

SVD: an Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \mathbf{A}\mathbf{A}^T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 3; \mathbf{u}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}; \quad \lambda_2 = 1; \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Top $n=2$ eigenvalues and eigenvectors for $\mathbf{A}\mathbf{A}^T$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 3; \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = 1; \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigenvalues and eigenvectors for $\mathbf{A}^T\mathbf{A}$

SVD: an Example (cont.)

Singular values: $\mathbf{A}\mathbf{v}_j = \sigma_j\mathbf{u}_j; j = 1, 2$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sigma_1 \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \sigma_1 = \sqrt{3}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \sigma_2 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \sigma_2 = 1$$

$$\begin{aligned} \mathbf{A} &\equiv \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}}_{\mathbf{U}} \underbrace{\text{diag}(\sigma_1, \sigma_2)}_{\mathbf{D}} \underbrace{\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix}}_{\mathbf{V}^T} \\ &\equiv \underbrace{\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{\mathbf{V}^T} \end{aligned}$$

SVD: an Example (cont.)

Matrix representation:

$$\begin{aligned}\hat{\mathbf{A}}_1 &= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T \\ &\equiv \sqrt{3} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} [1 \ 1] \equiv \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [1 \ 1] \equiv \begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \\ 0.5 & 0.5 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{A}}_2 &= \hat{\mathbf{A}}_1 + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T \\ &\equiv \begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \\ 0.5 & 0.5 \end{bmatrix} + 1 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} [-1 \ 1] \\ &\equiv \begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \\ 0.5 & 0.5 \end{bmatrix} + \begin{bmatrix} -0.5 & 0.5 \\ 0 & 0 \\ 0.5 & -0.5 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}}\end{aligned}$$