Monads – Part II

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1. LINQ Workflows vs. F# Workflows
2. Kleisli Operators
3. Monad Laws

Outline

LINQ defines Problem Specific Workflows

- C# Query Comprehensions
  – on sequences or sequence-like queryable collections

```
1  var x = new[] { 10, 20, 30, };
2
3  var y = from t in x
4          where t%4 != 0   // → .Where (...)
5          select t+1;      // → .Select (...)
```

- LINQ: Query Comprehensions → Enumerable Methods

```
1  var y = x.Where(t => t%4 != 0).Select (t => t+1);
```

```
1  var y = Enumerable.Select (
2              Enumerable.Where(x, t => t%4 != 0),
3              t => t+1);
```

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- C# Query Comprehensions
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F# Workflows

- F# pipeline – List module functions

```
1 let x = [ 10; 20; 30; ]
2
3 let y = x |> List.filter (fun t -> t%4 <> 0)
4           |> List.map (fun t -> t+1)
```

- F# Query Comprehensions → QueryBuilder Methods

```
1 let y = query {
2             for t in x do       // → .For (...)
3             where (t%4 <> 0)    // → .Where (...)
4             select (t+1)        // → .Select (...)
5         }
```

https://msdn.microsoft.com/en-us/visualfsharpdocs/conceptual/linq.querybuilder-class-%5Bfsharp%5D

F# Workflows

- F# pipeline – List module functions

```
1 let x = [ 10; 20; 30; ]
2
3 let y = x |> List.filter (fun t -> t%4 <> 0)
4           |> List.map (fun t -> t+1)
```

- F# Query Comprehensions → QueryBuilder Methods

```
1 let y = query {
2             for t in x do       // → .For (...)
3             where (t%4 <> 0)    // → .Where (...)
4             select (t+1)        // → .Select (...)
5           }
```

https://msdn.microsoft.com/en-us/visualfsharpdocs/conceptual/linq.querybuilder-class-%5Bfsharp%5D

F# Workflows

- F# are highly customizable
- F# are more powerful and cover a wide variety of domains:
  seq, query, async, actor, job, cloud, ...
  `http://mbrace.io/programming-model.html`

```
1 let first = cloud { return 15 }
2 let second = cloud { return 27 }
3
4 cloud {
5     let! x = first
6     let! y = second
7     return x + y
8 }
```

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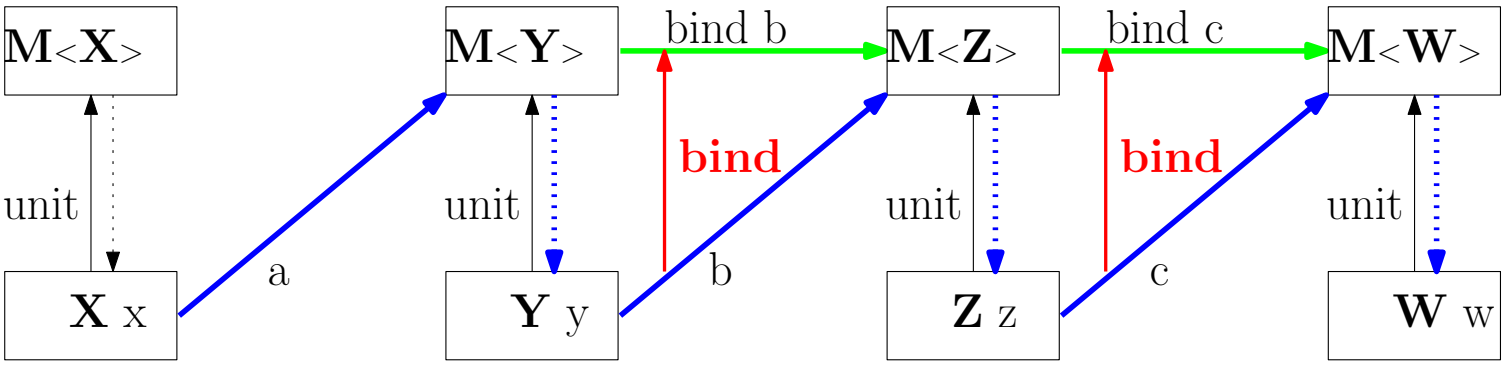
Outline

Composing Diagonal Arrows

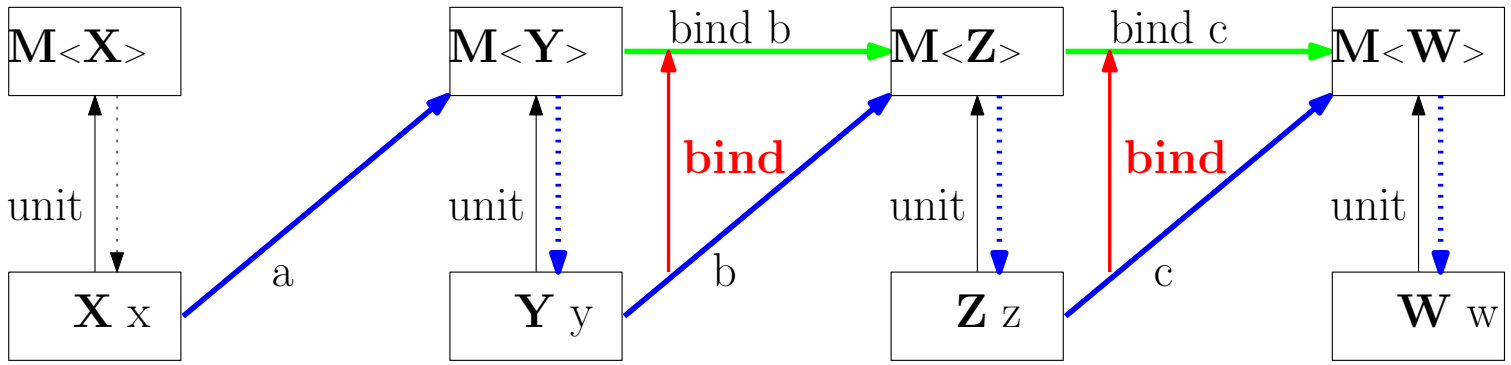


- How to combine several diagonal arrows – cf. the blue line

```
1  let a x = mytask { return x+1 }
2  let b y = mytask { return y+y }
3  let c z = mytask { return z*z }
```

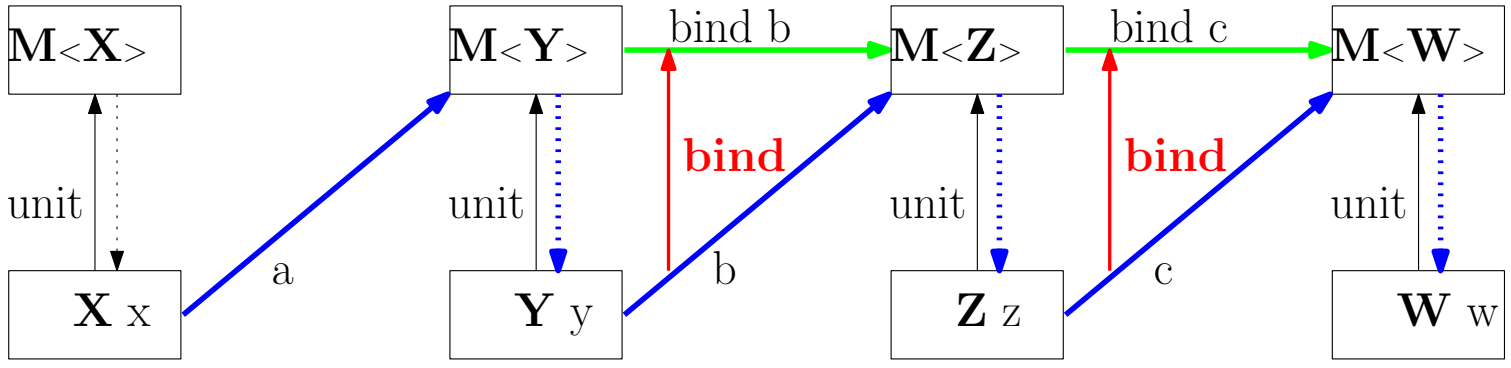
```
1  x = 3 ⇒ y = 4 ⇒ z = 8 ⇒ w = 64
```

Solution: Workflow




```
1  let d x = mytask {                                    // x = 3
2    let! y = a x // mytask.Bind (a x, ...)              // y = 4
3    let! z = b y // mytaks.Bind (b y, ...)              // z = 8
4    return! c z  // mytask.ReturnFrom (c z) // unit 64
5    }
6  let w = unit1 (d 3)
```

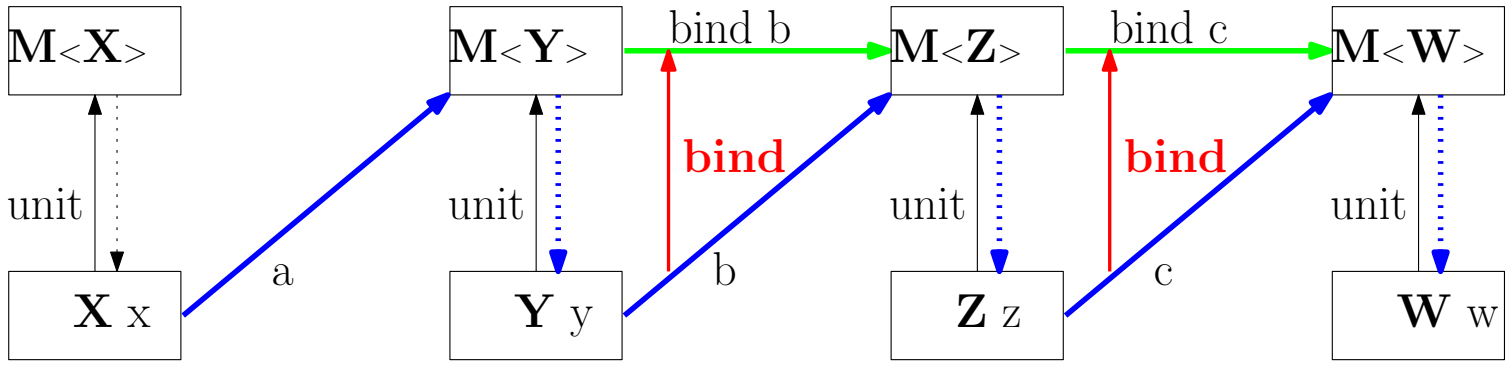
Solution: Builder Methods




```
1  let d x =
2      mytask.Bind (a x,
3          fun y ->
4              mytask.Bind (b y,
5                  fun z -> mytask.ReturnFrom c z))
```

```
1                  fun z -> c z))
2                  c))
```

Solution: Monad Functions




```
1  let d x = bind (fun y -> bind c (b y)) (a x)
2
3  let d x = a x |> bind (fun y -> (bind c << b) y)
4  let d x = a x |> bind (bind c << b)
5  let d x = a x |> bind (b >> bind c)
6
7  let d x = x |> a >> bind (b >> bind c)
```

Note the patterns: `m |> bind f` and `f >> bind g`

An Even "Better" Solution ? Kleisli Operators!

- Bind operator

```
1 // (>>=) : Task<'T> -> ('T->Task<'U>)
2 //          -> Task<'U>
3
4 let inline (>>=) m f = bind f m // m |> (bind f)
```

- Kleisli composition – aka Kleisli "fish" or "komposition"

```
1 // (>=>) : ('T->Task<'U>) -> ('U->Task<'V>)
2 //          -> ('T->Task<'V>)
3
4 let inline (>=>) f g = f >> (bind g)
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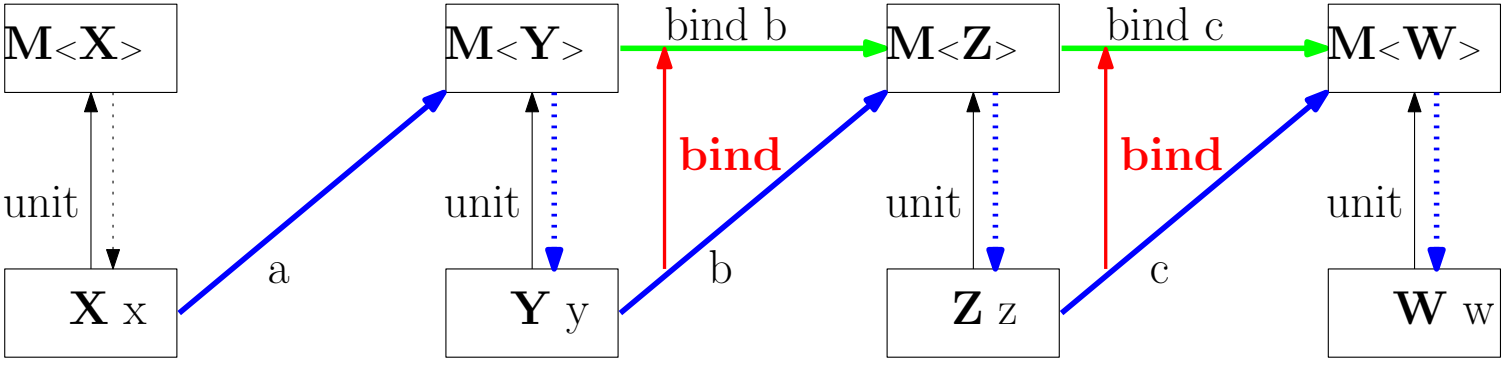



```
1 let k x = x |> (a >=> (b >=> c))
```

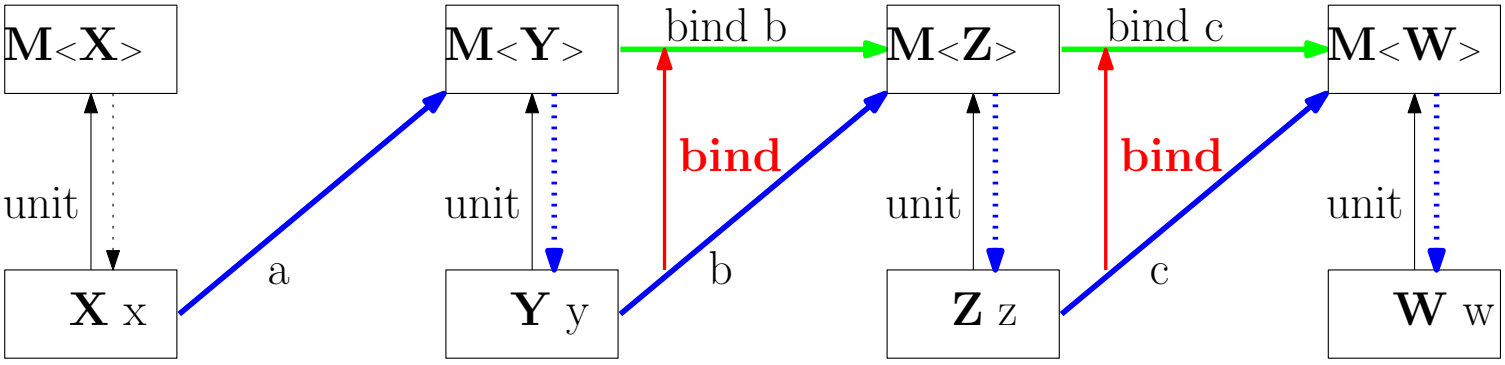
As we will see, (>=>) is associative, so

```
1 let k x = x |> ((a >=> b) >=> c)
2 let k x = x |> (a >=> b >=> c)
```

Hopac "komposition":
https://hopac.github.io/Hopac/Hopac.html

```
1 (>=>): ('x->#Job<'y>) -> ('y->#Job<'z>) -> 'x->Job<'z>
```

Solution: Kleisli Composition!

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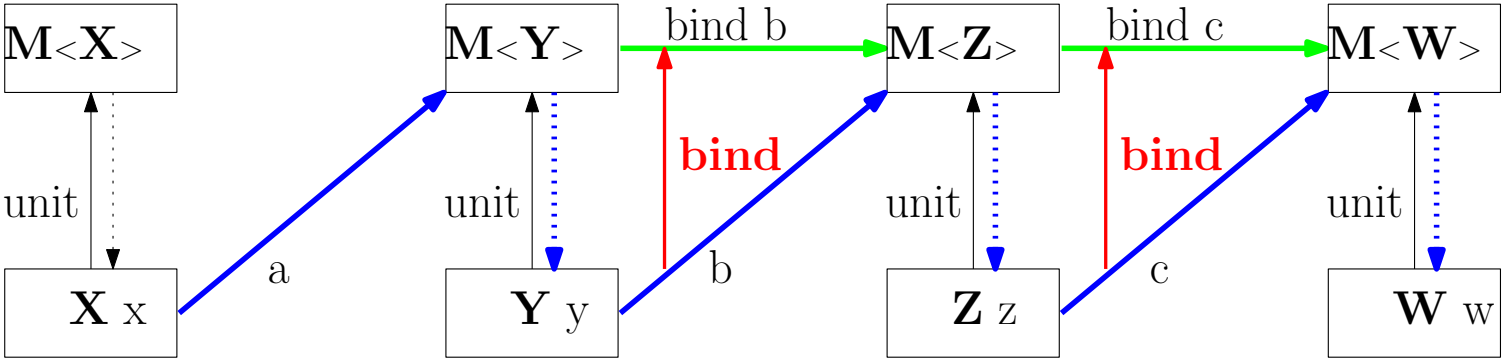


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```
1 (>=>): ('x->#Job<'y>) -> ('y->#Job<'z>) -> 'x->Job<'z>
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Outline

Three Laws for Classical Function Composition

- Left identity – id

```
1 id >> g ≡ g    // 0 + g = g,   1 * g = g
```

- Right identity – id

```
1 f >> id ≡ f     // g + 0 = g,   g * 1 = g
```

- Associativity

```
1 (f >> g) >> h ≡ f >> (g >> h) // (f+g)+h = f+(g+h)
```

- Note: NOT Commutative – in general

```
1 f >> g ≠ g >> f
```

Three Laws for Classical Function Composition

- Left identity – id

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1 f >> g ≢ g >> f
```

Three Monad Laws – w/ w/- Kleisli

- Left identity – unit

```
1 unit >> bind g ≡ g
2
3 unit >=> g ≡ g
```

- Right identity – unit

```
1 bind unit ≡ id
2
3 f >=> unit ≡ f
```

- Associativity

```
1 bind g >> bind h ≡ bind (g >> bind h)
2
3 (f >=> g) >=> h ≡ f >=> (g >=> h)
```

Three Monad Laws – Why ?

- Diagrams that are naturally expected to be commutative
- Anecdotal evidence for simple monads, such as List; recall:

```
1  bind g [u;v] ≡ flat (map g [u;v])
2               ≡ flat ([g u; g v])
```

```
1  let g t = [t+1]
2  map g [3; 7] ≡ [g 3; g 7] ≡ [[4]; [8]]
3  flat (map g [3; 7]) ≡ flat [[4]; [8]] ≡ [4; 8]
4  bind g [3; 7] ≡ [4; 8]
```

- Actual proof in the special case when unit is invertible (later)

```
1  (unit >> unit^-1) t = t
2  (unit^-1 >> unit) m = m
```

- Naturally expected workflow transformations (later)

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2  map g [3; 7] ≡ [g 3; g 7] ≡ [[4]; [8]]
3  flat (map g [3; 7]) ≡ flat [[4]; [8]] ≡ [4; 8]
4  bind g [3; 7] ≡ [4; 8]
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1  let g t = [t+1]
2  map g [3; 7] ≡ [g 3; g 7] ≡ [[4]; [8]]
3  flat (map g [3; 7]) ≡ flat [[4]; [8]] ≡ [4; 8]
4  bind g [3; 7] ≡ [4; 8]
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- Actual proof in the special case when unit is invertible (later)

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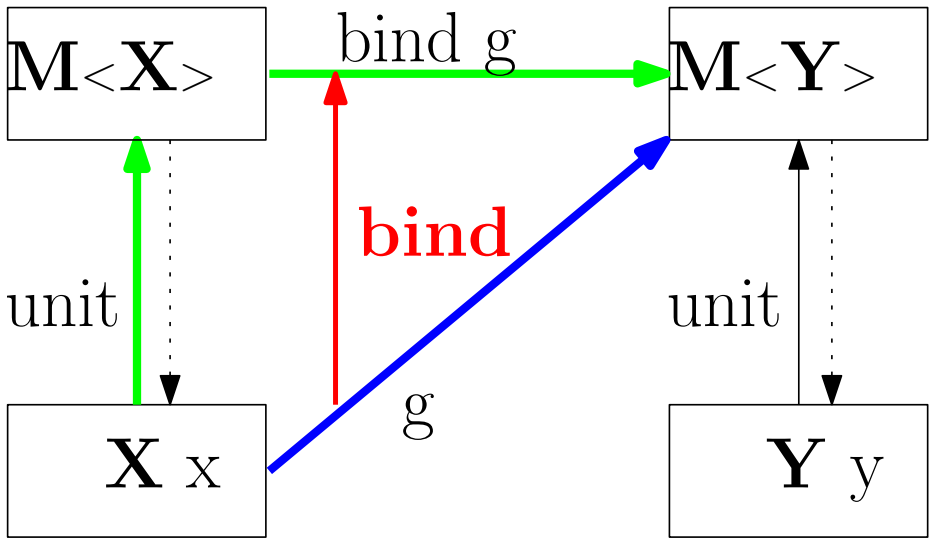
Left identity : unit >> bind g ≡ g



Left identity : unit >> bind g ≡ g

- Consider for example:

```
1 let g t = [t; t+1]
```

- Evaluate left-side on a given t:

```
1 (unit >> bind g) t
2     ≡ bind g (unit t)
3     ≡ bind g [t] ≡ flat (map g [t])
4     ≡ flat [g t] ≡ flat [[t; t+1]]
5     ≡ [t; t+1]
6     ≡ g t
```

- Thus:

```
1 unit >> bind g ≡ g
```

- Cf. definition of (>=>):

```
1 unit >=> g ≡ g
```

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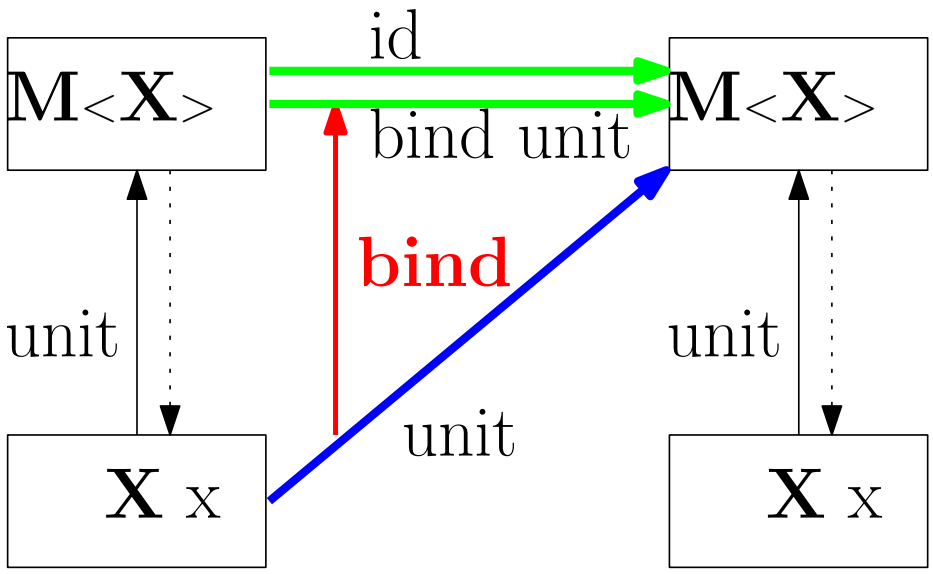
Right identity : bind unit ≡ id



Right identity : bind unit ≡ id

- Evaluate left-side on a given list, e.g. m = [3; 7]:

```
1 (bind unit) m
2     ≡ bind unit [3; 7] ≡ flat (map unit [3; 7])
3     ≡ flat [unit 3; unit 7] ≡ flat [[3]; [7]]
4     ≡ [3; 7]
5     ≡ id m
```

- Thus:

```
1 bind unit ≡ id
```

- Combine with an arbitrary f on the left:

```
1 f >> (bind unit) ≡ f >> id
2 f >=> unit ≡ f
```

Right identity : bind unit ≡ id

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```

- Combine with an arbitrary f on the left:

```
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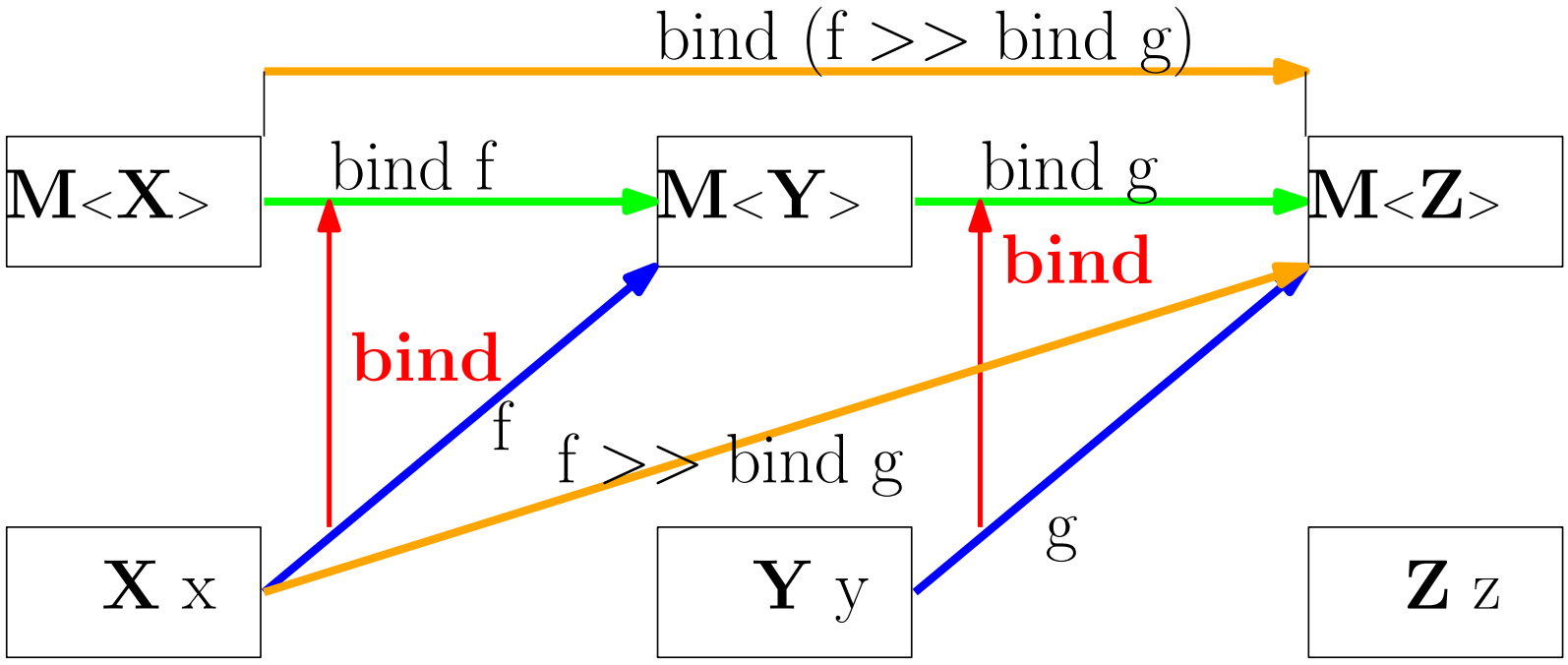
Associativity : bind g >> bind h ≡ bind (g >> bind h)



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- Consider for example:

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1 let g t = [t; t+1]
2 let h t = [t+2]
```

- Evaluate left-side on a given list, e.g. m = [3;7]:

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1 (bind g >> bind h) [3;7]
2     ≡ bind h (bind g [3;7])
3     ≡ bind h (flat [g 3; g 7])
4     ≡ bind h (flat [[3;4]; [7;8]])
5     ≡ bind h [3; 4; 7; 8]
6     ≡ flat [h 3; h 4; h 7; h 8]
7     ≡ flat [[5]; [6]; [9]; [10]]
8     ≡ [5; 6; 9; 10]
```

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- Consider for example:

```
1 let g t = [t; t+1]
2 let h t = [t+2]
```

- Evaluate left-side on a given list, e.g. m = [3;7]:

```
1 (bind g >> bind h) [3;7]
2     ≡ bind h (bind g [3;7])
3     ≡ bind h (flat [g 3; g 7])
4     ≡ bind h (flat [[3;4]; [7;8]])
5     ≡ bind h [3; 4; 7; 8]
6     ≡ flat [h 3; h 4; h 7; h 8]
7     ≡ flat [[5]; [6]; [9]; [10]]
8     ≡ [5; 6; 9; 10]
```

Associativity : bind g >> bind h ≡ bind (g >> bind h)

- Recall:

```
1 let g t = [t; t+1]
2 let h t = [t+2]
```

- Evaluate right-side on a given list, e.g. m = [3;7]:

```
1 bind (g >> bind h) [3;7]
2     ≡ flat [(g >> bind h) 3; (g >> bind h) 7]
3     ≡ flat [bind h (g 3); bind h (g 7)]
4     ≡ flat [bind h [3;4]; bind h [7;8]]
5     ≡ flat [flat [h 3; h 4]; flat [h 7; h 8]]
6     ≡ flat [flat [[5]; [6]]; flat [[9]; [10]]]
7     ≡ flat [[5;6]; [9;10]]
8     ≡ [5; 6; 9; 10]
```

Associativity : bind g >> bind h ≡ bind (g >> bind h)

- Recall:

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1 let g t = [t; t+1]
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5     ≡ flat [flat [h 3; h 4]; flat [h 7; h 8]]
6     ≡ flat [flat [[5]; [6]]; flat [[9]; [10]]]
7     ≡ flat [[5;6]; [9;10]]
8     ≡ [5; 6; 9; 10]
```

Associativity : bind g >> bind h ≡ bind (g >> bind h)

- Thus:

```
1 bind g >> bind h ≡ bind (g >> bind h)
```

- Combine with an arbitrary f on the left:

```
1 f >> (bind g >> bind h) ≡ f >> bind (g >> bind h)
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3 (f >> bind g) >> bind h ≡ f >> bind (g >> bind h)
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5 (f >=> g) >=> h ≡ f >=> (g >=> h)
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5 (f >=> g) >=> h ≡ f >=> (g >=> h)
```

What we get for map?

- Recall:

```
1 let map h = bind (h >> unit)
```

- Left identity:

```
1 unit >> bind g ≡ g
2 unit >> bind (h >> unit) ≡ h >> unit
3 unit >> map h ≡ h >> unit
```

- Right identity:

```
1 bind unit ≡ id
2 bind (id >> unit) ≡ id
3 map id ≡ id
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What we get for map?

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