Derivatives of Regular Expressions and an Application

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Derivatives $w^{-1}(E)$ Brzozowski, 1964

➔ (left) quotient of language $w^{-1}L(E)$

Berry and Sethi 's Result 1986

Derivatives of E → classes of *similar* derivatives

Why? *E* linear

Our work

A characterization of the structure of derivatives of linear E *implies Berry and Sethi 's Result*

Derivatives

Berry and Sethi's result

Structure of derivatives

Properties of repeating terms

An application

Conclusion

Regular expressions

$$E ::= \phi \mid \varepsilon \mid a \in \Sigma \mid E + E \mid EE \mid E^*$$

ACI-similar

$$E_1 \sim_{\text{aci}} E_2$$

Associativity $(E_1 + E_2) + E_3 = E_1 + (E_2 + E_3)$

Commutativity $E_1 + E_2 = E_2 + E_1$

Idempotence $E + E = E$

Marked expressions

$(a+b)^*ab(a+b)$ E $(a_1+b_1)^*a_2b_2(a_3+b_3)$ $\overline{E}$

The same notation used for dropping of subscripts:

$\overline{\overline{E}} = E$

Note

Marking is not unique

For example $(a_1+b_2)^*a_3b_4(a_5+b_6)$

(left) quotient set of a language L

$$w^{-1}(L)=\{u \mid wu\in L\}$$

$$L = wL(w^{-1}(L))$$

Derivatives (Brzozowski)

$$\begin{aligned}
a^{-1}(\emptyset) &= a^{-1}(\varepsilon) = \emptyset \\
a^{-1}(b) &= \begin{cases} \varepsilon, & \text{if } b = a \\ \emptyset, & \text{otherwise} \end{cases} \\
a^{-1}(F+G) &= a^{-1}(F) + a^{-1}(G) \\
a^{-1}(FG) &= \begin{cases} a^{-1}(F)G + a^{-1}(G), & \text{if } \varepsilon \in L(F) \\ a^{-1}(F)G, & \text{otherwise} \end{cases} \\
a^{-1}(F^*) &= a^{-1}(F)F^*
\end{aligned}$$

$$\varepsilon^{-1}(E) = E,\ (wa)^{-1}(E) = a^{-1}(w^{-1}(E))$$

$$L(w^{-1}(E)) = w^{-1}(L(E))$$

Derivatives

Berry and Sethi’s result

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Regular expressions with distinct symbols (linear):

One symbol occurs only once

Next we consider this kind of expressions

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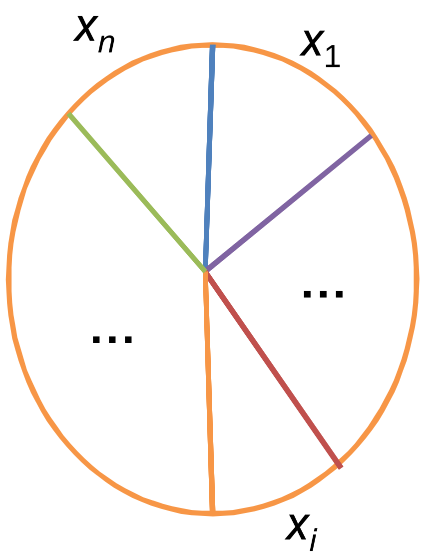
Berry and Sethi proved that

Let all symbols in E be distinct. Given a fixed $x \in \Sigma_E$, $(wx)^{-1}(E)$ is either $\emptyset$ or unique modulo $\sim_{aci}$ for all words w.



Non-null derivatives of *E*

$|\Sigma_E| = n$

Theorem 1 Let all symbols in E be distinct. Given a fixed $x \in \Sigma_E$, for all words w, each non-null $(wx)^{-1}(E)$ must be of one of the following forms: F or $F + \ldots + F$, where F is a non-null regular expression called the repeating term of $(wx)^{-1}(E)$ which does not contain + at the top level.

Example 1 *Let* $E = (a + b)(a^* + ba^* + b^*)^*$, *then*

$$\overline{E} = (a_1 + b_2)(a_3^* + b_4 a_5^* + b_6^*)^*,$$
$$a_1^{-1}(\overline{E}) = (a_3^* + b_4 a_5^* + b_6^*)^* = \tau_1,$$
$$(a_1 a_3)^{-1}(\overline{E}) = a_3^{-1}(\tau_1) = a_3^* \tau_1 = \tau_2,$$
$$(a_1 a_3 a_3)^{-1}(\overline{E}) = a_3^{-1}(\tau_2) = \tau_2 + \tau_2,$$
$$\ldots$$

repeating term for
$(wa_1)^{-1}(E)$: τ_1
$(wa_3)^{-1}(E)$: τ_2

Denote by $rt_x(E)$ the repeating term of $(wx)^{-1}(E)$

Corollary 1 Let all symbols in E be distinct. If $(wx)^{-1}(E)$ is non-null, then $(wx)^{-1}(E) \sim_{aci} rt_x(E)$.

a more precise version of Berry and Sethi's result

Q: For each $x \in \Sigma_E$, whether there is a non-null $(wx)^{-1}(E)$ containing one $rt_x(E)$, that is, $rt_x(E)$ is a derivative of E.

A: positive see below

The first appearance $F_x(E)$

$ind : \Sigma_E \to \{1, \ldots, \|E\|\}$: $ind(x) = d$ if x is the dth occurrence of symbols from left to right in E

$x < y$ iff $ind(x) < ind(y)$ $x, y \in \Sigma_E$

$w_1 \prec w_2$ if either $|w_1| < |w_2|$, or $|w_1| = |w_2|$ and

let $w_1 = x_1 \ldots x_n, w_2 = x'_1 \ldots x'_n, 1 \leq k \leq n$,

$x_t = x'_t$ for $t = 1, \ldots, k-1$, and $x_k < x'_k$

A non-null $(wx)^{-1}(E)$ is called the *first appearance* of derivative of E w.r.t. x if for any other non-nul $(w_1x)^{-1}(E)$ it has $w \prec w_1$

Example 2 *For* $E = (a+b)(a^* + ba^* + b^*)^*$, $\overline{E} = (a_1 + b_2)(a_3^* + b_4 a_5^* + b_6^*)^*$

$$\underline{a_1}^{-1}(\overline{E}) = (a_3^* + b_4 a_5^* + b_6^*)^* = \tau_1, \qquad \underline{b_2}^{-1}(\overline{E}) = (a_3^* + b_4 a_5^* + b_6^*)^* = \tau_1,$$
$$(a_1\underline{a_3})^{-1}(\overline{E}) = a_3^{-1}(\tau_1) = a_3^*\tau_1 = \tau_2, \qquad (a_1\underline{b_4})^{-1}(\overline{E}) = b_4^{-1}(\tau_1) = a_5^*\tau_1 = \tau_3,$$
$$(a_1\underline{b_6})^{-1}(\overline{E}) = b_6^{-1}(\tau_1) = b_6^*\tau_1 = \tau_4, \qquad (b_2a_3)^{-1}(\overline{E}) = a_3^{-1}(\tau_1) = \tau_2,$$
$$(b_2b_4)^{-1}(\overline{E}) = b_4^{-1}(\tau_1) = \tau_3, \qquad (b_2b_6)^{-1}(\overline{E}) = b_6^{-1}(\tau_1) = \tau_4,$$
$$(a_1a_3a_3)^{-1}(\overline{E}) = a_3^{-1}(\tau_2) = \tau_2 + \tau_2, \qquad (a_1a_3b_4)^{-1}(\overline{E}) = b_4^{-1}(\tau_2) = \tau_3,$$
$$(a_1a_3b_6)^{-1}(\overline{E}) = b_6^{-1}(\tau_2) = \tau_4, \qquad (a_1b_4a_3)^{-1}(\overline{E}) = a_3^{-1}(\tau_3) = \tau_2,$$
$$(a_1b_4b_4)^{-1}(\overline{E}) = b_4^{-1}(\tau_3) = \tau_3, \qquad (a_1b_4\underline{a_5})^{-1}(\overline{E}) = a_5^{-1}(\tau_3) = \tau_3.$$

Proposition 1 Let all symbols in E be distinct. Given a fixed $x \in \Sigma_E$, the first appearance $F_x(E)$ consists of only one repeating term.

The choice of the order is not significant.

Proposition 2 Let all symbols in E be distinct. Given any words $w_1, w_2 \in \Sigma_E^*$ and $x \in \Sigma_E$, if $| w_1 | = | w_2 |$ and $(w_1 x)^{-1}(E), (w_2 x)^{-1}(E) \neq \phi$, and there is no w, such that $|w| < | w_1 |$ and), $(w x)^{-1}(E) \neq \phi$, then $(w_1 x)^{-1}(E) = (w_2 x)^{-1}(E)$.

Proposition 3 Let all symbols in E be distinct. There exists a word $w \in \Sigma_E^*$ for each $x \in \Sigma_E$, such that $(w x)^{-1}(E) = rt_x(E)$.

Thus repeating terms are derivatives of E, and any non-null derivative of E is built from one of them.

Derivatives

Berry and Sethi’s result

Structure of derivatives

Properties of repeating terms

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Proposition 7 *Let all symbols in E be distinct. For each $x \in \Sigma_E$,*

(1) $rt_x(E)$ exists, and $rt_x(E) \neq \emptyset$.

(2) $rt_x(E)$ is unique.

There cannot be two $rt_x(E)$ for $(w\,x)^{-1}(E)$

Example 3 *For* $E = (a+b)(a^* + ba^* + b^*)^*$, $\overline{E} = (a_1 + b_2)(a_3^* + b_4 a_5^* + b_6^*)^*$.

$$
\begin{aligned}
rt_{a_1}(\overline{E}) &= rt_{a_1}(a_1 + b_2)(a_3^* + b_4 a_5^* + b_6^*)^* = rt_{a_1}(a_1)(a_3^* + b_4 a_5^* + b_6^*)^* \\
&= \varepsilon(a_3^* + b_4 a_5^* + b_6^*)^* = (a_3^* + b_4 a_5^* + b_6^*)^* = \tau_1, \\
rt_{b_2}(\overline{E}) &= \varepsilon(a_3^* + b_4 a_5^* + b_6^*)^* = \tau_1, \\
rt_{a_3}(\overline{E}) &= rt_{a_3}(a_3^* + b_4 a_5^* + b_6^*)^* = rt_{a_3}(a_3^* + b_4 a_5^* + b_6^*)\tau_1 = rt_{a_3}(a_3^*)\tau_1 \\
&= rt_{a_3}(a_3)a_3^*\tau_1 = a_3^*\tau_1 = \tau_2, \\
rt_{b_4}(\overline{E}) &= rt_{b_4}(a_3^* + b_4 a_5^* + b_6^*)^* = rt_{b_4}(b_4 a_5^*)\tau_1 = a_5^*\tau_1 = \tau_3, \\
rt_{a_5}(\overline{E}) &= rt_{a_5}(a_3^* + b_4 a_5^* + b_6^*)^* = rt_{a_5}(b_4 a_5^*)\tau_1 = a_5^*\tau_1 = \tau_3, \text{ and} \\
rt_{b_6}(\overline{E}) &= rt_{b_6}(a_3^* + b_4 a_5^* + b_6^*)^* = rt_{b_6}(b_6^*)\tau_1 = b_6^*\tau_1 = \tau_4.
\end{aligned}
$$

Proposition 8 Let all symbols in E be distinct. If there are non-null $(w_1 x_1)^{-1}(E)$ and $(w_2 x_2)^{-1}(E)$, such that $(w_1 x_1)^{-1}(E) \sim_{aci} (w_2 x_2)^{-1}(E)$, then $rt_{x1}(E) = rt_{x2}(E)$, and vice versa.

Corollary 2 Let all symbols in E be distinct. If $rt_{x1}(E) \sim_{aci} rt_{x2}(E)$, then $rt_{x1}(E)=rt_{x2}(E)$.

Remark $rt_x(E)$'s are `atomic' building blocks

(1) Each non-null $(w\,x)^{-1}(E)$ is uniquely decomposed into a sum of $rt_x(E)$, that is, $(w\,x)^{-1}(E) = \sum rt_x(E)$.

(2) $rt_x(E)$ and $rt_y(E)$ are either identical, or not equivalent modulo $\sim_{aci}$, if $x \neq y$.

Derivatives

Berry and Sethi's result

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Solves an issue in using Berry and Sethi's result: find a unique representative for $(w\,x)^{-1}(E)$

Glushkov automaton

$$M_{\text{pos}}(E) = (Q_{\text{pos}}, \Sigma, \delta_{\text{pos}}, q_E, F_{\text{pos}}),$$

where

1. $Q_{\text{pos}} = \Sigma_{\overline{E}} \cup \{q_E\}$, q_E *is a new state not in* $\Sigma_{\overline{E}}$
2. $\delta_{\text{pos}}(q_E, a) = \{x \mid x \in \mathit{first}(\overline{E}), \overline{x} = a\}$ *for* $a \in \Sigma$
3. $\delta_{\text{pos}}(x, a) = \{y \mid y \in \mathit{follow}(\overline{E}, x), \overline{y} = a\}$ *for* $x \in \Sigma_{\overline{E}}$ *and* $a \in \Sigma$
4. $F_{\text{pos}} = \begin{cases} \mathit{last}(\overline{E}) \cup \{q_E\}, & \text{if } \varepsilon \in L(E), \\ \mathit{last}(\overline{E}), & \text{otherwise} \end{cases}$

Berry and Sethi showed the class of derivatives $\{(wx)^{-1}(E)\}$ corresponds to a state x of $M_{\text{pos}}(E)$, $x \in \Sigma_{\overline{E}}$

In many cases, however, one needs a unique representative for the class of $\{(w\,x)^{-1}(E)\}$ to correspond to a state x

By the work, the representatives are obtained immediately

An improvement of Ilie and Yu‘s proof presented in (Ilie & Yu 2003)

A proof about the quotient relation between Glushkov and partial derivative automata

Partial derivatives

$$\partial_a(\emptyset) = \partial_a(\varepsilon) = \emptyset$$

$$\partial_a(b) = \begin{cases} \{\varepsilon\}, \text{if } b = a \\ \emptyset, \quad \text{otherwise} \end{cases}$$

$$\partial_a(F + G) = \partial_a(F) \cup \partial_a(G)$$

$$\partial_a(FG) = \begin{cases} \partial_a(F)G \cup \partial_a(G), \text{ if } \varepsilon \in L(F) \\ \partial_a(F)G, \qquad\qquad \text{otherwise} \end{cases}$$

$$\partial_a(F^*) = \partial_a(F)F^*$$

$$\partial_\varepsilon(E) = \{E\},\ \partial_{wa}(E) = \bigcup_{p \in \partial_w(E)} \partial_a(p)$$

$$PD(E) = \cup_{w \in \Sigma^*} \partial_w(E)$$

Partial derivative automaton

$$M_{\mathrm{pd}}(E) = (PD(E), \Sigma, \delta_{\mathrm{pd}}, E, \{q \in PD(E) \mid \varepsilon \in L(q)\}),$$

where $\delta_{\mathrm{pd}}(q, a) = \partial_a(q)$, *for any* $q \in PD(E), a \in \Sigma$.

$M_{\mathrm{pd}}(E)$ is a quotient of $M_{\mathrm{g}}(E)$

Ilie and Yu's proof

- The central issue is to find a unique representative for a class of derivatives
- The proof fails to find the correct representatives

It is claimed in the proof that, by using the rules (ϕε-rules), for a fixed $x \in \Sigma_{\overline{E}}$ and for all words *w*, $(wx)^{-1}(\overline{E})$ is either ϕ or unique.
incorrect

$$E + \emptyset = \emptyset + E = E$$
$$E\emptyset = \emptyset E = \emptyset$$
$$E\varepsilon = \varepsilon E = E$$

Rules (ϕ-rules)

Example. In Example 1, $(a_1a_3)^{-1}(\overline{E})$ and $(a_1a_3a_3)^{-1}(\overline{E})$ are distinct

An improved proof

Use $rt_x(\overline{E})$ as the unique representative.

See our paper

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Berry and Sethi’s result

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A characterization of the structure of derivatives

Several properties

An application

A useful technique

Thanks!