Part 3: Image Processing
Basics of Mathematical Morphology

Georgy Gimel’farb

COMPSCI 373 Computer Graphics and Image Processing
1 Main concepts
2 Structuring element
3 Erosion
4 Dilation
5 Opening
6 Closing
7 Hit-and-Miss transform
8 Greyscale Morphology
Basic Concepts

**Morphology**: a study of structure or form.  
(https://www.merriam-webster.com/dictionary/morphology)

**Morphological image processing** may remove imperfections of a binary image.

- Regions in binary images produced by simple thresholding are typically distorted by noise.

**Morphological operations** are non-linear and account for structure and forms of regions (objects) to improve an image.

- These operations can be extended to greyscale images.
For morphological processing, the foreground (i.e. object) and background pixels are treated as 1s and 0s, respectively, independently of their visual black-white coding.
Morphological Operations: Several Examples

White objects on black background: “1” / “0” as white / black

Image | Erosion | Dilation | Closing | Opening | Gradient

Black objects on white background: “1” / “0” as black / white
Morphological Operations: Basic Properties

Morphological operations rely on relative ordering of pixel values, not on their numerical values.

• Thus the operations are especially suited to binary image processing.

• These operations can be applied also to greyscale images such that their absolute pixel values are of no or minor interest.
  • E.g. images with unknown light transfer functions.

Morphological operations probe an image with a small shape or template called a **structuring**, or structure **element** (SE).

• SE resembles a convolution kernel in linear filtering.
A small binary image, i.e. a small matrix of pixels, each with a value of zero (0) or one (1).

- **Zero-valued pixels of the SE are ignored.**
- **Size** of the SE: the matrix dimensions.
- **Shape** of the SE: the pattern of ones and zeros.
- **Origin** of the SE: usually, one of its pixels.
  - Generally, the origin can be also outside the matrix.
Structuring Element

When an SE is placed in a binary image, each its pixel is associated with the pixel of the area under the SE:

- The SE **fits** the image if **for each** of its pixels set to 1 the corresponding image pixel is also 1.
- The SE **hits** (intersects) the image if **at least for one** of its pixels set to 1 the corresponding image pixel is also 1.

\[
\begin{array}{c|c|c|c}
\text{fit} & s_1 & s_2 & \text{hit} \\
A & yes & no & s_1 \\
B & yes & yes & yes \\
C & yes & no & s_2 \\
\end{array}
\]
Probing with a Structuring Element (SE)

A: SE fits the foreground (object);
B: SE hits (intersects with) the object;
C: SE neither fits, nor hits the object.

- SE is positioned at all possible locations in an image and compared with the pixel neighbourhood at each location.
- Operations test whether the SE “fits” within the neighbourhood or “hits” the neighbourhood.
- An operation on a binary image creates a new binary image in which the pixel has a foreground value only if the test at that place in the input image is successful.
Probing with a Structuring Element (SE)
Fundamental Operation: Erosion

Erosion $f \ominus s$ of a binary image $f$ by an SE $s$ produces a new binary image $g = f \ominus s$.

**Eroded image** $g$ has ones in all locations $(x, y)$ of an origin of the SE $s$ at which $s$ **fits** the input image $f$.

For all pixel coordinates $(x, y)$, $g(x, y) = 1$ if $s$ fits $f$ and 0 otherwise.

Erosion of a Binary Image with $2 \times 2$ and $3 \times 3$ SE

- Erosion with small (e.g. $2 \times 2 - 5 \times 5$) square SEs shrinks an image by stripping away a layer of pixels from both inner and outer region boundaries.
- Holes in and gaps between foreground objects become larger, and small details are eliminated.

www.cs.princeton.edu/ pshilane/mosaic/  
www.inf.u-szeged.hu/ ssip/1996/morpho/morphology.html  
Erosion of a Binary Image

Larger SEs have a more pronounced effect.

- Erosion with a large SE is similar to an iterated erosion with a smaller SE of the same shape. If a pair of SEs $s_1$ and $s_2$ are identical in shape, with $s_2$ twice the size of $s_1$, then $f \ominus s_2 \approx (f \ominus s_1) \ominus s_1$.

Erosion removes fine (small-scale) details, as well as noise from a binary image.

- Simultaneously, erosion reduces the size of regions of interest (objects), too.
- Background areas are growing, i.e. an image with the black or white background becomes blacker or whiter, respectively.
Boundary Detection by Erosion

**Internal gradient** $b = f - (f \ominus s)$ of each region:

by subtracting the eroded image from an original image.

Gradient is computed with the $3 \times 3$ square SE:

$$
\begin{array}{c}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
$$

**Binary image** $f$

**Eroded image** $f \ominus s$

**Boundary** $b = f - (f \ominus s)$
Boundary Detection by Erosion

Internal gradient of each region:

\[ f - (f \ominus s) \]

Fundamental Operation: Dilation

Dilation $f \oplus s$ of a binary image $f$ by an SE $s$ produces a new binary image $g = f \oplus s$.

Dilated image $g$ has ones in all locations $(x, y)$ of an origin of the SE $s$ at which $s$ hits the input image $f$.

For all pixel coordinates $(x, y)$, $g(x, y) = 1$ if $s$ hits $f$ and 0 otherwise.

Dilation of a Binary Image with $2 \times 2$ and $3 \times 3$ SE

- Dilation is opposite to erosion: it adds a layer of pixels to the inner and outer boundaries of regions.
- Holes enclosed by a single region and gaps between different regions become smaller; small intrusions into boundaries of a region are filled in.

www.cs.princeton.edu/ pshilane/mosaic/
www.inf.u-szeged.hu/ ssip/1996/morpho/morphology.html
Dilation and Erosion: Duality

Dilation and erosion are **dual operations** in that they have opposite effects: \( f \oplus s = f^c \ominus s_{\text{rot}} \).

- \( f^c \) – the complement of \( f \) produced by replacing “1”s / “0”s with “0”s / “1”s.

- \( s_{\text{rot}} \) – the SE \( s \) rotated by 180°.

  - If a SE is symmetric with respect to rotation, then \( s_{\text{rot}} \) does not differ from \( s \).

If a binary image is considered as a collection of connected regions of “1”s on background of “0”s:

- **Erosion** is the fitting of the SE to these regions.

- **Dilation** is the fitting of the SE, rotated if necessary, into the background, followed by inversion of the result.
External gradient $b = (f \oplus s) - f$ of each region:

by subtracting an original image from the dilated image.

Gradient is computed with the $3 \times 3$ square SE:

$$1\ 1\ 1$$

$$1\ 1\ 1$$

$$1\ 1\ 1$$
Morphological Gradient $= \text{Dilation} - \text{Erosion}$

$$\nabla f = (f \oplus s) - (f \ominus s) \equiv [(f \oplus s) - f] + [f - (f \ominus s)]$$

Dilation  Erosion  External gradient  Internal gradient

Gradient
Set-theoretic Binary Operations

- Many morphological operations are combinations of erosion, dilation, and simple set-theoretic operations.
- Set-theoretic *complement* of a binary image:

\[
f^c(x, y) = \begin{cases} 
1 & \text{if } f(x, y) = 0 \\
0 & \text{if } f(x, y) = 1 
\end{cases}
\]
Set-theoretic Binary Operations: Intersection

**Intersection** $h = f \cap g$ of two binary images $f$ and $g$:

$$h(x, y) = \begin{cases} 
1 & \text{if } f(x, y) = 1 \text{ AND } g(x, y) = 1 \\
0 & \text{otherwise}
\end{cases}$$
Set-theoretic Binary Operations: Union

**Union** \( h = f \cup g \) of two binary images \( f \) and \( g \):

\[
h(x, y) = \begin{cases} 
1 & \text{if } f(x, y) = 1 \text{ OR } g(x, y) = 1 \\
0 & \text{otherwise}
\end{cases}
\]
Opening $f \circ s$ of an image $f$ by a structuring element $s$ is an erosion followed by a dilation:

$$f \circ s = (f \ominus s) \oplus s$$

Opening of a Binary Image with Square $2 \times 2$ and $3 \times 3$ SE

Opening is so called because it can open up a gap between objects connected by a thin bridge of pixels.

www.cs.princeton.edu/ pshilane/mosaic/
www.inf.u-szeged.hu/ ssip/1996/morpho/morphology.html
Opening a Binary Image with a Square $3 \times 3$ SE

Initial image $f$:
white “1”s and black “0”s

SE $s$

Erosion $f \ominus s$

Opening $f \circ s = (f \ominus s) \oplus s$
Opening a Binary Image

Any regions survived the erosion are restored to their original size by the dilation:

Idempotent operation: \((f \circ s) \circ s = f \circ s\)

- Once an image is opened, next openings with the same structuring element have no further effect.
Opening a Binary Image with Square $5 \times 5$ and $9 \times 9$ SE

Binary image

Opening with a $5 \times 5$ SE

Opening with a $9 \times 9$ SE

Closing $f \bullet s$ of an image $f$ by a structuring element $s$ is a **dilation** followed by an **erosion**:

$$f \bullet s = (f \oplus s) \ominus s$$

Closing of a Binary Image with Square $2 \times 2$ and $3 \times 3$ SE

Closing is so called because it can fill holes in the regions while keeping the initial region sizes.

References:

www.cs.princeton.edu/ pshilane/mosaic/
www.inf.u-szeged.hu/ ssip/1996/morpho/morphology.html
Closing a Binary Image with a Square $3 \times 3$ SE

Initial image $f$:
white “1”s and black “0”s

$SE\; s$

Dilation $f \oplus s$

Closing $f \circ s = (f \oplus s) \ominus s$

Closing a Binary Image

Dilation and erosion with a rotated by $180^\circ$ SE.

- Typical symmetric SE: the rotated and initial SE do not differ.

Idempotent operation: $(f \ast s) \ast s = f \ast s$

- Once an image is closed, next closings with the same structuring element have no further effect.

Closing Vs. Opening

Closing is the dual operation of opening.

- Just as opening is the dual operation of closing.

**Closing** of a binary image (*dual implementation*):
  - Take the complement of that image ("1 / 0" $\Rightarrow$ "0 / 1").
  - Perform opening with the structuring element.
  - Take the complement of the result.

**Opening** of a binary image (*dual implementation*):
  - Take the complement of that image ("1 / 0" $\Rightarrow$ "0 / 1").
  - Perform closing with the structuring element.
  - Take the complement of the result.
Closing Vs. Opening

Closing with a square or disk SE:
- Fills thin connections within an object.
- Eliminates small holes and fills dents in contours.
- Fills small gaps in parts of an object.

Opening with a square or disk SE:
- Breaks thin connections within an object.
- Eliminates small islands and sharp protrusions.
Binary Image: Closing + Opening

Initial image $f$

Closed, then opened image $g = (f \bullet s) \circ s$

Binary Image: Opening + Closing

Initial image $f$

Opened, then closed image

$g = (f \circ s) \bullet s$

Opening + Closing vs. Closing + Opening

- erode - dilate - dilate - erode (open)
- close

- dilate - erode - erode - dilate (close)
- open

Opened

Opened + Closed

Closed

Closed + Opened
Hit-and-Miss Transform

The hit-and-miss transform tests how objects in a binary image relate to their surroundings.

Matched pair of SEs, \( \{s_1, s_2\} \), probes the inside and outside, respectively, of objects in the image:

\[
f \odot \{s_1, s_2\} = (f \ominus s_1) \cap (f^c \oplus s_2)
\]

Hit and miss transform with an elongated \( 2 \times 5 \) prototype formed by the two SEs.

Hit-and-Miss Transform

Preserves a pixel of a region if and only if $s_1$ for that pixel fits inside the region AND $s_2$ for that pixel fits outside the region.

- Structuring elements $s_1$ and $s_2$ do not intersect, otherwise it would be impossible for both fits to occur simultaneously.
Hit-and-Miss Transform

Easier description: by considering $s_1$ and $s_2$ as a single element with “1”s for pixels of $s_1$ and “0”s for pixels of $s_2$:

0 0 0 0 0 0 0 0 0 0
0 1 1 1 1 1 1 1 1 0
0 0 0 0 0 0 0 0 0 0

- An output pixel is set to 1 only if the 1s and 0s in the SE exactly match the 1s and 0s, respectively, in the input image.
- Otherwise, that pixel is set to 0.

If the two SEs together present a specific shape (spatial pattern of foreground 1s and background 0s), the transform can detect the desired shapes.

- The transform can be used for thinning or thickening of linear elements of objects.
Morphological Filtering

Compound operations (e.g. opening and closing) act as non-linear filters of shape in a binary image.

- Opening and closing with a disc SE smooth corners from the inside and the outside, respectively.
- Details smaller in size than the disc are also filtered out.
  - Opening is filtering at a scale of the size of the SE.
  - Only those portions of the image that fit the SE are passed by the filter.
  - Smaller structures are blocked and excluded.

The size of the SE is most important in order to eliminate noisy details but not to damage objects.

- If the SE is too large, the object could be degraded by the operation.
Greyscale Morphology

Definitions of operations:

Similar to the binary case but with the additional dimension of pixel grey level.

An image is considered as a 3D surface \( f(x, y) \).

- The height at any point represents the non-negative integer grey level \( f \) at that point.
- The structuring element \( s(\xi, \eta) \) is also 3D surface, but its pixels take any integer value \( s \), including zero and negative values.
Greyscale Morphology

\[ f(x, y) \quad \text{and} \quad s(\xi, \eta) \]

http://www.cs.wits.ac.za/ michael/Lab5.html
The SE is sometimes referred in the greyscale morphology to as a **structuring function**.

Zero value is now significant: pixels that do not participate in morphological operations have to be indicated by some other means.
Greyscale erosion \( f \ominus s \) of an image \( f \) by the SE \( s \):

Replacing the grey level in the pixel aligned with the origin of \( s \) with the minimum difference between each pixel grey level in \( f \) and the corresponding value in \( s \) over the domain \( S_s \) of the SE:

\[
f_{\text{ero}}(x, y) = \min_{(\xi, \eta) \in S_s} \{f(x + \xi, y + \eta) - s(\xi, \eta)\}
\]
Greyscale Erosion

Greyscale erosion \( f_{\text{ero}} = f \ominus s \) of an image \( f \) by the SE \( s \):

\[
f_{\text{ero}}(x, y) = \min_{(\xi, \eta) \in S_s} \{ f(x + \xi, y + \eta) - s(\xi, \eta) \}
\]

where \((\xi, \eta)\) — 2D indices of a pixel in the SE \( s \), i.e. pixel coordinates with respect to the origin of the SE \( s \).

- For a \((2k + 1) \times (2k + 1)\) square SE \( s \), the indices \( \xi, \eta \in [-k, -k + 1, \ldots, 0, 1, \ldots, k] \).

\[
\begin{align*}
\begin{array}{cccc}
220 & 210 & 120 & 45 & 50 \\
225 & 200 & 130 & 67 & 53 \\
202 & 199 & 100 & 73 & 45 \\
189 & 190 & 110 & 68 & 49 \\
190 & 200 & 134 & 71 & 57 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
-1 & -9 & -1 \\
11 & 11 & 11 \\
-1 & -9 & -1 \\
\end{array}
\end{align*}
\]

\[
\min \begin{Bmatrix}
220 + 1, 210 + 9, 120 + 1, \\
225 - 11, 200 - 11, 130 - 11, \\
202 + 1, 199 + 9, 100 + 1
\end{Bmatrix}
\]

\[
= \min \{221, 219, 121, 214, 189, 119, 203, 208, 101\} = 101
\]
Greyscale Erosion

Flat structuring element: \( s(\xi, \eta) = 0 \) for all \( (\xi, \eta) \in S_s \).

- Erosion with the flat SE: \( f_{\text{ero}}(x, y) = \min_{(\xi, \eta) \in S_s} \{ f(x + \xi, y + \eta) \} \)

- The same effect as the **minimum rank**, or **minimum filter**:

  Greyscale image \[\xrightarrow{\text{Eroded image}}\] Eroded image (3 × 3 minimum filter)

www.inf.u-szeged.hu/ ssip/1996/morpho/morphology.html
Flat SE: The Minimum Filter

Selects the minimum (bottom-ranked) grey level from the neighbourhood as the output value.

Greyscale image → Eroded image (3 × 3 minimum filter)

www.inf.u-szeged.hu/SSIP/1996/morpho/morphology.html
Greyscale dilation $f \oplus s$ of an image $f$ by the SE $s$:

Replacing the grey level in the pixel aligned with the origin of $s$ with the maximum sum of each pair of the pixel grey level in $f$ and the corresponding value in $s$ over the domain $S_s$ of the SE:

$$f_{\text{dil}}(x, y) = \max_{(\xi, \eta) \in S_s} \left\{ f(x - \xi, y - \eta) + s(\xi, \eta) \right\}$$
Greyscale Dilation

Greyscale dilation $f_{dil} = f \oplus s$ of an image $f$ by the SE $s$:

$$f_{dil}(x, y) = \max_{(\xi, \eta) \in S_s} \{ f(x - \xi, y - \eta) + s(\xi, \eta) \}$$

where $(\xi, \eta)$ — 2D indices of a pixel in the SE $s$.

- Greyscale dilation is a dual operation with respect to erosion.

\[
\begin{align*}
\text{max} \left\{ \begin{array}{c}
220-1, 210-9, 120-1, \\
225+11, 200+11, 130+11, \\
202-1, 199-9, 100-1
\end{array} \right\} &= \max \{219, 201, 119, 236, 211, \\
141, 201, 190, 99\} = 236
\end{align*}
\]
Greyscale Dilation

Flat structuring element: \( s(\xi, \eta) = 0 \) for all \((\xi, \eta) \in \mathbb{S}_s\).

- Dilation with the flat SE: 
  \[
  f_{\text{dil}}(x, y) = \max_{(\xi, \eta) \in \mathbb{S}_s} \{ f(x - \xi, y - \eta) \}
  \]

- The same effect as the maximum rank, or maximum filter:

  Greyscale image → Dilated image (3 × 3 maximum filter)

www.inf.u-szeged.hu//ssip/1996/morpho/morphology.html
Flat SE: The Maximum Filter

Selects the maximum (top-ranked) grey level from the neighbourhood as the output value.

![Greyscale image](left)  ➔  ![Dilated image](right) (3 × 3 maximum filter)

www.inf.u-szeged.hu/ssip/1996/morpho/morphology.html
Opening and closing for greyscale images are defined just as those for binary images:

**Opening** \( f \circ s = (f \ominus s) \oplus s \)

**Closing** \( f \bullet s = (f \oplus s) \ominus s \)

With an appropriate structuring element, these operations can smooth the image.

- However, it is the surface of grey levels that is smoothed, rather than the contours of shapes in a binary image.
Greyscale Opening / Closing

- Opening tends to smooth away small-scale **bright** details in an image.
- Closing tends to smooth away the small-scale **dark** details in an image.

Greyscale Opening

Greyscale image → Opened image (3 × 3 square flat filter)

Smoothing out bright small-scale noisy details of the initial image.

www.inf.u-szeged.hu/SSIP/1996/morpho/morphology.html
Erosion - Opening / Dilation - Closing

The pairs differ like the corresponding binary operations.

- **Erosion** shrinks bright features and enlarges dark features.
- **Opening** removes small bright features, but does not enlarge dark ones.
- Similar considerations apply to **dilation** and **closing**.

Greyscale image

3 × 3 square flat SE

Morphological smoothing: an opening-closing iteration.
- Removes small-scale bright and dark details.
- Resembles extreme forms of the median filter.

Top-hat transform: \( g = f - (f \circ s) \).
- Opening \( f \circ s \) removes small-scale bright details.
- Top-hat transform selects only these details.
- Therefore the top-hat transform acts as a detector of peaks and ridges of the grey level surface.
The dual of the top-hat transform, \((f \bullet s) - f\), detects pits and valleys in the grey level surface.
Top-hat Transform

Correcting uneven illumination if the background is dark with a 12 pixel wide disk-shaped SE: