Constructing the infimum of two projections

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Constructive mathematics?

Three ways to approach computability in mathematics:

1) Use classical computability theory.

The logic allows “decisions” that cannot be made by any real computer, so we need a clearly specified type of algorithm.

This is the approach of recursive analysis and Weihrauch’s TTE theory.

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2) Use classical proof mining (Kohlenbach).

This requires a heavy logical analysis in order to extract (admittedly often good) constructive estimates from classical proofs. Moreover, it is not clear that this technique would work well in functional analysis or measure theory, let alone for deep, highly nonconstructive, results of operator algebra theory.

3) Use intuitionistic logic (Brouwer, Markov, Bishop, Martin-Löf, ...).

This

automatically takes care of the problem of noncomputational "decisions", and

enables us to work, with any mathematical objects, in the familiar style of the analyst, algebraist, geometer, ...

Bishop-style constructive mathematics (**BISH**) is just

mathematics with intuitionistic logic,

together with some appropriate foundation such as

- the constructive set theory of Myhill, Aczel, and Rathjen, or
- Martin-Löf type theory,

and dependent choice.

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> "Intuitionistic logic is richer than classical logic, since the former makes distinctions that the latter fails to make."
>
> J.L. Bell & M. Machover

Projections on a Hilbert space

Let H be a (real or complex) Hilbert space, and V a (linear) subspace of H.

The following are equivalent:

(i) V is located—that is,

$$\rho(x, V) \equiv \inf \{\|x - y\| : y \in V\}$$

exists for each $x \in H$.

(ii) For each $x \in H$, there exists a unique $v \in V$ such that $\|x - v\| \leq \|x - y\|$ for all $y \in V$.

(iii) For each $x \in H$, there exists a unique $v \in V$ such that $x - v$ is orthogonal to V.

If any, and hence each, of these conditions hold, then the vector v in (ii) and (iii) is the same and is called the **projection of the vector** x on V. The maping that takes x to the corresponding v is **the projection of** H **onto** V. If V is nontrivial (that is, contains a vector of positive norm), then P is normed, with $\|P\| = 1$.

We define a partial order on the set of projections on H by

$$E \leq F \Leftrightarrow \forall_{x \in H} \left(\langle Ex, x \rangle \leq \langle Fx, x \rangle \right).$$

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We wish to construct the **infimum** $E \wedge F$ **of the projections** E and F of H to be the unique projection P (if it exists) that satisfies the following two conditions:

(a) $P \leqslant E$ and $P \leqslant F$.

(b) If Q is a projection with $Q \leqslant E$ and $Q \leqslant F$, then $Q \leqslant P$.

Classically, $E \wedge F$ always exists and is the projection on the intersection of $\mathsf{ran}\, E$ (the range of E) and $\mathsf{ran}\, F$.

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Proposition 1. *The projection* $E \wedge F$ *exists if and only if* $E(H) \cap F(H)$ *is located, in which case* $E \wedge F$ *is the projection on* $E(H) \cap F(H)$.

It can be shown classically (see **Halmos**, page 257) that the decreasing sequence $((EFE)^n)_{n \geqslant 1}$ of projections converges strongly to a projection P on H, in the sense that

$$Px = \lim_{n \to \infty} (EFE)^n x$$

for each $x \in H$. It can then be shown that P is the infimum of the projections E and F.

We shall examine, within **BISH**, the connection between the existence of P and the strong convergence of the sequence $((EFE)^n)_{n \geqslant 1}$.

The algorithm

Specker's theorem shows that the monotone convergence theorem in $\mathbf{R}$ is false in the recursive constructive mathematics (**RUSS**) of the Markov School; see either **Specker** or **BR** (Chapter 3).

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Hence the corresponding monotone convergence theorem for projections of a Hilbert space (**KR**, Lemma 5.1.4) is also false in **RUSS**, and therefore, since **RUSS** is consistent with **BISH**, not constructively provable.

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Hence the corresponding monotone convergence theorem for projections of a Hilbert space (**KR**, Lemma 5.1.4) is also false in **RUSS**, and therefore, since **RUSS** is consistent with **BISH**, not constructively provable.

In consequence, Halmos's classical proof of the statement

$$(E \wedge F)\,x = \lim_{n\to\infty} (EFE)^n\,x \quad (x \in H) \tag{1}$$

fails constructively.

We can adapt Halmos's argument to obtain the following constructive result.

Proposition 2. *Let E, F be projections on H such that the strong limit P of the sequence $((EFE)^n)_{n \geqslant 1}$ exists. Then $P = E \wedge F$. Moreover, the sequence $((FEF)^n)_{n \geqslant 1}$ converges strongly to $E \wedge F$.*

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Can we prove, conversely, that if $E \wedge F$ exists, then $((EFE)^n x)_{n \geqslant 1}$ converges for each $x \in H$?

Consider the case where

$H = \mathbf{R}^2$,

E is the projection of H on the subspace $\mathbf{R}e$, where $e = (0, 1)$,

and F is the projection on $\mathbf{R}(\cos\theta, \sin\theta)$, where $\neg(\theta = 0)$.

We have $E \wedge F = 0$.

Also,

$$(EFE)^n e = \left(\cos^{2n}\theta, 0\right)$$

for each n; but this converges to $(0, 0)$ if and only if $\cos\theta \neq 1$ (that is, $|1 - \cos\theta| > 0$) and therefore $\theta \neq 0$.

Conclusion: if the existence of $E \wedge F$ entails the strong convergence of $((EFE)^n x)_{n \geqslant 1}$, then we can prove

$$\forall_{\theta \in \mathbf{R}} \left(\neg (\theta = 0) \Rightarrow \theta \neq 0 \right),$$

which is easily shown to be equivalent to **Markov's Principle**:

MP: For each binary sequence $(a_n)_{n \geqslant 1}$, if it is impossible that $a_n = 0$ for all n, then there exists n such that $a_n = 1$.

MP is not a part of **BISH** (though it is consistent with **BISH** and is accepted **RUSS**).

To make further progress, we need some estimates.

Lemma 1. *Let H be a Hilbert space, and E, F projections on H such that $E \wedge F$ exists. Then for each $x \in H$, there exists a strictly increasing sequence $(n_k)_{k \geqslant 1}$ of positive integers such that*

$$\|(E - F)\,(EFE)^{n_k}\,x\| < 2^{-k} \quad (k \geqslant 1)\,.$$

Sketch proof. For each positive integer n,

$$\|(I - F)(EFE)^n x\|^2 = \left\langle \left((EFE)^{2n} - (EFE)^{2n+1} \right) x, x \right\rangle .$$

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Given $\varepsilon > 0$, pick N such that $N\varepsilon > \left\langle (EFE)^2 x, x \right\rangle$. Then we cannot have

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So there exists $n \leqslant N$ such that

$$\begin{aligned} \|(E-F)(EFE)^n x\| &= \|(I-F)(EFE)^n x\|^2 \\ &= \left\langle \left((EFE)^{2n} - (EFE)^{2n+1}\right) x, x \right\rangle < 2\varepsilon. \end{aligned}$$

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Since $\varepsilon > 0$ is arbitrary, we can now construct, inductively, the desired strictly increasing sequence $(n_k)_{k \geqslant 1}$ of positive integers.

Recall that a sequence $(v_n)_{n\geq 1}$ in H is weakly convergent to $v \in H$ if for each $x \in H$, $\langle v_n, x\rangle \to \langle v, x\rangle$ as $n \to \infty$.

An operator T on H is **weak-sequentially open** if

for each sequence $(x_n)_{n\geqslant 1}$ such that $Tx_n \to 0$, there exists a sequence $(y_n)_{n\geqslant 1}$ in $\ker T$ such that $x_n + y_n \xrightarrow{w} 0$.

Proposition 3. *If T is a jointed operator—that is, one with an adjoint—on H, then the following conditions are equivalent:*

- *T has located range.*

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In particular, if $\ker T$ is located, then a necessary and sufficient condition for $\operatorname{ran}(T)$ to be located is that T is weak-sequentially open.

Theorem 1. *Let H be a separable Hilbert space, and E, F projections on H such that $E \wedge F$ exists and $E - F$ is weak-sequentially open. Then $((EFE)^n)_{n \geqslant 1}$ converges strongly to $E \wedge F$.*

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Sketch proof. By Proposition 1, $P \equiv E \wedge F$ is the projection on $E(H) \cap F(H)$.

Fix x in H. Using Lemma 1, construct a strictly increasing $(n_k)_{k \geqslant 1}$ of positive integers such that

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Since $E - F$ is weak-sequentially open, there exists $(y_k)_{k \geqslant 1}$ in $\ker(E - F)$ such that $(EFE)^{n_k}\,x + y_k \xrightarrow{w} 0$ as $k \to \infty$.

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Since $E - F$ is weak-sequentially open, there exists $(y_k)_{k \geqslant 1}$ in $\ker(E - F)$ such that $(EFE)^{n_k} x + y_k \xrightarrow{w} 0$ as $k \to \infty$. For each $z \in H$,

$$\begin{aligned} \langle (I - P) E [(EFE)^{n_k} x + y_k], z \rangle &= \langle (EFE)^{n_k} x + y_k, E (I - P) z \rangle \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Now, $(E - F)\, y_k = 0$, so

$$E y_k = F y_k \in E(H) \cap F(H) = \mathsf{ran}\,(P)\,.$$

For $n \geq n_k$ we therefore have

$$\begin{aligned} 0 &\leqslant \langle (I-P)\,(EFE)^n\, x, x\rangle \leqslant (I-P)\,(EFE)^{n_k}\, x \\ &= (I-P)\,E\,[(EFE)^{n_k} x + y_k] \xrightarrow{w} 0 \text{ as } k \longrightarrow \infty. \end{aligned}$$

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Since $PE = P = EP$ and $PF = F = FP$, we obtain

$$\begin{aligned} \|(EFE)^n x - Px\|^2 &= \langle (I-P)\,(EFE)^n\,x, (I-P)\,(EFE)^n\,x\rangle \\ &= \left\langle (EFE)^n\,(I-P)^2\,(EFE)^n\,x, x\right\rangle \\ &= \left\langle (I-P)\,(EFE)^{2n}\,x, x\right\rangle \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

as desired.

Corollary 1. *Let H be a separable Hilbert space, and E, F projections on H such that $E \wedge F$ exists and $E - F$ has located range. Then $((EFE)^n)_{n \geqslant 1}$ converges strongly to $E \wedge F$.*

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Corollary 2. *Let H be a separable Hilbert space, and E, F projections on H such that $E \wedge F$ exists and $E - F$ has closed range. Then $((EFE)^n)_{n \geqslant 1}$ converges strongly to $E \wedge F$.*

Proof. By the closed range theorem (Theorem 6.5.9 of **BV**), the range of $E - F$ is located; so we can immediately invoke Corollary 1.

An elementary geometric result:

Lemma 2. *Let* E, F *be* 1*-dimensional projections in* $\mathbf{R}^2$ *such that* $E - F \neq 0$*.*
Then $\operatorname{ran}(E - F) = \mathbf{R}^2$.

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Consider again our Markovian example, where $H = \mathbf{R}^2$; E is the projection of H on $\mathbf{R}e$, where $e = (0, 1)$; and F is the projection on $\mathbf{R}(\cos\theta, \sin\theta)$, where $\neg(\theta = 0)$.

In this example, the following are equivalent:

- The range of $E - F$ is located.
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In that case,

$$\text{ran}\,(E - F) = \ker(E - F) + \text{ran}\,(E - F)$$

is dense in $H \equiv \mathbf{R}^2$, so $\text{ran}\,(E - F)$ is 2-dimensional and equals H.

If $(EFE)^n$ converges strongly to 0, then $\theta \neq 0$, so $E - F \neq 0$ and therefore, by Lemma 2, $E - F$ has closed range equal to $\mathbf{R}^2$.

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Then $\operatorname{ran}(E - F)$ is located, by the closed range theorem, and so is finite-dimensional.

If $\theta \neq 0$, then, as above, the dimension of $\operatorname{ran}(E - F)$ is 2. Since $\neg\neg(\theta \neq 0)$, that dimension must indeed be 2.

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If $\theta \neq 0$, then, as above, the dimension of $\mathrm{ran}\,(E - F)$ is 2. Since $\neg\neg\,(\theta \neq 0)$, that dimension must indeed be 2.

Elementary computations show that $\cos\theta < 1$; whence $\theta \neq 0$, and therefore $(EFE)^n$ converges strongly to 0.

Putting all this together, we actually have the following result:

Proposition 5. *The following are equivalent:*

(i) *For all* 1*-dimensional projections* E, F *in* $\mathbf{R}^2$ *such that* $E \wedge F = 0$*, the sequence* $((EFE)^n)_{n \geqslant 1}$ *converges strongly to* 0*.*

(ii) *For all* 1*-dimensional projections* E, F *in* $\mathbf{R}^2$ *such that* $E \wedge F = 0$*, the supremum* $E \vee F$ *exists.*

(iii) **MP.**

We end with a more general theorem that brings in Markov's principle

Theorem 2. **MP** $\vdash$ *Let H be a finite-dimensional Hilbert space, and E, F projections on H such that $E \wedge F$ and $E \vee F$ exist. Then $((EFE)^n)_{n \geqslant 1}$ converges strongly to $E \wedge F$.*

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The proof of this theorem requires a few nontrivial preliminaries, the most interesting of which is this:

Proposition 6. **MP** $\vdash$ *Let T be a bounded operator mapping H into a finite-dimensional subspace of itself such that* $\ker T^*$ *is located. Then the range of T is located.*

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The preceding proposition cannot be established without Markov's principle. Can all reference to Markov's principle be removed from Theorem 2?

We thank Cristian Calude for many years of friendship and encouragement of our researches, and we wish him many fruitful, happy, and healthy years beyond 60!

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