THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2013 — MID-SEMESTER TEST
Campus: City

COMPUTER SCIENCE

Computational Science

(Time allowed: 50 minutes)

NOTE: Attempt *all* questions

Use of calculators is NOT permitted.

Put your answers in the answer boxes provided below each question. You may use the blank pages at the end of the exam script for scratch work, which will not be marked.

52 Marks in total. For grading purposes, 50 marks will be considered 100%.

SURNAME:

FIRSTNAME:

ID:

ID: ____________________

1. The bisection method and Newton's method are both methods for finding the roots of a function, $f(x)$. [7 marks total]

   (a) What does it mean for x to be a root of f? [**1** mark]

   (b) What is one reason to choose the bisection method over Newton's method? [**1** mark]

   (c) What is one reason to choose Newton's method over the bisection method? [**1** mark]

   (d) Suppose we are using Newton's method to find the roots of $f(x) = x^2 - 2$. If $x_0 = 0.5$, calculate x_1, the approximation found after one iteration of Newton's method. [**4** marks]

2. Suppose $\mathbf{A}$ is the square matrix $\mathbf{A} = \mathrm{diag}(a_1, \ldots, a_n)$. [5 marks total]

   (a) What is the determinant of $\mathbf{A}$, $\det(\mathbf{A})$? [**1** mark]

   (b) What condition on $\det(\mathbf{A})$ tells us that $\mathbf{A}^{-1}$ exists? If $\mathbf{A}^{-1}$ exists, what can we say about each a_i? [**2** marks]

ID: ______________________________

(c) Assuming $\mathbf{A}^{-1}$ exists, what is it? [**2** marks]

3. Suppose $\mathbf{A}$ is a square matrix. [4 marks total]

(a) What does it mean for $\mathbf{A}$ to be orthonormal? [**2** marks]

(b) If $\mathbf{A}$ is orthonormal, what is the solution to $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x}$ is unknown? [**2** marks]

4. Suppose we wish to solve the over-determined linear system represented by $\mathbf{Au} = \mathbf{b}$ where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{u}$ is an unknown $n \times 1$ vector and $\mathbf{b}$ is known. [12 marks total]

(a) Given that $\mathbf{A}$ is over-determined, what is the relationship between m and n? [**1** mark]

(b) Write down the normal equation that we solve to find the least squares solution to this problem. [**2** marks]

(c) Let $\mathbf{A} = \mathbf{QR}$ be a QR-decompostion of $\mathbf{A}$. What are the dimensions and properties of $\mathbf{Q}$ and $\mathbf{R}$? [**4** marks]

ID: ____________________

(d) Show how the QR decomposition is used to simplify the normal equation. Why is the resulting equation easy to solve? [**5** marks]

5. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$. [14 marks total]

(a) The eigenvectors of $\mathbf{A}^\top\mathbf{A}$ are

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find the corresponding eigenvalues. [**6** marks]

ID: ______________________________

(b) The eigenvectors of $\mathbf{A}\mathbf{A}^{\mathsf{T}}$ corresponding to the two largest eigenvalues are (in order of descending eigenvalues)

$$\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

What are the corresponding eigenvalues? *Hint: don't calculate them directly!* [**2** marks]

(c) Find the singular value decomposition of $\mathbf{A}$. Ensure that the singular values in matrix $\mathbf{D}$ are ordered from largest to smallest. [**3** marks]

(d) Write an expression for an approximation, $\widehat{\mathbf{A}}_1$, of $\mathbf{A}$ using only its largest singular vectors and singular value (you can leave it in the form of a vector product). [**3** marks]

6. Suppose we have a method of generating Bernoulli random variables with probability of success p for any $0 < p < 1$. [10 marks total]

(a) How would we generate geometric random variables with parameter p? [**2** marks]

(b) How would we simulate the position of a random walker after 200 steps where the walk started at 0 and steps in the positive direction with probability p? [**2** marks]

ID: ______________________________

(c) How would we generate binomial random variables? [**3** marks]

(d) If we also had a method of simulating the Poisson process with rate $\lambda_1 = 5$, how could we couple this with our Bernoulli generator to simulate the Poisson process with rate $\lambda_2 = 2$? [**3** marks]

ID: ____________________

Blank page 1 — will not be marked

ID: ____

Blank page 2 — will not be marked