Lecture 7: Insertion Sort

Analysis of Complexity

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Worst-case Complexity of Insertion Sort

Iterative growth of a head ("sorted" sublist) of a list $\mathbf{A}$:

$$\underbrace{\boxed{a[0]}\boxed{a[1]}\boxed{\ldots}\boxed{a[i-1]}}_{\text{Head of size } i}\underbrace{\boxed{a[i]}\boxed{a[i+1]}\boxed{\ldots}\boxed{a[n-1]}}_{\text{Tail of size } n-i}$$

$n-1$ iterations (stages) $i = 1, 2, \ldots, n-1$;
j; $1 \leq j \leq i$, comparisons and j or $j-1$ moves per stage.

- Insertion sort is **correct**, since the head sublist is always sorted, and eventually expands to include all elements of $\mathbf{A}$.
- The worst-case complexity is $\Theta(n^2)$.
  - The worst-case inputs $\mathbf{A}$ consist of distinct items in reverse sorted order: $a[0] > a[1] > \ldots > a[n-1]$.
  - The total worst-case number of comparisons is $1 + 2 + \ldots + n - 1 = \frac{(n-1)n}{2} = \frac{1}{2}(n^2 - n) \in \Theta(n^2)$.

Average-case Complexity of Insertion Sort

Lemma 2.3, p.30

The average-case time complexity of insertion sort is $\Theta(n^2)$

The proof's outline:

- Assuming all possible inputs are equally likely, evaluate the average number $\overline{C}_i$ of comparisons at each stage $i = 1, \ldots, n-1$.
- Calculate the average total number $\overline{C} = \sum\limits_{i=1}^{n-1} \overline{C}_i$.
- Evaluate the average-case complexity of insertion sort by taking into account that the total number of data moves is at least zero and at most the number of comparisons.

Average Complexity of Insertion Sort at Stage i

$i+1$ positions in the already ordered head $a[0],\ldots,a[i-1]$ of a list $\mathbf{A}$ to insert the next unordered yet item $a[i]$:

$a[0]$	$a[1]$	$a[2]$	$\cdots$	$a[i-1]$	$a[i]$

$\uparrow_0 \quad \uparrow_1 \quad \uparrow_2 \quad \uparrow_{\ldots} \quad \uparrow_{i-1} \quad \uparrow_i$

For placing $[a[i]$ into each preceding position $j = i, i-1, \ldots, 1$, the algorithm performs $i-j+1$ comparisons and $i-j$ moves

- For position $j = 0$, it performs i comparisons and i moves.

Therefore, the average number of comparisons at stage i:

$$\overline{C}_i = \frac{1+2+\ldots+i+i}{i+1} = \frac{\frac{i(i+1)}{2}+i}{i+1} = \frac{i}{2}+\frac{i}{i+1} \equiv \frac{i}{2}+\left(1-\frac{1}{i+1}\right)$$

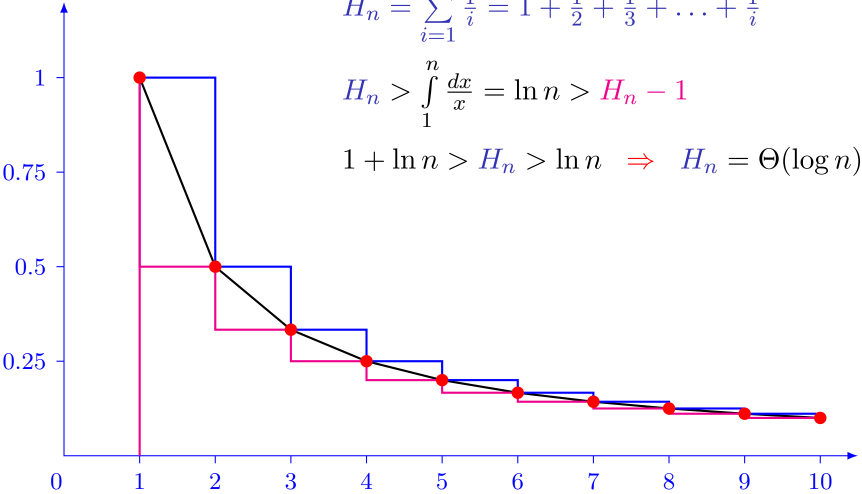
Total Average Complexity for n Input Items

The total average number of comparisons for $n-1$ stages:

$$\begin{aligned}
\overline{C} &= \overbrace{\left(\frac{1}{2}+\left(1-\frac{1}{2}\right)\right)}^{\overline{C}_1} + \overbrace{\left(\frac{2}{2}+\left(1-\frac{1}{3}\right)\right)}^{\overline{C}_2} + \ldots + \overbrace{\left(\frac{n-1}{2}+\left(1-\frac{1}{n}\right)\right)}^{\overline{C}_{n-1}} \\
&= \frac{1}{2}\underbrace{\left(1+2+\ldots+(n-1)\right)}_{\frac{(n-1)n}{2}} + \\
&\quad \underbrace{\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{3}\right)+\ldots+\left(1-\frac{1}{n}\right)}_{n-H_n} \\
&= \frac{(n-1)n}{4} + n - H_n \in \Theta(n^2)
\end{aligned}$$

where $H_n = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n$ when $n \to 0$ is the n-th harmonic number.

Math Appendix: Evaluating Harmonic Numbers



$$H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{i}$$

$$H_n > \int_1^n \frac{dx}{x} = \ln n > H_n - 1$$

$$1 + \ln n > H_n > \ln n \quad \Rightarrow \quad H_n = \Theta(\log n)$$

Analysis of Inversions

The running time of insertion sort is strongly related to **inversions** in a list $\mathbf{A}$ to be sorted.

Definition 2.5 (p.30): An inversion in a list $\mathbf{A} = [a_1, a_2, \ldots, a_n]$

is any ordered pair of positions (i, j) such that $i < j$ but $a_i > a_j$.

Examples of inversions: $[\ldots, 2, \ldots, 1]$ or $[100, \ldots, 35, \ldots]$.

List $\mathbf{A}$	Number of inversions	Reverse list $\mathbf{A}_{\text{rev}}$	Number of inversions	Total
$[3, 2, 5]$	1	$[5, 2, 3]$	2	3
$[3, 2, 5], 1]$	4	$[1, 5, 2, 3]$	2	6
$[1, 2, 3, 5, 7]$	0	$[7, 5, 3, 2, 1]$	10	10

The number of inversions of a list is a measure of how far it is from being sorted.

Analysis of Inversions

Number of inversions I_i, comparisons C_i and data moves M_i for each element $a[i]$ in $\mathbf{A}$:

Element i	0	1	2	3	4	5	6	
$\mathbf{A}$	44	13	35	18	15	10	20	
I_i		1	1	2	3	5	2	$I = 14$
C_i		1	2	3	4	5	3	$C = 18$
M_i		1	1	2	3	5	2	$M = 14$

It is always true that $I_i = M_i$, so the total number $I = \sum\limits_{i=1}^{n-1} I_i$ of inversions is equal to the total number $M = \sum\limits_{i=1}^{n-1} M_i$ of backward moves of elements $a[i]$ during the sort.

Analysis of Inversions

The total number of data comparisions $C = \sum_{i=1}^{n-1} C_i$ is also equal to the total number of inversions plus at most $n - 1$.

Total number of inversions in both an arbitrary list $\mathbf{A}$ and its reverse $\mathbf{A}_{\text{rev}}$ is equal to the **total number of the ordered pairs** $(i < j)$ of integers $i, j \in \{1, \ldots, n-1\}$:

$$\binom{n-1}{2} = \frac{(n-1)n}{2}$$

- A sorted list has no inversions.
- A reverse sorted list of size n has $\frac{(n-1)n}{2}$ inversions.
- In the average, all lists of size n have $\frac{(n-1)n}{4}$ inversions.

Complexity of Insertion Sort by Analysis of Inversions

Exactly **one inversion** is removed by swapping two neighbours $a_{i-1} > a_i$.

- If an original list has I inversions, insertion sort has to swap I pairs of neighbours.
- A list with I inversions results in $\Theta(n + I)$ running time of `insertionSort` because of $\Theta(n)$ other operations in the algorithm.
  - In the very rare cases of nearly sorted lists for which I is $\Theta(n)$, insertion sort runs in linear time.
  - The worst-case time: $c\frac{n^2}{2}$, or $\Theta(n^2)$.
  - The average-case time: $c\frac{n^2}{4}$, or $\Theta(n^2)$.

More efficient sorting algorithms must eliminate more than just one inversion between neighbours per swap.

Implementation of Insertion Sort

The number of comparisons does not depend on how the list is implemented, but the number of moves does.

- Backward moves in an array implementation of a list:
  - Shifting elements to the right (linear time per stage) in the worst and average case, or
  - Successive swaps to move the element backward.
- Insertion operation in a linked list implementation of a list:
  - Constant-time insertion of an element.
  - Fewer swaps by simply scanning backward (but it may take time for a singly linked list).

None of the implementation issues affect the asymptotic Big-Theta running time of the algorithm, just the hidden constants and lower order terms, due to too many comparisons in the worst and average case.

One More Quadratic $\Theta(n^2)$ Sorting Algorithm (p.181)

```
// Selection sort of an input array a of size n
// (building a tail by successive minima selection from a head)
public static void selectionSort( int [ ] a ) {
  for ( int i = 0; i < a.length - 1; i++ ) {
     int posMin = i;
     for ( int k = i + 1; k < a.length; k++ ) {
       if ( a[ posMin ] > a[ k ] ) posMin = k;
     }
     if ( posMin != i ) {          swap a[i] and a[posMin]
       int tmp = a[ i ];
       a[ i ] = a[ posMin ];
       a[ posMin ] = tmp;
     }
  }
}
```