THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2014 — MID-SEMESTER TEST
Campus: City

COMPUTER SCIENCE

Computational Science

(Time allowed: 50 minutes)

NOTE: Attempt *all* questions. There are 51 Marks in total.

Use of calculators is NOT permitted.

Put your answers in the answer boxes provided below each question.

SURNAME:

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1. Consider the bisection method for finding the roots of $f(x) = x^2 + \frac{x}{2} - 7$. [5 marks total]

   (a) Check that $a = 2$ and $b = 6$ are valid starting points for the algorithm. [**3** marks]

   (b) What are the values of a and b after one iteration of the bisection method? [**2** marks]

2. Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is an $n \times n$ matrix. [7 marks total]

   (a) The system is considered easy to solve when $\mathbf{A}$ is diagonal, lower triangular or orthonormal. Define the terms diagonal, lower triangular and orthonormal in this context. [**3** marks]

   (b) Suppose $\mathbf{A}$ is diagonal. What simple test determines whether a solution to $\mathbf{Ax} = \mathbf{b}$ exists? [**1** mark]

   (c) Suppose $\mathbf{A}$ is orthonormal. What is the solution to $\mathbf{Ax} = \mathbf{b}$? [**2** marks]

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(d) Name a method that can coerce the equation $\mathbf{Ax} = \mathbf{b}$ into the form $\mathbf{Cx} = \mathbf{d}$ where $\mathbf{C}$ is triangular. [**1** mark]

3. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$. [15 marks total]

(a) The eigenvectors of $\mathbf{A}^\top\mathbf{A}$ are

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find the corresponding eigenvalues. [**6** marks]

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(b) Given that the eigenvectors of $\mathbf{A}\mathbf{A}^\mathsf{T}$ corresponding to the two largest eigenvalues are (in order of descending eigenvalues)

$$\frac{1}{\sqrt{22}}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

find the singular value decomposition of $\mathbf{A}$. [**3** marks]

(c) Write an expression for an approximation, $\widehat{\mathbf{A}}_1$, of $\mathbf{A}$ (you can leave it in the form of a matrix product). [**3** marks]

(d) Write down the pseudo-inverse of $\mathbf{A}$, $\mathbf{A}^+$ (you can leave it in the form of a matrix product). [**3** marks]

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4. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$. [8 marks total]

   (a) Why do we call the linear system $\mathbf{A}\mathbf{u} = \mathbf{b}$, where $\mathbf{u}$ is unknown, over-determined for the given matrix $\mathbf{A}$? [**1** mark]

   (b) Let $\mathbf{A} = \mathbf{Q}\mathbf{R}$ be a QR-decompostion of $\mathbf{A}$. What are the dimensions and properties of $\mathbf{Q}$ and $\mathbf{R}$? [**3** marks]

   (c) Find the first column of the matrix $\mathbf{Q}$ using the Gram-Schmidt process. [**2** marks]

   (d) Given $\mathbf{Q}$, describe how you could find the matrix $\mathbf{R}$. [**2** marks]

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5. Suppose we wish to study the annual mutation rate in some basic organism so we run five independent trials for one year each and count the number of mutations we see. The resulting data vector is $\mathbf{x} = (x_1, \ldots, x_5) = (15, 9, 16, 19, 16)$. [8 marks total]

   (a) Explain why a Poisson distribution could be an appropriate model for each trial. [**2** marks]

   (b) Given the data and the probability density function for a Poisson random variable with rate parameter λ is

   $$f(x) = \exp(-\lambda)\frac{\lambda^x}{x!},$$

   write down the likelihood for λ. [**3** marks]

   (c) Explain why we often prefer to work with the log-likelihood and write down an expression for the log-likelihood. [**3** marks]

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6. Suppose we have a method, called `randunif()`, that returns a uniform random variate between 0 and 1. [6 marks total]

   (a) Write the body of pseudo-code method, `randbern(p)`, that returns a Bernoulli random variate with parameter p. Include a check that the input parameter is legal. [**3** marks]

   (b) Write the body of pseudo-code method, `randbin(n,p)`, that returns a binomial random variate with parameters n and p (you can assume the `randbern(p)` method). Include checks that the input parameters are legal. [**3** marks]

7. Suppose that the sequence of random variables $X_0, X_1, X_2, \ldots$ has the Markov property. Then simplify the following: [**2** marks]

$$P(X_{n+1}|X_n, X_{n-1}, \ldots, X_0) =$$

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