Example: Taylor Approximations of $f(x) = \exp(x) \equiv \mathrm{e}^x$

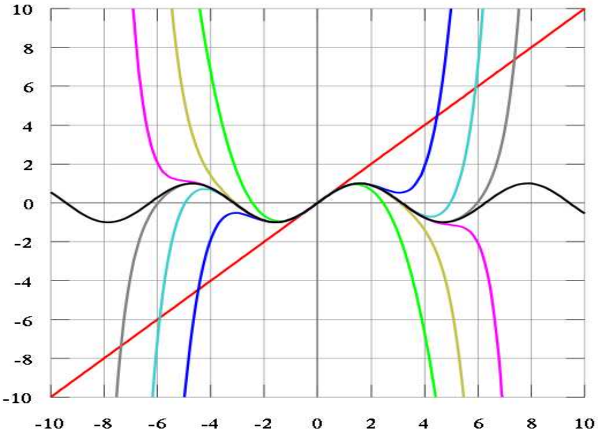
At $a = 0$, all the derivatives $\mathrm{f}^{(k)}(a) \equiv \left.\frac{d^k}{dx^k}\mathrm{e}^x\right|_{x=0} = \mathrm{e}^x|_{x=0} = 1$ so the Taylor series is

$$\widehat{f}_n(x;0) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Table below shows quality of the approximation for different values of n and x.

n	1	2	3	4	...	true value e^x
$\widehat{f}_n(x=1;0)$	2.0000	2.5000	2.6667	2.7083	...	2.7183
Relative error	0.26	0.08	0.019	0.0037		
$\widehat{f}_n(x=2;0)$	3.0000	5.0000	6.3333	7.0000	...	7.3891
Relative error	0.59	0.32	0.14	0.053		

Example: Taylor Approximations of $f(x) = \sin x$



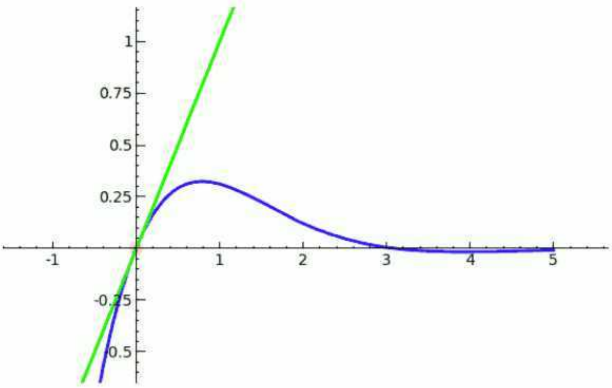
$f(x) = \sin x$ and Taylor approximations $\widehat{f}_n(x; 0)$ for $n = 1, 3, 5, 7, 9, 11$, and 13

As the degree n rises, the Taylor polynomial approaches the correct function

Taylor Series Example (by H. Schilly)

order 1

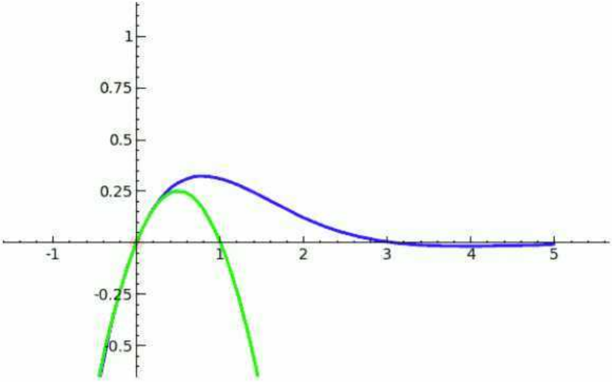
$$f(x) = e^{-x} \sin(x)$$
$$\hat{f}(x;0) = x + \mathcal{O}(x^2)$$



Taylor Series Example (by H. Schilly)

order 2

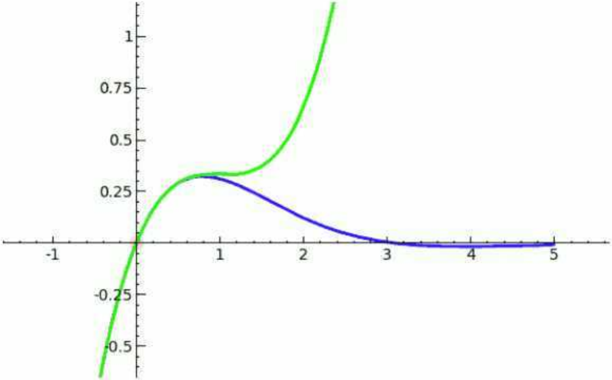
$$f(x) = e^{-x} \sin(x)$$
$$\hat{f}(x;0) = x - x^2 + \mathcal{O}(x^3)$$



Taylor Series Example (by H. Schilly)

order 3

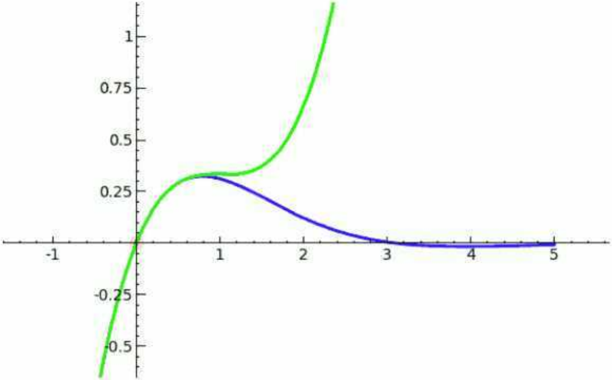
$$f(x) = e^{-x}\sin(x)$$
$$\hat{f}(x;0) = x - x^2 + \frac{x^3}{3} + \mathcal{O}(x^4)$$



Taylor Series Example (by H. Schilly)

order 4

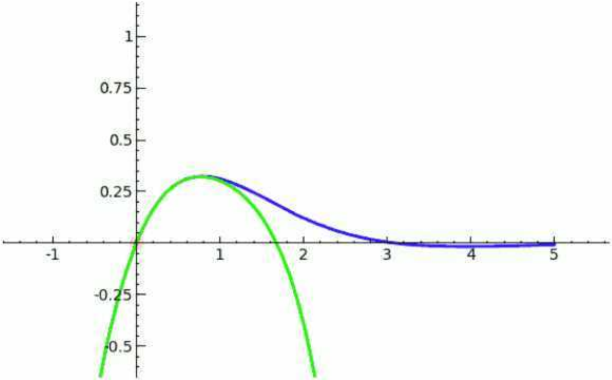
$$f(x) = e^{-x}\sin(x)$$
$$\hat{f}(x;0) = x - x^2 + \frac{x^3}{3} + \mathcal{O}(x^5)$$



Taylor Series Example (by H. Schilly)

order 5

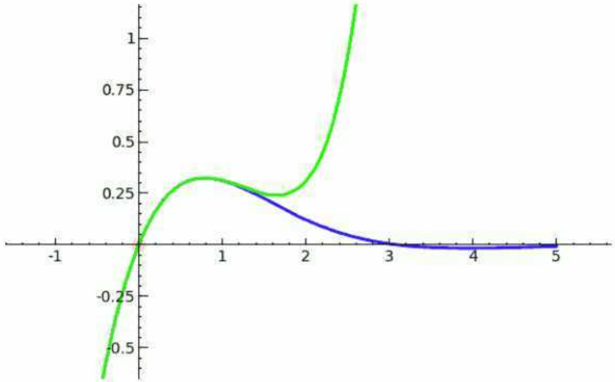
$$f(x) = e^{-x}\sin(x)$$
$$\hat{f}(x;0) = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \mathcal{O}(x^6)$$



Taylor Series Example (by H. Schilly)

order 6

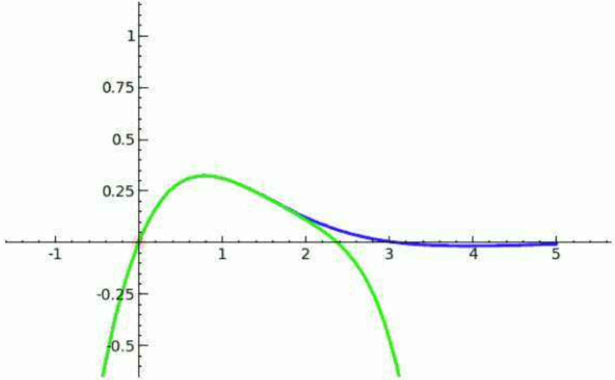
$$f(x) = e^{-x}\sin(x)$$
$$\hat{f}(x;0) = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^6}{90} + \mathcal{O}(x^7)$$



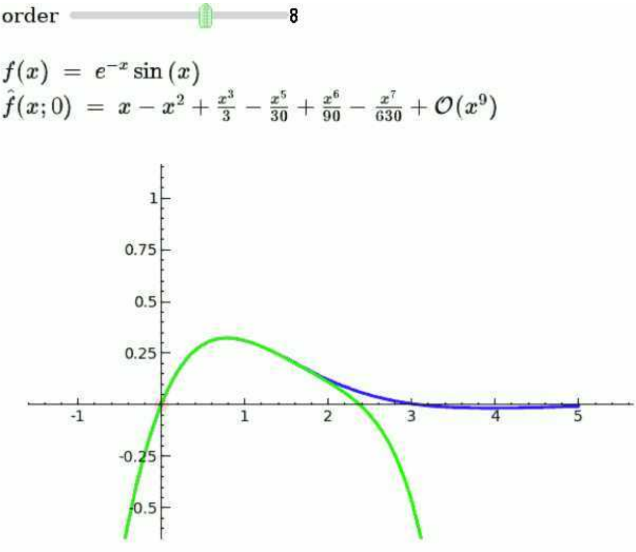
Taylor Series Example (by H. Schilly)

order 7

$$f(x) = e^{-x}\sin(x)$$
$$\hat{f}(x;0) = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^6}{90} - \frac{x^7}{630} + \mathcal{O}(x^8)$$



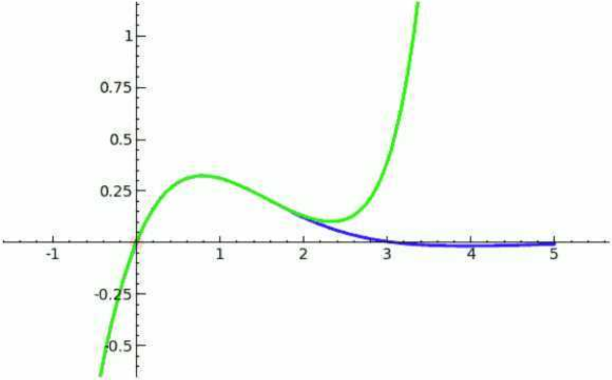
Taylor Series Example (by H. Schilly)



Taylor Series Example (by H. Schilly)

order 9

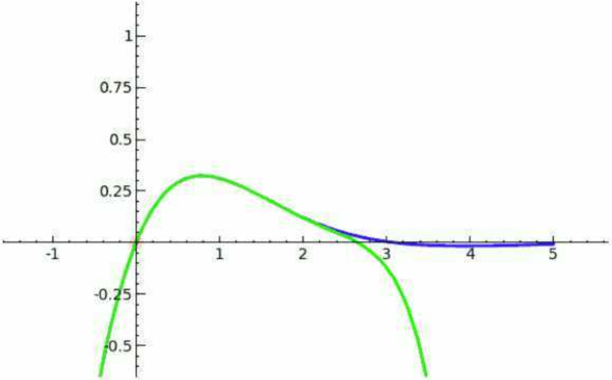
$$f(x) = e^{-x}\sin(x)$$
$$\hat{f}(x;0) = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^6}{90} - \frac{x^7}{630} + \frac{x^9}{22680} + \mathcal{O}(x^{10})$$



Taylor Series Example (by H. Schilly)

order 10

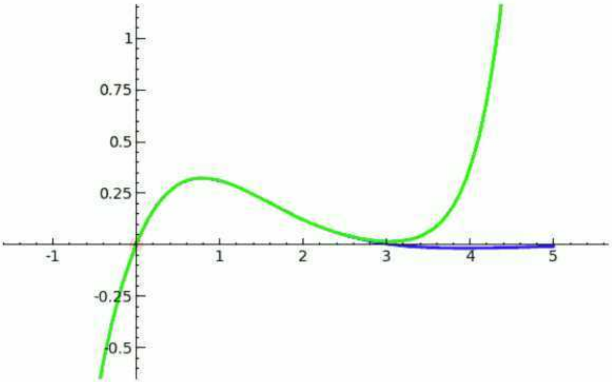
$$f(x) = e^{-x}\sin(x)$$
$$\hat{f}(x;0) = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^6}{90} - \frac{x^7}{630} + \frac{x^9}{22680} - \frac{x^{10}}{113400} + \mathcal{O}(x^{11})$$



Taylor Series Example (by H. Schilly)

order 11

$$f(x) = e^{-x}\sin(x)$$
$$\hat{f}(x;0) = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^6}{90} - \frac{x^7}{630} + \frac{x^9}{22680} - \frac{x^{10}}{113400} + \frac{x^{11}}{1247400} + \mathcal{O}(x^{12})$$

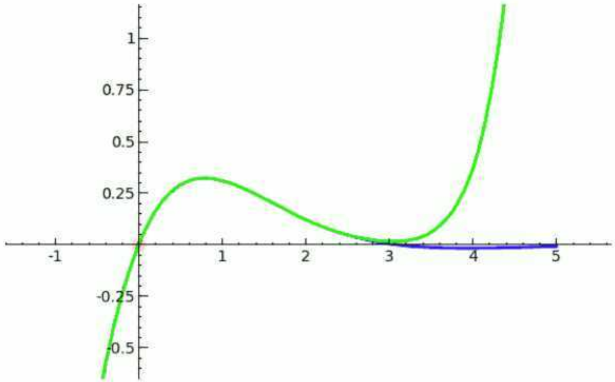


Taylor Series Example (by H. Schilly)

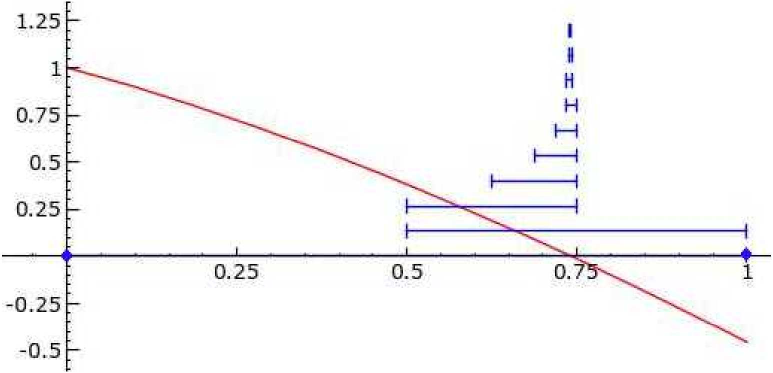
order 12

$$f(x) = e^{-x}\sin(x)$$

$$\hat{f}(x;0) = x - x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^6}{90} - \frac{x^7}{630} + \frac{x^9}{22680} - \frac{x^{10}}{113400} + \frac{x^{11}}{1247400} + \mathcal{O}(x^{13})$$



Example: Bisection Method on $f(x) = \cos(x) - x$



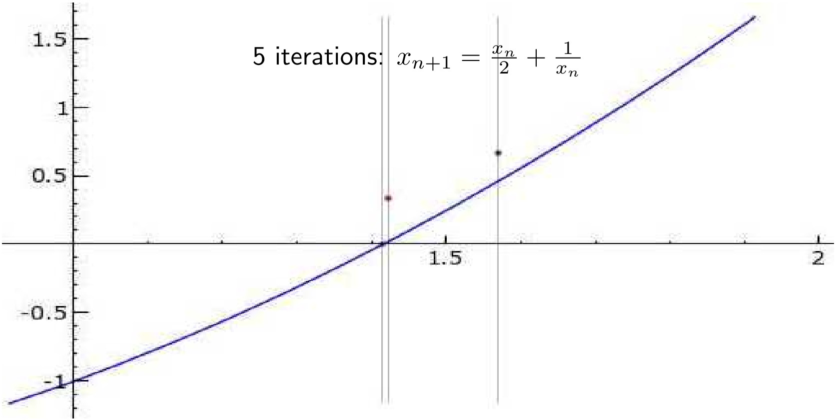
Initial guesses: $a = 0$ and $b = 1$.

$$f(a) = 1 \text{ and } f(b) = \cos(1) - 1 = -0.4597$$

After 10 iterations:

root $c_9 = 0.7392578$; $f(0.7392578) = -0.0002890091\ldots$

Example: Newton's Method on $f(x) = x^2 - 2$



x_n	0.5	$\to$ 2.25	$\to$ 1.569444	$\to$ 1.421890	$\to$ 1.414234
$f(x_n)$	-1.75	3.065	0.463154	0.0217711	0.00005861552

True root value: $x_{\text{root}} = 1.41421356237\ldots$

Newton's Method Vs. Bisection on $f(x) = x^2 - 2$

Step n	1	2	3	4	5
a_n	0.5	1.25	1.25	1.25	1.34375
$f(a_n)$	-1.75	-0.4375	-0.4375	-0.4375	-1943360
b_n	2.0	2.0	1.625	1.4375	1.4375
$f(b_n)$	2.0	2.0	0.640625	0.06640625	0.06640625
$\frac{a_n+b_n}{2}$	1.25	1.625	1.4375	1.34375	1.390625

- After 5 steps of bisection, $a = 1.34375$ and $b = 1.4375]$ so best guess at root is 1.390625.
  - Absolute error: 0.0268834562.
- After 5 steps of Newton, best guess at root is 1.414234
  - Absolute error: $0.0002043762\ldots$ (two orders smaller!)