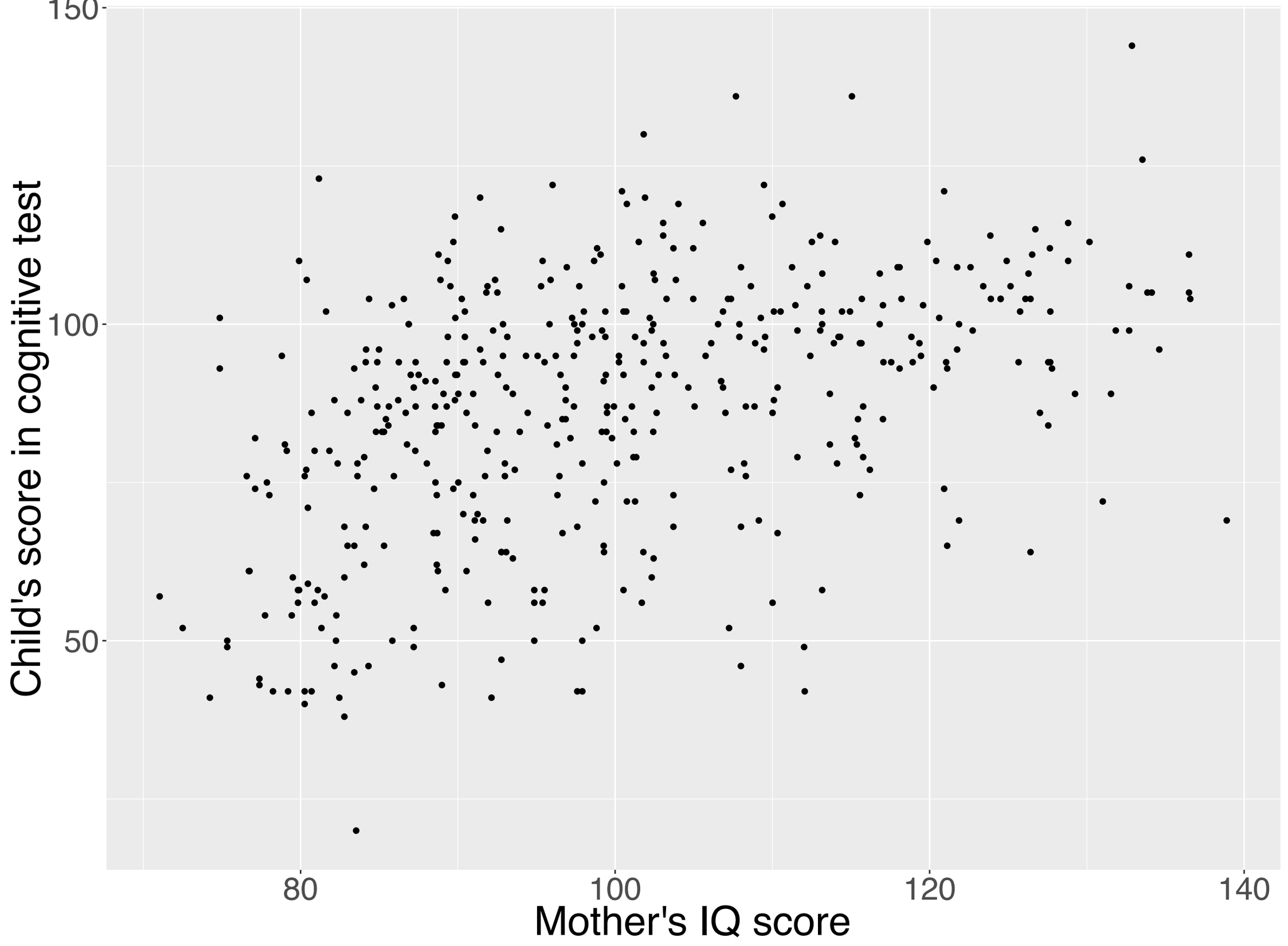
Child's score in cognitive test
150
100
50
80
100
120
140
Mother's IQ score


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Toy Example: Linear regression in statistics

Find the straight line that best fits given points (x, b_x): (0,1), (1,9), (3, 9) and (4,21)

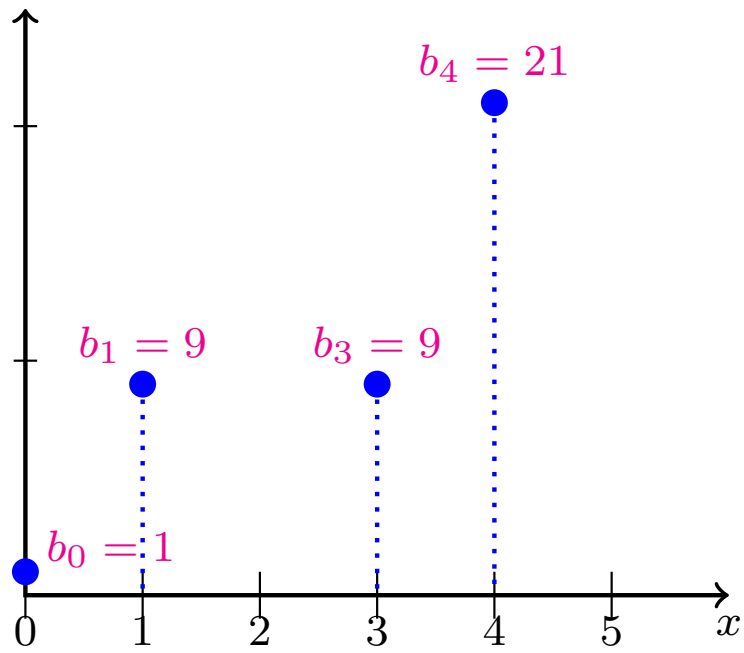
The equation for a straight line is

$$b_x = u_1 + u_2 x$$

so have 4 equations and two unknowns, u_1 and u_2.

$$\mathbf{Ax} = \mathbf{u} \Leftrightarrow \begin{array}{rcl} u_1 + u_2 \cdot 0 & = & 1 \\ u_1 + u_2 \cdot 1 & = & 9 \\ u_1 + u_2 \cdot 2 & = & 9 \\ u_1 + u_2 \cdot 3 & = & 21 \end{array} \Leftrightarrow$$

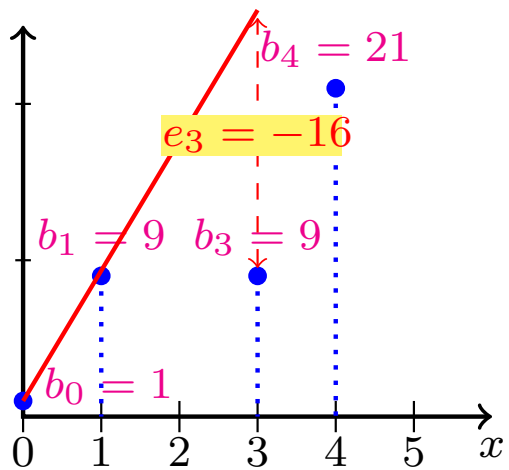
$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 21 \end{bmatrix}$$



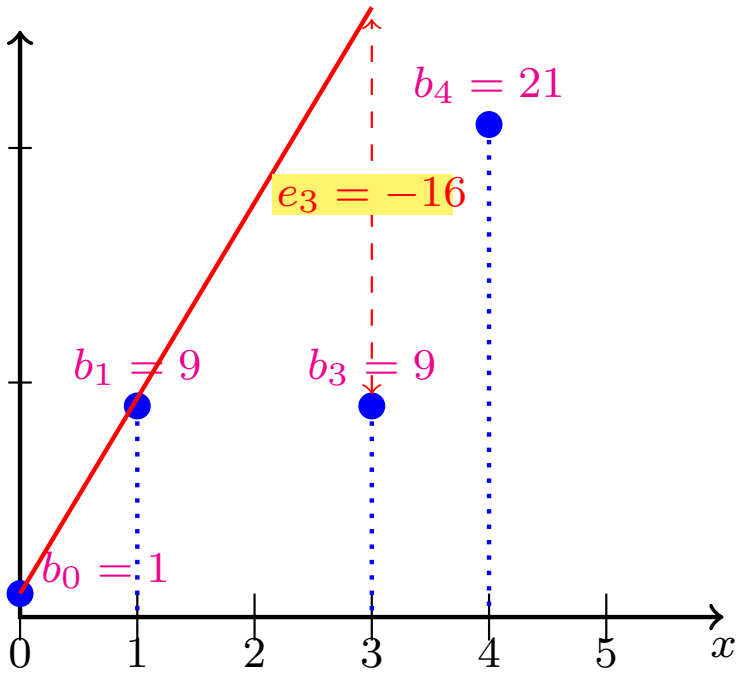
No line fits all the points since $\mathbf{b}$ is not a linear combination of the two column vectors from $\mathbf{A}$:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 9 \\ 9 \\ 21 \end{bmatrix}$$

Try candidate line through first two points, $1 + 8x = b_x$, i.e., $u_1 = 1, u_2 = 8$.



- Let $e_x = b_x - (1 + 8x)$ between the observed point and position of line at x



$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 1 \\ 9 \\ 9 \\ 21 \end{bmatrix}}_{\mathbf{b}} \Rightarrow \underbrace{\mathbf{e} = \mathbf{b} - \mathbf{A}\mathbf{u}}_{\text{residual error}}$$

Total squared error $E(\mathbf{u}) = \mathbf{e}^{\mathsf{T}}\mathbf{e} \equiv \| \mathbf{e} \|^2$

$$= (\mathbf{b} - \mathbf{A}\mathbf{u})^{\mathsf{T}}(\mathbf{b} - \mathbf{A}\mathbf{u})$$

For this line squared error is $E(\mathbf{u}) = 0 + 0 + 256 + 144 = \mathbf{400}$.

Finding best fit line

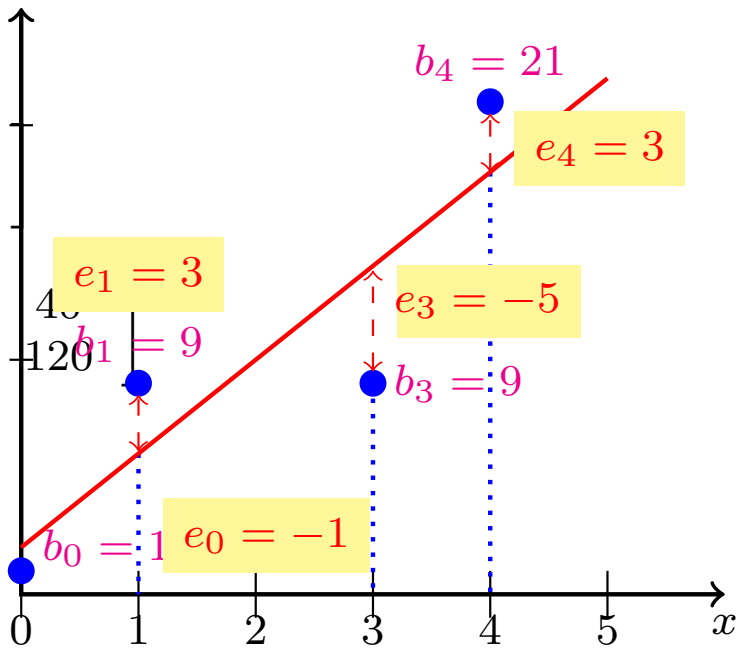
The normal equation $\mathbf{A}^\mathsf{T}\mathbf{A}\mathbf{u}^* = \mathbf{A}^\mathsf{T}\mathbf{b}$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \\ 9 \\ 21 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} = \begin{bmatrix} 40 \\ 120 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} 40 \\ 120 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



Error for the best line: $E(\mathbf{u}^*) = 1 + 9 + 25 + 9 = 44$