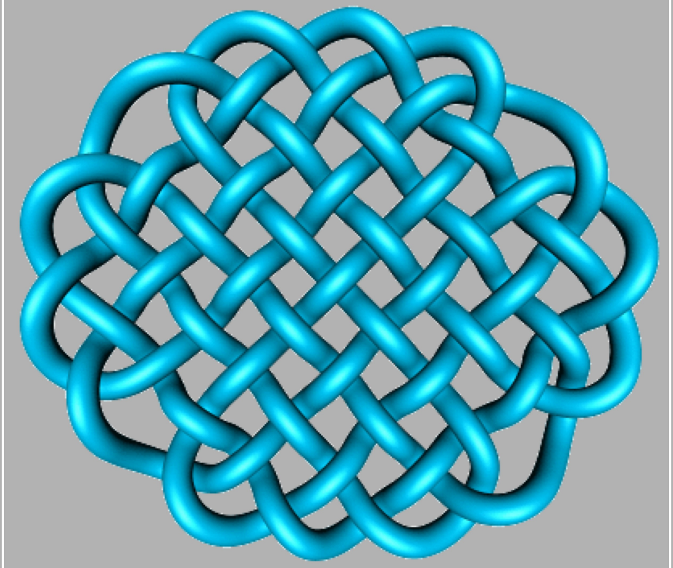

# Hartmanis-Stearns conjecture on real time and transcendence

**Rūsiņš Freivalds**

The most interesting results in mathematics, computer science and elsewhere are those which expose unexpected relations between seemingly unrelated objects. One of the most famous examples is the Cauchy-Hadamard theorem relating radius of convergence of a power series to the properties of the complex variable function defined by the power series.

The radius of convergence of a power series f centered on a point $a$ is equal to the distance from $a$ to the nearest point where $f$ cannot be defined in a way that makes it holomorphic.

This work was extended by Alan Baker (1939–) in 1966 by proving a result on linear forms in any number of logarithms (of algebraic numbers).

**Theorem.** (Alan Baker [1966]) Let $\alpha_1, \alpha_2, \cdots, \alpha_M$ be nonzero algebraic numbers such that the numbers $\log \alpha_1, \log \alpha_2, \cdots, \log \alpha_M$ are linearly independent over rational numbers. Then for any algebraic numbers $\beta_1, \beta_2, \cdots, \beta_M$, not all zero, the number

$$\beta_0 + \sum_{m=1}^{M} \beta_m \log \alpha_m$$

is transcendental.

The nearest point means the nearest point in the complex plane, not necessarily on the real line, even if the center and all coefficients are real. For example, the function

$$f(z) = \frac{1}{1+z^2}$$

has no singularities on the real line, since $1+z^2$ has no real roots. Its Taylor series about 0 is given by

$$\sum_{n=0}^{\infty}(-1)^n z^{2n}.$$

Juris Hartmanis and Richard Edwin Stearns in their paper awarded by the ACM Turing Award asked do there exist irrational algebraic numbers which are computable in real time.

More precisely, a real number is said to be computable in time $T(n)$ if there exists a multitape Turing machine which gives the first $n$-th terms of its binary expansion in (at most) $T(n)$ operations. *Real time* means that $T(n) = n$. All rational numbers clearly share this property. On the other hand, there are some transcendental numbers that can be computed in real time.

Why Hartmanis-Stearns conjecture is interesting? First of all, because mathematicians have had and they still have enormous difficulties to prove transcendence of numbers. Had Hartmanis-Stearns conjecture been proved, this would have been a very powerful tool to obtain new transcendence proofs.

A *rational number* is a number of the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q$ is not zero. An *irrational number* is any complex number which is not rational. A *transcendental number* is a number (possibly a complex number) that is not algebraic - that is, it is not a root of a non-constant polynomial equation with rational coefficients.

The name *transcendental* comes from Gottfried Wilhelm von Leibniz (1646 – 1716) in his 1682 paper where he proved $sin x$ is not an algebraic function of $x$. Leonhard Euler (1707 - 1783) was probably the first person to define transcendental numbers in the modern sense.

Joseph Liouville (1809 – 1882) first proved the existence of transcendental numbers in 1844, and in 1851 gave the first decimal examples such as the Liouville constant

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000\ldots$$

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000\ldots$$

We call an irrational number $\alpha$ *well-approximable* if for all positive integers $N, n$, there is a rational number $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Nq^n}.$$

It is easy to see that the Liouville constant is well-approximable.

**Theorem 1.** *(Joseph Liouville [1851]) No well-approximable number can be algebraic.*

A completely different proof was given three decades later by Georg Ferdinand Ludwig Philipp Cantor (1845 – 1918). He proved that there are more real numbers than algebraic numbers. According to the intuitionist school in the philosophy of mathematics (originated by Luitzen Egbertus Jan Brouwer, 1881 – 1966), such a pure existence proof is not valid unless it explicitly provides an algorithm for the construction of the object whose existence is asserted. However, even much less radical mathematicians felt that Cantor's theorem does not eliminate the need for explicit proofs of transcendence for specific numbers.

Charles Hermite (1822 –1901) proved the transcendence of the number $e$ in 1873. In 1882, Ferdinand von Lindemann published a proof that the number $\pi$ is transcendental. He first showed that e to any nonzero algebraic power is transcendental, and since $e^{i\pi} = -1$ is algebraic $i\pi$ and therefore $\pi$ must be transcendental. This approach was generalized by Karl Theodor Wilhelm Weierstrass (1815 – 1897) to the Lindemann–Weierstrass theorem. The transcendence of $\pi$ allowed the proof of the impossibility of several ancient geometric constructions involving compass and straightedge, including the most famous one, squaring the circle.

In 1900, David Hilbert (1862 – 1943) posed an influential question about transcendental numbers, Hilbert's seventh problem: If $\alpha$ is an algebraic number, that is not zero or one, and $\beta$ is an irrational algebraic number, is $\alpha^\beta$ necessarily transcendental? The affirmative answer was provided in 1934 by the Gelfond – Schneider theorem (Alexander Osipovich Gelfond, 1906 – 1968, Theodor Schneider, 1911 – 1988).

Alan Baker (1939–) was awarded the Fields Medal in 1970 for this result. Baker's theorem can make an impression that there is no more any difficulty to prove transcendence of numbers widely used in mathematics. Unfortunately, we are still very far from such a situation. Even for many numbers constructed from $e$, $\pi$ and similar ones, we do not know much.

It is known that $e^{\pi}$ is transcendental (implied by Gelfond – Schneider theorem), but for the number $\pi^{e}$ it is not known whether it is rational. At least one of $\pi \times e$ and $\pi + e$ (and probably both) are transcendental, but transcendence has not been proven for either number on its own. It is not known if $e^{e}, \pi^{\pi}, \pi^{e}$ are transcendental.

However, not only Liouville's result but also his method was important. It was later generalized at a great extent. For Liouville the most important lemma was as follows.

**Lemma.** Let $\alpha$ be an irrational algebraic number of degree $d$. Then there exists a positive constant depending only on $\alpha$, $c = c(\alpha)$, such that for every rational number $\frac{p}{q}$, the inequality

$$\frac{c}{q^d} \leq \left|\alpha - \frac{p}{q}\right|$$

is satisfied.

This lemma produced a notion of Liouville number. We say that $L$ is a *Liouville number* if there exists an infinite sequence of rational numbers $\frac{p_n}{q_n}$ satisfying

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^n}.$$

Liouville's theorem asserts that all Liouville numbers are transcendental.

Many mathematicians including Axel Thue (1863 – 1922), Carl Ludwig Siegel (1896 – 1981), Freeman Dyson (1923–) made important improvements to Liouville's theorem. In 1955 Klaus Friedrich Roth (1925 –) provided the best possible improvement.

**Theorem.** (K.F.Roth [1955]) Let $\alpha$ be an irrational algebraic number of degree $d \geq 2$ and let $\epsilon > 0$. Then there exists a positive constant $c = c(\alpha, \epsilon)$, such that for all $\frac{p}{q}$,

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha, \epsilon)}{q^{2+\epsilon}}.$$

Roth's result is the best possible, because this statement would fail on setting $\epsilon = 0$ (by Dirichlet's theorem on diophantine approximation there are infinitely many solutions in this case). K.F.Roth was awarded Fields Medal for this result in 1958.

Roth's theorem easily implies transcendence of Champernowne's number $0.1234567891011121314\cdots$ (obtained by concatenating the decimal expansions of all natural numbers)

Roth's theorem continued research started by Adolf Hurwitz ( 1859 – 1919). Hurwitz's theorem asserted that for arbitrary irrational number $\alpha$ there are infinitely many rationals $\frac{m}{n}$ such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{\sqrt{5}\,n^2},$$

and $\sqrt{5}$ cannot be substituted by a smaller number. Hurwitz's theorem is often used to classify irrational numbers according to the rate of the well-approximability. For example, for the number $\xi{=}(1{+}\sqrt{5})/2$ (the golden ratio) then there exist only finitely many rational numbers $\frac{m}{n}$ such that the formula above holds. Unfortunately for us, the rate of the well-approximability has no direct relation to number's being or not being transcendental.

Transcendental numbers initially were supposed to be more complicated rather than algebraic numbers. At least, the choice of the term "transcendental" suggests so. On the other hand, the Liouville constant has a rather simple description

$$0.11000100000000000000000001000\ldots$$

while the algebraic number

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317667973799...$$

seems to be quite "random". Of course, all rational numbers are algebraic and decimal (and all the other $b$-adic expansions) of them are periodic and hence simple. It turns out that all irrational algebraic numbers are rather complicated.

In 1909 Emile Borel (Félix *E*douard Justin *É*mile Borel, 1871 – 1956) asked whether it is possible to tell transcendental numbers from algebraic ones by statistics of their digits in some $b$-adic expansion. He introduced the notion of a *normal number*.

Let $x$ and $b \geq 2$. Consider the sequence of digits of the expansion of $x$ in base $b$. We are interested in finding out how often a given digit $s$ shows up in the above representation of $x$ . If we denote by $N(s, n)$ the number of occurrences of $s$ in the first $n$ digits of $x$ , we can calculate the ratio $\frac{N(s,n)}{n}$ As $n$ approaches $\infty$ , this ratio may converge to a limit, called the frequency of $s$ in $x$ . The frequency of $s$ in $x$ is necessarily between 0 and 1. If all base $b$ digits are equally frequent, i.e. if the frequency of each digit $s$ , $0 \leq s < b$, is $\frac{1}{b}$ , then we say that $x$ is *simply normal* in base $b$ . For example, in base 5 , the number $01234012340123401234 \cdots$ is simply normal.

In 1909 Emile Borel (Félix $E$douard Justin Émile Borel, 1871 – 1956) asked whether it is possible to tell transcendental numbers from algebraic ones by statistics of their digits in some $b$-adic expansion. He introduced the notion of a *normal number*

Let $x$ and $b \geq 2$. Consider the sequence of digits of the expansion of $x$ in base $b$. We are interested in finding out how often a given digit $s$ shows up in the above representation of $x$ . If we denote by $N(s,n)$ the number of occurrences of $s$ in the first $n$ digits of $x$ , we can calculate the ratio $\frac{N(s,n)}{n}$ As $n$ approaches $\infty$ , this ratio may converge to a limit, called the frequency of $s$ in $x$ . The frequency of $s$ in $x$ is necessarily between 0 and 1. If all base $b$ digits are equally frequent, i.e. if the frequency of each digit $s$ , $0 \leq s < b$, is $\frac{1}{b}$ , then we say that $x$ is *simply normal* in base $b$ . For example, in base 5 , the number $01234012340123401234 \cdots$ is simply normal.

If we allow $s$ to be any finite string of digits (in base $b$), then we have the notion of a normal number.

Let $x$ be a real number. Let $s$ be a string of digits of length $k$ , in base $b$ : $s = s_1 s_2 \cdots s_k$ where $0 \leq s_j < b$. Define $N(s,n)$ to be the number of times the string $s$ occurs among the first $n$ digits of $x$ in base $b$ . For example, if $x = 21131112$ in base $4$ , then $N(1,8) = 5, N(11,8) = 3$ , and $N(111,8) = 1$ . We say that $x$ is normal in base $b$ if

$$\lim_{n\to\infty} \frac{n}{N(s,n)} = \frac{1}{b^k}$$

for every finite string $s$ of length $k$ . We see that if $k = 1$ , we are back to the definition of a simply normal number, so every number normal in base $b$ is in particular simply normal in base $b$ .

Intuitively, $x$ is normal in base $b$ if all digits and digit-blocks in the base $b$ digit sequence of $x$ occur just as often as would be expected if the sequence had been produced completely randomly. Unlike simply normal numbers, normal numbers are necessarily irrational.

Normal numbers are not as easy to find as simply normal numbers. One example is Champernowne's number

$$0.1234567891011121314\cdots$$

(obtained by concatenating the decimal expansions of all natural numbers), which is normal in base 10. It is not known whether Champernowne's number is normal in other bases. Champernowne's number can be written as

$$C_{10} = \sum_{n=1}^{\infty} \sum_{k=10^{n-1}}^{10^n-1} \frac{k}{10^{kn-9\sum_{k=0}^{n-1} 10^k(n-k)}}.$$

There exist numbers which are normal in all bases $b = 2, 3, 4, \cdots$. They are called *absolutely normal*. The first absolutely normal number was constructed by Wacław Franciszek Sierpiński (1882 – 1969) in 1917. Verónica Becher and Santiago Figueira proved

**Theorem.** (Becher, Figueira (2002)) There exists a computable absolutely normal number.

The construction of computable absolutely normal numbers is an innovative and complicated recursive function theoretical adaptation of Sierpiński's construction. (By the way, the authors of this paper acknowledge valuable comments from Cristian Calude.)

In 1950 Borel asked whether all irrational algebraic numbers are absolutely normal. It is still not known. The mere existence of this open problem shows that absolute normality of numbers is a property that can be possessed only by numbers whose decimal (and other $b$-adic) expansions are very complicated. Unfortunately, no one has been able to use this observation to tell transcendental numbers from algebraic ones.

In contrast to Borel's conjecture, it is needed to say that all algebraic numbers about whom we know that they are absolutely normal, are highly artificial. They are specially constructed to prove their absolute normality.

However, is the notion of absolutely normal numbers the notion we need to prove the Hartmanis-Stearns conjecture?

# Continued fractions

Now we are looking for another way to describe irrational numbers with a hope that this new description could be used to distinguish transcendental numbers. One such potentially useful description is continued fractions.

Continued fractions is a natural notion. Most people believe that there cannot exist a way how to memorize good approximations for the number

$$\pi = 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848111745 02\ldots$$

However, they exist:

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\cfrac{9^2}{6+\cdots}}}}}$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\cdots}}}}} = \cfrac{4}{1 + \cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\cdots}}}}}}.$$

A *finite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}}$$

where $a_0$ is an integer, any other $a_i$ members are positive integers, and $n$ is a non-negative integer. An *infinite continued fraction* can be written as

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots}}}}}$$

One can abbreviate a continued fraction as $x = [a_0; a_1, a_2, a_3,$

The decimal representation of real numbers has some problems. One problem is that many rational numbers lack finite representations in this system. For example, $\frac{1}{3}$ is represented by the infinite sequence $(0, 3, 3, 3, 3, \ldots)$.

Another problem is that the constant 10 is an essentially arbitrary choice, and one which biases the resulting representation toward numbers that have some relation to the integer 10. Continued fraction notation is a representation of the real numbers that avoids both these problems.

Continued fractions provide regular patterns for many important numbers. For example, the golden ratio

$$\frac{1+\sqrt{5}}{2} = \varphi = 1 + \frac{1}{\varphi}$$

has a continued fraction representation $\varphi = [1; 1, 1, 1, 1, \cdots]$. Notably,
$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, \cdots]$,
$e^2 = [7; 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, 42, 11, \cdots, 3k, 12k+6, 1, 1, \cdots]$,
$e^{\frac{1}{n}} = [1; n-1, 1, 1, 3n-1, 1, 1, 5n-1, 1, 1, 7n-1, 1, 1, \cdots]$
$\tan(1) = [1; 1, 1, 3, 1, 5, 1, 7, 1, 9, 11, 1, 13, 1, 15, 1, 17, 1, \cdots]$.

If arbitrary values and/or functions are used in place of one or more of the numerators the resulting expression is a *generalized continued fraction*. The 3 distinct fractions above for $\pi$ were generalized continued fractions. Every real number has exactly one standard continued fraction. The continued fraction for $\pi$ is not as regular as the generalized continued fractions shown above.

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, \cdots].$$

From results of Leonhard Euler (1707 – 1783) and Joseph-Louis Lagrange (1736 – 1813) we know that the regular continued fraction expansion of $x$ is periodic if and only if $x$ is a quadratic irrational.

Continued fractions may give us many still not discovered criteria for properties of numbers. For example, if $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ are positive integers with $a_k \leq b_k$ for all sufficiently large $k$, then the generalized continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}}$$

converges to an irrational limit.

Aleksandr Yakovlevich Khinchin (1894 – 1959) expressed a conjecture in 1949 which is now widely believed that the continued fraction expansion of any irrational algebraic number $\alpha$ is either eventually periodic (and we know that this is the case if and only if $\alpha$ is a quadratic irrational), or it contains arbitrarily large partial quotients.

J.P. Allouche [2000] conjectures that the continued fraction expansion of any algebraic irrational number that is not a quadratic number is normal.

# Automata in number theory

J. Hartmanis and R. Stearns asked do there exist irrational algebraic numbers which are computable in real time. Hence it seems natural that we start by proving that it is not possible to compute an irrational algebraic number by a finite automaton. Indeed, a finite automaton with no input information can produce only a periodic sequence but every number whose $b$-adic expansion is periodic, is inevitably rational. True but too simple for a good result.

However, there is a possibility for nontrivial results.

**Definition.** Let $b \geq 2$ be an integer. A sequence $(a_n)$ is called $b$-automatic if there exists a finite automaton taking the base-$b$ expansion of $n$ as input and producing the term $a_n$ as output.

It is not hard to prove that all periodic sequences are $b$-automatic for every integer $b \geq 2$. But is every 2-automatic sequence also 3-automatic? Alan Cobham (1927-) published two influential papers [1969,1972] on this topic.

**Theorem.** A sequence is $b$-automatic if and only if it is $b^r$-automatic for all positive integers $r$.

**Definition.** Two positive integers $b$ and $d$ are multiplicatively independent if the equation $b^a = d^b$ has no nontrivial integer solution $(a, b)$, that is, $\frac{\log b}{\log d}$ is irrational.

**Theorem.** A nonperiodic sequence cannot be both $b$-automatic and $d$-automatic for two multiplicatively independent positive integers $b$ and $d$.

For continued fractions somewhat similar results were proved in 1997 by Ferenczi and Mauduit.

**Theorem.** (Cobham [1972]) An infinite word is $k$-automatic if and only if it is the image by a coding of a word that is generated by a $k$-uniform morphism.

**Definition.** The $k$-kernel of a sequence $a = (a_n)_{n\geq 0}$ is defined as the set

$$N_k(a) = \{(a_{k^r n+i})_{n\geq 0} \mid i \geq 0, 0 \leq i < k^r\}.$$

**Theorem.** (Eilenberg [1974]) A sequence is $k$-automatic if and only if its $k$-kernel is finite.

In 1977 John Loxton and Alf van der Poorten proved transcendence results on values of Mahler functions. In consequence the matter of the transcendence of irrational automatic numbers became known as the conjecture of Loxton and van der Poorten.

This line of research resulted in 2004 by a result by Boris Adamczewski, Yann Bugeaud and Florian Luca.

**Theorem.** Let $b \geq 2$ be an integer. The $b$-ary expansion of any irrational algebraic number cannot be generated by a finite automaton.

In other words, irrational automatic numbers are transcendental.

Two real numbers $\alpha$ and $\alpha'$ are said to be equivalent if their $b$-adic expansions have the same tail.

We say that $a$ is a *stammering sequence* if there exist a real number $w > 1$ and two sequences of finite words $(W_n)_{n\geq 1}, (X_n)_{n\geq 1}$ such that:

(i) For any $n \geq 1$, the word $W_n X_n^w$ is a prefix of the word $a$;

(ii) The sequence $(\mid W_n \mid / \mid X_n \mid)_{n\geq 1}$ is bounded from above;

(iii) The sequence $(\mid X_n \mid)_{n\geq 1}$ is increasing.

**Theorem.** (Adamczewski, Bugeaud, Luca [2007]) Let $a = (a_k)_{k\geq 1}$ be a stammering sequence of integers from $\{0, 1, \cdots, b-1\}$. Then, the real number

$$\alpha = \sum_{k=1}^{+\infty} \frac{a_k}{b^k}$$

is either rational or transcendental.

Let $a = (a_k)_{k\geq 1}$ and $a' = (a'_k)_{k\geq 1}$ be sequences of elements from $A$, that we identify with the infinite words $a_1 a_2 \cdots$ and $a'_1 a'_2 \cdots$, respectively. We say that the pair $(a, a')$ satisfies Condition $(*)$ if there exist three sequences of finite words $(U_n)_{n\geq 1}, (U'_n)_{n\geq 1}$, and $(V_n)_{n\geq 1}$ such that:
(i) For any $n \geq 1$, the word $U_n V_n$ is a prefix of the word $a$;
(ii) For any $n \geq 1$, the word $U'_n V_n$ is a prefix of the word $a'$;
(iii) The sequences $(\mid U_n \mid / \mid V_n \mid)_{n\geq 1}$ and $(\mid U'_n \mid / \mid V_n \mid)_{n\geq 1}$ are bounded from above;
(iv) The sequence $(\mid V_n \mid)_{n\geq 1}$ is increasing.
If, moreover, we add the condition
(v) The sequence $(\mid U_n \mid - \mid U'_n \mid)_{n\geq 1}$ is unbounded,
then, we say that the pair $(a, a')$ satisfies Condition $(**)$.

**Theorem.** (Adamczewski, Bugeaud [2010]) )

Let $a = (a_k)_{k\geq 1}$ and $a' = (a'_k)_{k\geq 1}$ be sequences of integers from $\{0, 1, \cdots, b-1\}$. If the pair $(a, a')$ satisfies Condition $(*)$, then at least one of the real numbers

$$\alpha = \sum_{k=1}^{+\infty} \frac{a_k}{b^k}\ ,\ \alpha' = \sum_{k=1}^{+\infty} \frac{a'_k}{b^k}$$

is transcendental, or $\alpha$ and $\alpha'$ are equivalent. Furthermore, if the pair $(a, a')$ satisfies Condition $(**)$, then at least one of the real numbers $\alpha, \alpha'$ is transcendental, or they are equivalent and both rational.

Theorems by Adamczewski and Bugeaud show that the solution to Hartmanis-Stearns conjecture may be coming soon.

One may say that the number-theoretical background is already provided and now it remains to prove properties of real-time Turing machines to ensure the properties (i)–(v).

The problem to prove Hartmanis-Stearns conjecture has changed its nature. From a number-theoretical problem it has become a problem in theory of computation.

I thank Cristian Calude for many years of friendship, and I wish him many happy and healthy years.