine an $\mathfrak{s} \in \mathfrak{S}$ and a $\mathfrak{d} \in \mathfrak{D}$ such that $\mathfrak{s}(\mathfrak{d}) = \mathfrak{o}$ and $\nu(\mathfrak{s}, \mathfrak{d})$ is minimal. For these values of $\mathfrak{s}$ and $\mathfrak{d}$, the value of $\nu(\mathfrak{s}, \mathfrak{d})$ is the *complexity of* $\mathfrak{o}$ *in the framework.*

- Desirable Properties:
  - *Invariance:* For every object $\mathfrak{o}$ and every specification method $\mathfrak{s}$, the complexity of $\mathfrak{o}$ with respect to $\mathfrak{s}$ is "essentially" not less than the complexity of $\mathfrak{o}$ with respect to the whole framework.
  - *Universality:* There is a specification method $\mathfrak{u} \in \mathfrak{S}$ such that, for every object $\mathfrak{o}$, the complexity of $\mathfrak{o}$ with respect to $\mathfrak{u}$ is "essentially" the same as the complexity of $\mathfrak{o}$ with respect to the whole framework.

- Invariance and Universality hold true in the realm of Turing computable functions. Complexity is not computable.
- Invariance holds true for finite state computability (Calude et al. 2011); universality cannot be expected. Complexity is computable.

Take snow flakes as objects:

SnowCrystals.com

x041219b055.jpg

- Snow in Canada or elsewhere.
- *Frøken Smillas fornemmelse for sne* by Peter Høeg.
- Two snowflakes are alike with a probability very close to 0.

What is the complexity of “snow”?

Plan for this Presentation:

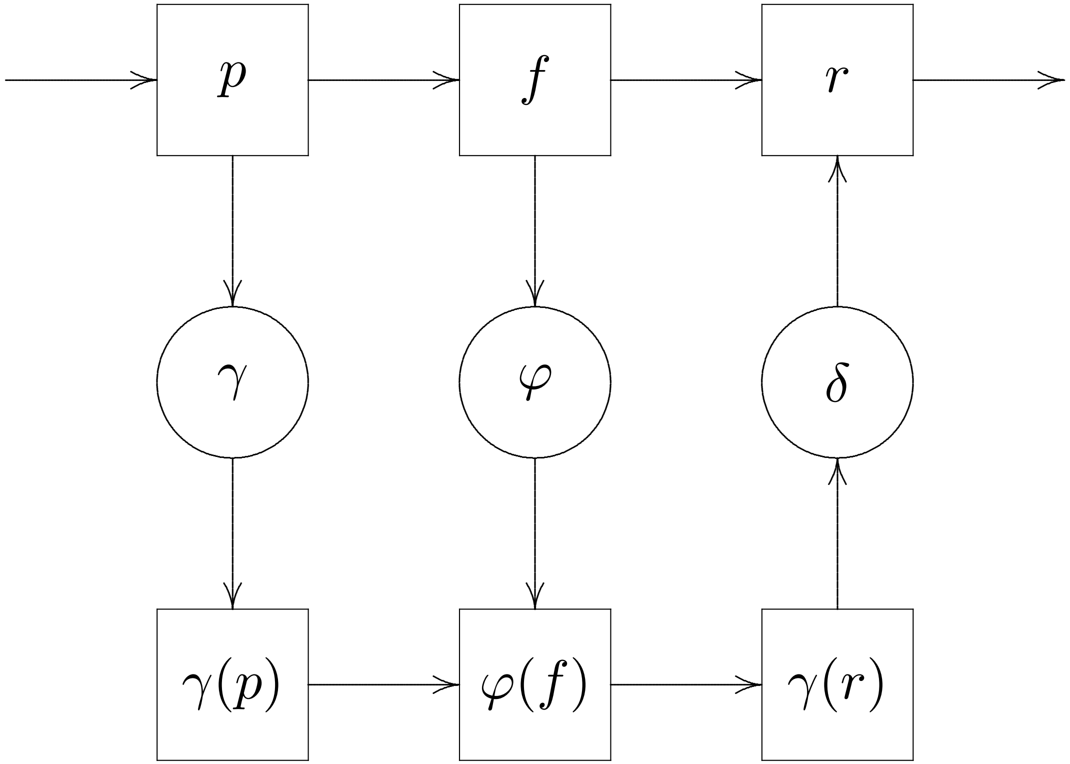
- Formal definition of the framework as *encoded function spaces* and *complexity* in such spaces. No computability conditions are imposed.
- Invariance theorem.
- When is complexity computable?
- Universal functions and the rôle of pairing functions. When can we expect a universality theorem? Some special cases.
- Summary, conclusions, questions.
- Personal historical remarks.

Notation:

- Standard notation for sets, automata, languages, functions.
- Alphabets have at least two symbols. Empty word is ε.
- $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- For real-valued partial functions f and g: $f \lesssim g$ if and only if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and there is $c \geq 0$ such that $f(x) \leq g(x) + c$ for all $x \in \mathrm{dom}(g)$.
- $f \sim g$ if and only if $f \lesssim g \lesssim f$.
- Convention: $\min \emptyset = \inf \emptyset = \infty$.
- Pairing function: Injective mapping

$$\pi : A \times B \to C : (a, b) \mapsto= \pi(a, b) = \langle a, b \rangle$$

with projections p_A and p_B.


p
f
r
γ
φ
δ
$\gamma(p)$
$\varphi(f)$
$\gamma(r)$

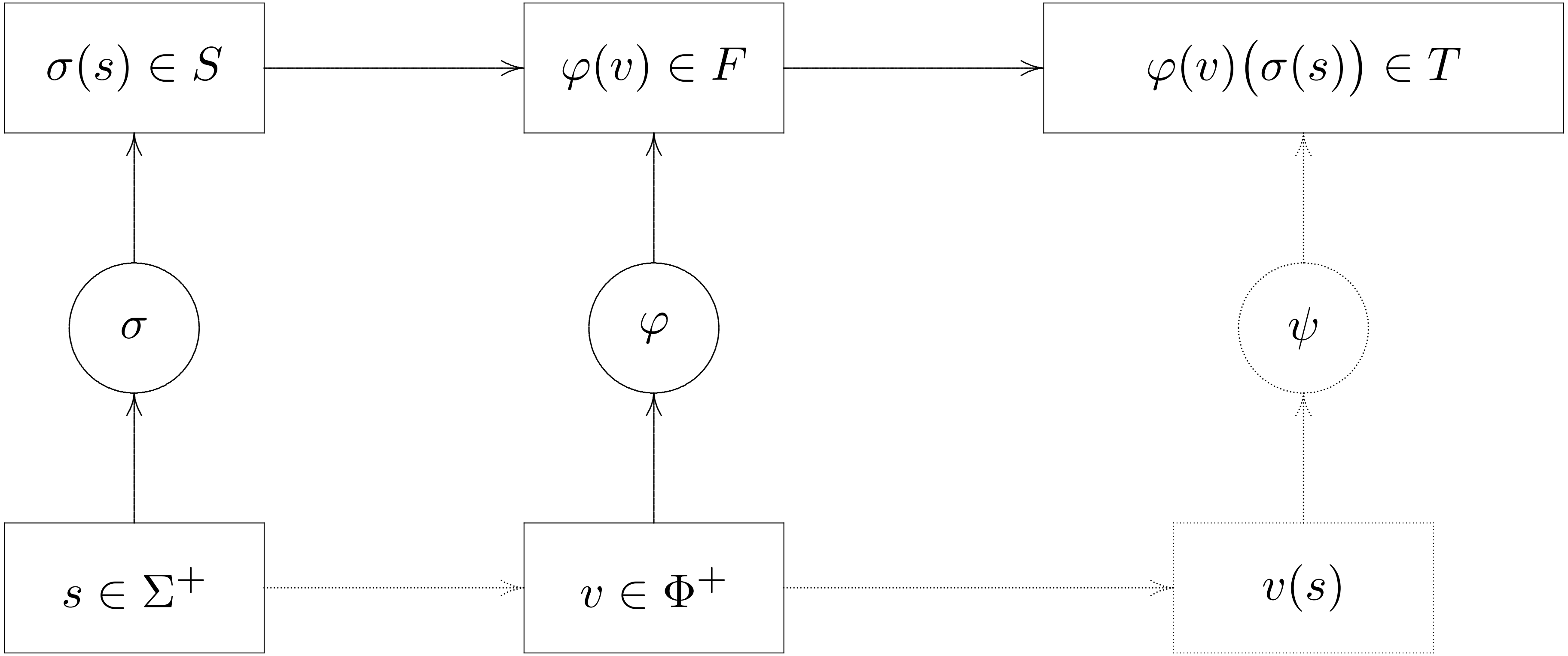
Definition 1 An *encoded function space* is a construct

$$\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$$

with the following properties.

- S is a non-empty countable set, the *source;*
- T is a non-empty set, the *target;*
- F is a non-empty countable set of partial mappings of S into T;
- Φ and Σ are alphabets;
- $\varphi : \Phi^+ \overset{\circ}{\rightarrow} F$ is a surjective partial mapping, the *encoding of* F.
- $\sigma : \Sigma^+ \overset{\circ}{\rightarrow} S$ is a surjective partial mapping, the *encoding of* S;

The encoded function space $\mathfrak{F}$ is said to be *effective* if all items in the construct are effectively given and the mappings σ and φ are computable.


$\sigma(s) \in S$
$\varphi(v) \in F$
$\varphi(v)\big(\sigma(s)\big) \in T$
σ
φ
ψ
$s \in \Sigma^+$
$v \in \Phi^+$
$v(s)$

Definition 2 Let $\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$ be an encoded function space and let $t \in T$.

- For $f \in F$, the f*-complexity* of t in $\mathfrak{F}$ is defined as

$$c_f^{\mathfrak{F}}(t) = \inf\bigl\{|\, s \,| \bigm| s \in \Sigma^+, f(\sigma(s)) = t\bigr\}$$

- The *complexity* of t in $\mathfrak{F}$ is defined as

$$c^{\mathfrak{F}}(t) = \inf\bigl\{|\, v \,| + |\, s \,| \bigm| v \in \Phi^+, s \in \Sigma^+, \varphi(v)(\sigma(s)) = t\bigr\}.$$

In the definition of complexity, one can replace "inf" by "min" as only subsets of $\mathbb{N}_0$ are concerned, using the convention of $\min\emptyset = \infty$.

Theorem 3 (General Invariance Theorem)

Let $\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$ *be an encoded function space. Then, for all* $f \in F$ *and* $v \in \varphi^{-1}(f)$,

$$c^{\mathfrak{F}}(t) \leq c^{\mathfrak{F}}_{f}(t) + |\, v \,|$$

for all $t \in T$.

Examples:

- Turing computable functions;
- finite state transducers (Calude et al.)

In general, complexity need not be computable!

Theorem 4 (Computability of Complexity Theorem)

Let $\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$ *be an encoded function space and let* $t \in T$.

1. *For* $f \in F$ *the complexity* $c_f^{\mathfrak{F}}(t)$ *is computable, if the following conditions are satisfied:*

   (a) f *is effectively defined and computable.*

   (b) $\mathrm{dom}(f)$ *is decidable.*

   (c) S *is effectively defined,* $\mathrm{dom}(\sigma)$ *is decidable and* σ *is computable.*

   (d) t *is effectively defined and equality is decidable in* T.

Theorem 4 continued:

2. *The complexity* $c^{\mathfrak{F}}(t)$ *is computable, if, in addition to Conditions a–d, also the following conditions are satisfied:*

   (e) *Emptiness of the set* $\{f \mid f \in F, t \in \mathrm{codom}(f)\}$ *is decidable.*

   (f) F *is effectively given and enumerable,* $\mathrm{dom}(\varphi)$ *is decidable and* φ *is computable.*

   (g) *For every* $c \in \mathbb{N}$*, emptiness of the set*

   $$F_{c,t} = \{f \mid f \in F, \exists s \in \Sigma^+ : |\, s \,| < c, f(\sigma(s)) = t\}$$

   is decidable.

Corollary 5 (Calude et al.)

If $\mathfrak{F}$ *is the space of finite-state computable functions with transducers as the computer model then* $c^{\mathfrak{F}}$ *is computable.*

Definition 6 Let S and T be non-empty sets and let F be a non-empty set of partial mappings of S into T. Let $\pi : F \times S \to S$ be a pairing function.

A partial function $g : S \overset{\circ}{\to} T$ is said to *universal for F by π*, if $\langle f, s\rangle \in \mathrm{dom}(g)$ and $g(\langle f, s\rangle) = f(s)$ for all $f \in F$ and all $s \in \mathrm{dom}(f)$.

Let $\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$ be an encoded function space. A partial function $g : S \overset{\circ}{\to} T$ is said to *universal for $\mathfrak{F}$ by π*, if it is universal for F by π.

Remark 7 *There are universal Turing-computable functions.*

Remark 8 *There is no finite-state computable function which is universal for all finite-state computable functions.*

Remark 9 *Let S and T be non-empty sets, let F be a non-empty set of mappings of S into T, and let π be a pairing function of $F \times S$ into S. If g is universal for F by π, then g is uniquely defined by*

$$g(s) = p_F(s)\big(p_S(s)\big)$$

for all $s \in \mathrm{codom}(\pi)$ with $p_S(s) \in \mathrm{dom}(p_F)(s)$. For all other values of s, g can be left undefined or defined in an arbitrary way. In particular, a universal function for F exists, if and only if there is a pairing function π.

Definition 10 Let $\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$ be an encoded function space, let $\pi : F \times S \to S$ be a pairing function, let $g : S \overset{\circ}{\to} T$ be universal for $\mathfrak{F}$ by π, and let $t \in T$. The *complexity of t in $\mathfrak{F}$ according to g (or π)* is defined as

$$C_g^{\mathfrak{F}}(t) = \min \left\{ |\, s \,| \;\middle|\; \begin{array}{l} s \in \Sigma^+, \exists f \in F, \exists s' \in S : \\ \sigma(s) = \langle f, s' \rangle, s' \in \mathrm{dom}(f), f(s') = t \end{array} \right\}.$$

Using the framework of Definition 10, there could be $s \in \Sigma^+$ such that $g\big(\sigma(s)\big) = t$ and $|\,s\,| < C_g^{\mathfrak{F}}(t)$. In this case $\sigma(s) \notin \operatorname{codom}(\pi)$ or $\sigma(s) = \langle f, s' \rangle$ for some $f \in F$ and $s' \in S$ such that $s' \notin \operatorname{dom}(f)$. Hence, one has $C_g^{\mathfrak{F}}(t) \leq |\,s\,|$ for all $s \in \Sigma^+$ with $g\big(\sigma(s)\big) = t$ satisfying the following condition:

$$\sigma(s) \in \operatorname{codom}(\pi) \wedge p_S\big(\sigma(s)\big) \in \operatorname{dom}\big(p_F(\sigma(s))\big).$$

To compare $C_g^{\mathfrak{F}}$ and $c^{\mathfrak{F}}$ we need a connection between the lengths of encodings of pairs $\langle f, s' \rangle$ and the sum of the lengths of encodings of f and s'.

Remark 11 *Let $\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$ be an encoded function space and let $\hat{\pi} : \Phi^+ \times \Sigma^+ \overset{\circ}{\to} \Sigma^+$ be a pairing function. There is a unique pairing function $\pi : F \times S \to S$ such that the following diagram commutes:*

$$\begin{tikzcd}
\Phi^+ \arrow[d, "\varphi"'] & \times & \Sigma^+ \arrow[d, "\sigma"'] \arrow[r, "\hat{\pi}"] & \Sigma^+ \arrow[d, "\sigma"'] \\
F & \times & S \arrow[r, "\pi"] & S
\end{tikzcd}$$

One has $\pi(f, s) = \sigma(\hat{\pi}(v, s'))$ with $f \in F$, $s \in S$, $\varphi(v) = f$ and $\sigma(s') = s$. We say that π is derived from $\hat{\pi}$.

Theorem 12 *Let* $\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$ *be an encoded function space, let* $\hat{\pi} : \Phi^{+} \times \Sigma^{+} \to \Sigma^{+}$ *be a pairing function, and let* $\pi : F \times S \to S$ *be the pairing function derived from* $\hat{\pi}$*. Let* g_{π} *be a universal function for* $\mathfrak{F}$ *by* π*. Then*

$$C_{g_{\pi}}^{\mathfrak{F}}(t) = \min\Big\{ \big| \hat{\pi}(v, s) \big| \;\Big|\; v \in \Phi^{+}, s \in \Sigma^{+}, \varphi(v)\big(\sigma(s)\big) = t \Big\}$$

for all $t \in T$*.*

Definition 13 An injective partial function $\hat{\pi} : \Phi^+ \times \Sigma^+ \overset{\circ}{\to} \Sigma^+$ is *length-bounded* if

$$|\,\hat{\pi}(u,v)\,| \lesssim |\,u\,| + |\,v\,|$$

for all $(u,v) \in \mathrm{dom}(\hat{\pi})$. It is said to be *length-preserving* if

$$|\,\hat{\pi}(u,v)\,| \sim |\,u\,| + |\,v\,|$$

for all $(u,v) \in \mathrm{dom}(\hat{\pi})$.

Theorem 14 (Universal Complexity Theorem)

Let $\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$ *be an encoded function space. Let* $\hat{\pi} : \Phi^+ \times \Sigma^+ \to \Sigma^+$ *be a length-bounded pairing function and let* $\pi : F \times S \to S$ *be derived from* $\hat{\pi}$*. Let* g_π *be a universal function for* $\mathfrak{F}$ *defined by* π*. Then* $C_{g_\pi}^{\mathfrak{F}} \lesssim c^{\mathfrak{F}}$*. Moreover,* $C_{g_\pi}^{\mathfrak{F}} \sim c^{\mathfrak{F}}$*, if* $\hat{\pi}$ *is length-preserving.*

Theorem 15 *Let* $\mathfrak{F} = (F, S, T, \Phi, \Sigma, \varphi, \sigma)$ *be an encoded function space. For* $i = 1, 2$, *let* $\hat{\pi}_i : \Phi^+ \times \Sigma^+ \to \Sigma^+$ *be length-preserving pairing functions and let* $\pi_i : F \times S \to S$ *be derived from* $\hat{\pi}_i$. *Let* g_{π_i} *be a universal function for* $\mathfrak{F}$ *defined by* π_i. *Then* $C^{\mathfrak{F}}_{g_{\pi_1}} \sim C^{\mathfrak{F}}_{g_{\pi_2}}$.

None of the results so far require that the universal function be in the space $\mathfrak{F}$. The results follow solely from properties of the pairing functions involved.

Definition 16 For $i = 1, 2$ let $\mathfrak{F}_i = (F_i, S, T, \Phi, \Sigma, \varphi_i, \sigma_i)$ be two encoded function spaces.

(1) We say that $\mathfrak{F}_1$ is a *subspace of* $\mathfrak{F}_2$, written as $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$, if $F_1 \subseteq F_2$, $\varphi_1 \subseteq \varphi_2$ and $\sigma_1 \subseteq \sigma_2$.

(1) We say that $\mathfrak{F}_1$ is a *conservative* subspace of $\mathfrak{F}_2$, if $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ and additionally,

$$\varphi_1^{-1}(f) = \varphi_2^{-1}(f) \text{ and } \sigma_1^{-1}(s) = \sigma_2^{-1}(s)$$

for all $f \in F_1$ and all $s \in S_1$.

Intuition:

The function space of interest is the space $\mathfrak{F}_1$. The universal functions for $\mathfrak{F}_1$ to be considered are in $\mathfrak{F}_2$, and $\mathfrak{F}_1$ is a conservative subspace of $\mathfrak{F}_2$. Thus, the universal functions have encodings in $\mathfrak{F}_2$, and the functions of $\mathfrak{F}_1$ and their arguments have exactly the same encodings in both spaces.

Special situations:

(1) $\mathfrak{F}_1 = \mathfrak{F}_2$, as is the case for Turing-computable functions;

(2) for $\mathfrak{F}_1$ being the finite-state computable functions, $\mathfrak{F}_2$ could be the space of functions computable by deterministic two-way pushdown automata (Ring, 1973) or the space of functions computable by deterministic linearly bounded Turing machines (Boucher, 1971).

Theorem 17 *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be encoded function spaces such that $\mathfrak{F}_1$ is a conservative subspace of $\mathfrak{F}_2$. For $i = 1, 2$, let $\hat{\pi}_i : \Phi^+ \times \Sigma^+ \to \Sigma^+$ be length-bounded pairing functions, and let $\pi_i : F_2 \times S \to S$ be derived from $\hat{\pi}_i$. Let g_{π_i} be universal for $\mathfrak{F}_2$ by π_i. If $g_{\pi_1}, g_{\pi_2} \in F_2$, then $C^{\mathfrak{F}_1}_{g_{\pi_1}} \sim C^{\mathfrak{F}_1}_{g_{\pi_2}}$.*

Distinguish universality according to how the arguments are presented to the universal function g:

- The arguments are:
  - representation v of the function $\varphi(v)$;
  - representation s of the input $\sigma(s)$.
- Value to be computed: $\varphi(v)\big(\sigma(s)\big)$.
- *Weak universality:* Use one injective function ψ to obtain $g\big(\psi(v,s)\big)$.
- *Universality:* Use two injective partial functions χ and ψ into Σ^+ to obtain $g\big(\chi(v)\psi(s)\big)$.
- *Strong Universality:* As before, but with ψ a homomorphism.

This distinction is suggested by Ring and Boucher. I am using it to illustrate the problem.

Theorem 18 (Ring, 1973)

1. *For every effectively defined set of automata recognizing only decidable languages there is a weakly universal finite automaton.*

2. *There is no universal finite automaton for the deterministic finite automata with a fixed non-empty alphabet.*

3. *For every effectively defined set of automata recognizing only decidable languages there is a universal pushdown automaton.*

4. *For every effectively defined set of finite automata there is a strongly universal deterministic two-way pushdown automaton.*

5. *There is no strongly universal one-way push-down automaton for the deterministic finite automata with a fixed non-empty alphabet.*

Corollary 19 *Let $\mathfrak{F}$ be an encoded function space of finite-state computable functions. There is a length-preserving pairing function $\hat{\pi}$ such that the universal function g_π for $\mathfrak{F}$ can be computed by a deterministic two-way pushdown automaton. Up to an additive constant, the complexity with respect to g_π is independent of the choice of π and, hence, of the corresponding deterministic two-way pushdown automaton.*

One can attempt to continue along this line of thought:

Theorem 20 (Boucher, 1971)

There is a deterministic linearly bounded Turing machine which is universal for the space of finite automata.

Theorem 21 (Boucher, 1971)

1. *There is a deterministic linearly bounded Turing machine, which is weakly universal for the class of all deterministic linearly bounded Turing machines.*

2. *There is no deterministic linearly bounded Turing machine, which is universal for the class of all deterministic linearly bounded Turing machines.*

The second statement is a consequence of the need to simulate deterministic linearly bounded Turing machines with arbitrarily large alphabets. When the alphabet is fixed, deterministic linearly bounded Turing machines satisfy the assumptions of Theorem 17: Any two such machines which are universal for the space of finite-state functions give rise to the same complexity up to an additive constant.

Summary, Conclusions, Questions:

- A general framework suffices to derive essential properties of complexity. Computing models are not needed.
- Key parameters have been identified. They including the encodings and the pairing functions.
- There seems to be a general approach to defining conditional complexity and connecting it to information theoretic concepts without computability theory. The crucial idea is to use pairing functions as parameters.

Some Personal History:

I knew Cris's work from being a reviewer who read mathematical Romanian for Zentralblatt and Mathematical Reviews since the seventies. We had letter contact to the extent possible (or rather impossible) since the late seventies.

He sent me his book on complexity:

[1] C. S. Calude: *Complexitatea Calculului: Aspecte Calitative*. Editura ştiinţifică şi enciclopedică, Bucureşti, 1982.

We first met in April of 1990 (there could have been other occasions in Eastern Europe, but they did not happen). He was on UNESCO work in the USA. A colleague from Boston informed me. I managed to facilitate a visit in London (Canada) for him. We started to work on a question inspired by the following papers:

[2] C. S. Calude, G. Păun: Independent instances for some undecidable problems. *RAIRO Inform. Théor.* **17** (1983), 49–54.

[3] C. S. Calude, I. Chiţescu: Random sequences: Some topological and measure-theoretical properties. *An. Univ. Bucureşti, Mat.-Inf.* **2** (1988), 27–32.

The question was: Is independence an aberration based on a syntactical trick or is it a common phenomenon?

First answer: It is common in a topological sense!

[4] C. S. Calude, H. Jürgensen, M. Zimand: Is independence an exception? *Applied Mathematics and Computation* **66** (1994), 63–76.

Second answer (much later): It is common in a probabilistic sense!

[5] C. S. Calude, H. Jürgensen: Is complexity a source of incompleteness? *Advances in Appl. Math.* **35** (2005), 1–15.

After 1990 I travelled a few times to Romania and (later) to New Zealand, and Cris visited me in Canada and Germany. Most importantly, he and his family spent several months in London (Canada) during their move from Romania to New Zealand.

I have used his book on complexity for courses both in Canada and Germany. For a course on recursive functions and abstract complexity, I should use it still as the primary reference for my students.

[6] C. S. Calude: *Theories of Computational Complexities.* North-Holland, Amsterdam, 1988.

We wrote a paper on number representations, which added to a hot and partisan (and not always professional) controversy concerning the publication of Cris's book on algorithmic information theory (the traces of the controversy can be found between the lines of the preface):

[7] C. S. Calude, H. Jürgensen: Randomness as an invariant for number representations. In H. Maurer, J. Karhumäki, G. Rozenberg (editors): *Results and Trends in Theoretical Computer Science. Lecture Notes in Computer Science* **812**, 44–66. Springer-Verlag, Berlin, 1994.

[8] C. S. Calude: *Information and Randomness – An Algorithmic Perspective.* Springer-Verlag, Berlin, 1994.

Independence is a common phenomenon in a topologial sense. Moreover, this is not an artificial result as it holds true under very weak conditions on the topology.

[9] C. S. Calude, H. Jürgensen, L. Staiger: Topology on words. *Theoret. Comput. Sci.* **410**(24–25) (2009), 2323–2335. A collection of papers in honor of Sheng Yu.

Some further joint work:

[10] C. S. Calude, P. Hertling, H. Jürgensen, K. Weihrauch: Randomness on full shift spaces. *Chaos, Solitons & Fractals* **12**(3) (2001), 491–503.

[11] C. S. Calude, H. Jürgensen, S. Legg: Solving finitely refutable problems. In C. S. Calude, G. Păun (editors): *Finite versus Infinite – Contributions to an Eternal Dilemma.* 39–52, Springer-Verlag, London, 2000.

[12] C. S. Calude, H. Jürgensen: Randomness and coding. In M. Marinov, D. Ivanchev (editors): *Proceedings, 20th Summer School on Applications of Mathematics in Engineering, Varna, 1994.* 53–57, Technical University of Sofia, Institute of Applied Mathematics and Informatics, Sofia, 1995.

[13] H. Jürgensen, C. Calude (editors): Algorithmic complexity and applications. *Fund. Inform.* **83** (2008). Special Issue in Celebration of Ludwig Staiger's 60th Birthday.

And some joint work we started, but never finished:

[14] C. S. Calude, H. Jürgensen: Randomness-preserving transformations. Manuscript, 32 pp., 1995.

[15] C. S. Calude, H. Jürgensen, A. Salomaa: Coding without tears. 1994. Manuscript.

And there are many new ideas!

My and my family's best wishes to you, Cris!