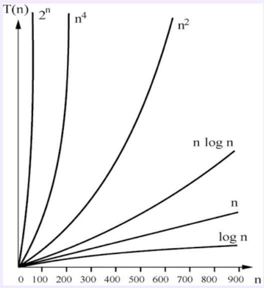
Typical Complexity Curves

$T(n) \propto \log n$ logarithmic
$T(n) \propto n$ linear
$T(n) \propto n \log n$
$T(n) \propto n^2$ quadratic
$T(n) \propto n^3$ cubic
$T(n) \propto n^k$ polynomial
$T(n) \propto 2^n$ exponential



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Relative growth: $g(n) = f(n)/f(5)$

Function $f(n)$		Input size n: 5	25	125	625
Constant	1	1	1	1	1
Logarithm	$\log_5 n$	1	2	3	4
Linear	n	1	5	25	125
"$n \log n$"	$n \log_5 n$	1	10	75	500
Quadratic	n^2	1	25 (5^2)	625 (5^4)	15,625 (5^6)
Cubic	n^3	1	125 (5^3)	15,625 (5^6)	1,953,125 (5^9)
Exponential	2^n	1	$2^{20} \approx 10^6$	$2^{120} \approx 10^{36}$	$2^{620} \approx 10^{187}$

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"Big-Oh" $O(\ldots)$: Formal Definition

Let $f(n)$ and $g(n)$ be nonnegative-valued functions defined on nonnegative integers n

The function $g(n)$ is $O(f)$ (read: $g(n)$ is **Big Oh** of $f(n)$) **iff** there exists a positive real constant c and a positive integer n_0 such that $g(n) \le cf(n)$ for all $n > n_0$

– Notation "***iff***" is an abbreviation of "**if and only if**"

– *Example* 1.9 (p.15): $g(n) = 100\log_{10} n$ is $O(n)$

$\Leftarrow g(n) < n$ if $n > 238$ or $g(n) < 0.3n$ if $n > 1000$

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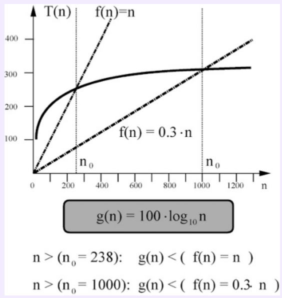
$g(n)$ is $O(f(n))$, or $g(n) = O(f(n))$

$g(n)$ is $O(f(n))$ if:

a constant $c > 0$ exists such that $cf(n)$ grows faster than $g(n)$ for all $n > n_0$

To prove that some function $g(n)$ is $O(f(n))$ means to show for g and f such constants c and n_0 exist

The constants c and n_0 are interdependent



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"Big-Oh" $O(\ldots)$: Informal Meaning

- If $g(n)$ is $O(f(n))$, an algorithm with running time $g(n)$ runs **asymptotically** (i.e. for large n), **at most** as fast, to within a constant factor, as an algorithm with running time $f(n)$

  $O(f(n))$ specifies an asymptotic upper bound, i.e. $g(n)$ for large n may approach closer and closer to $cf(n)$

  Notation $g(n) = O(f(n))$ means actually $g(n) \in O(f(n))$, i.e. $g(n)$ is a member of the set $O(f(n))$ of functions increasing with the same or lesser rate if $n \to \infty$

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Big Omega $\Omega(\ldots)$

- The function $g(n)$ is $\Omega(f(n))$ **iff** there exists a positive real constant c and a positive integer n_0 such that $g(n) \ge cf(n)$ for all $n > n_0$

  $\Omega(\ldots)$ is opposite to $O(\ldots)$ and specifies an asymptotic lower bound: if $g(n)$ is $\Omega(f(n))$ then $f(n)$ is $O(g(n))$

  Example 1: $5n^2$ is $\Omega(n) \Leftarrow 5n^2 \ge 5n$ for $n \ge 1$

  Example 2: $0.01n$ is $\Omega(\log n) \Leftarrow 0.01n \ge 0.5\log_{10} n$ for $n \ge 100$

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Big Theta Θ(...)

- The function $g(n)$ is $\Theta(f(n))$ **iff** there exists two positive real constants c_1 and c_2 and a positive integer n_0 such that $c_1 f(n) \le g(n) \le c_2 f(n)$ for all $n > n_0$

$g(n)$ is $\Theta(f(n)) \Rightarrow$

$g(n)$ is $O(f(n))$ AND $f(n)$ is $O(g(n))$

Ex.: the same rate of increase for $g(n) = n + 5n^{0.5}$ and $f(n) = n$

$\Rightarrow n \le n + 5n^{0.5} \le 6n$ for $n > 1$

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Comparisons: Two Crucial Ideas

- Exact running time function is unimportant since it can be multiplied by an arbitrary positive constant.
- Two functions are compared ***asymptotically***, for large n, and not near the origin
  - If the constants c involved are very large, then the asymptotical behaviour is of no practical interest!
  - To prove that $g(n)$ is **not** $O(f(n))$, $\Omega(f(n))$, or $\Theta(f(n))$ we have to show that the desired constants do not exist, i.e. lead to a contradiction

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Example 1.12, p.17

Linear function $g(n) = an + b; a > 0$, is $O(n)$

To prove, we form a chain of inequalities:

$g(n) \le a\ n + |b| \le g(n) \le (a + |b|) \cdot n$ for all $n \ge 1$

Do not write $O(2n)$ **or** $O(an + b)$ **as this means still** $O(n)$**!**

$O(n)$ - running time:

$T(n) = 3n + 1$ $T(n) = 10^8 + n$

$T(n) = 50 + 10^{-8} n$ $T(n) = 10^6 n + 1$

Remember that "Big-Oh" describes an "asymptotic behaviour" for large problem sizes

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Example 1.13, p.17

Polynomial $P_k(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0;\ a_k > 0$, is $O(n^k) \Leftarrow P_k(n) \le (a_k + |a_{k-1}| + \ldots + |a_0|)\ n^k;\ n \ge 1$

Do not write $O(P_k(n))$ **as this means still** $O(n^k)$**!**

$O(n^k)$ - running time:

- $T(n) = 3n^2 + 5n + 1$ is $O(n^2)$ Is it also $O(n^3)$?
- $T(n) = 10^{-8} n^3 + 10^8 n^2 + 30$ is $O(n^3)$
- $T(n) = 10^{-8} n^8 + 1000n + 1$ is $O(n^8)$

$T(n) = P_k(n) \Rightarrow O(n^m), m \ge k;\ \Theta(n^k);\ \Omega(n^m); m \le k$

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Example 1.14, p.17

Exponential $g(n) = 2^{n+k}$ is $O(2^n)$: $2^{n+k} = 2^k \cdot 2^n$ for all n

Exponential $g(n) = m^{n+k}$ is $O(l^n)$, $l \ge m > 1$:

$m^{n+k} \le l^{n+k} = l^k \cdot l^n$ for all n, k

A "brute-force" search for the best combination of n interdependent binary decisions by exhausting all the 2^n possible combinations has exponential time complexity! Therefore, **try to find a more efficient way of solving the decision problem** with $n \ge 20 \ldots 30$

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Example 1.15, p.17

- Logarithmic function $g(n) = \log_m n$ has the same rate of increase as $\log_2 n$ because

$\log_m n = \log_m 2 \cdot \log_2 n$ for all $n, m > 0$

Do not write $O(\log_m n)$ **as this means still** $O(\log n)$!

You will find later that the most efficient search for data in an ordered array has logarithmic time complexity

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