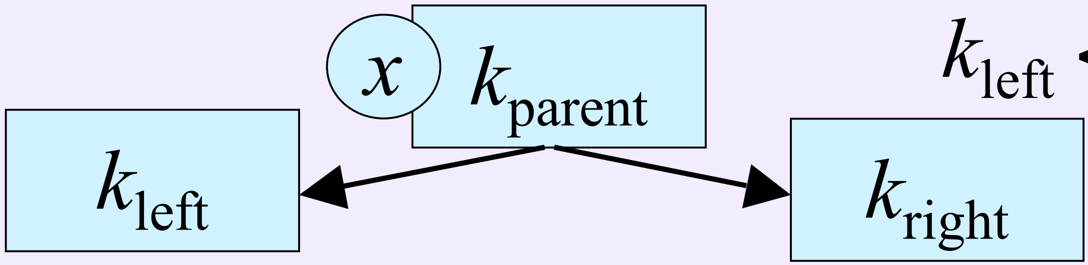
Binary Search Tree

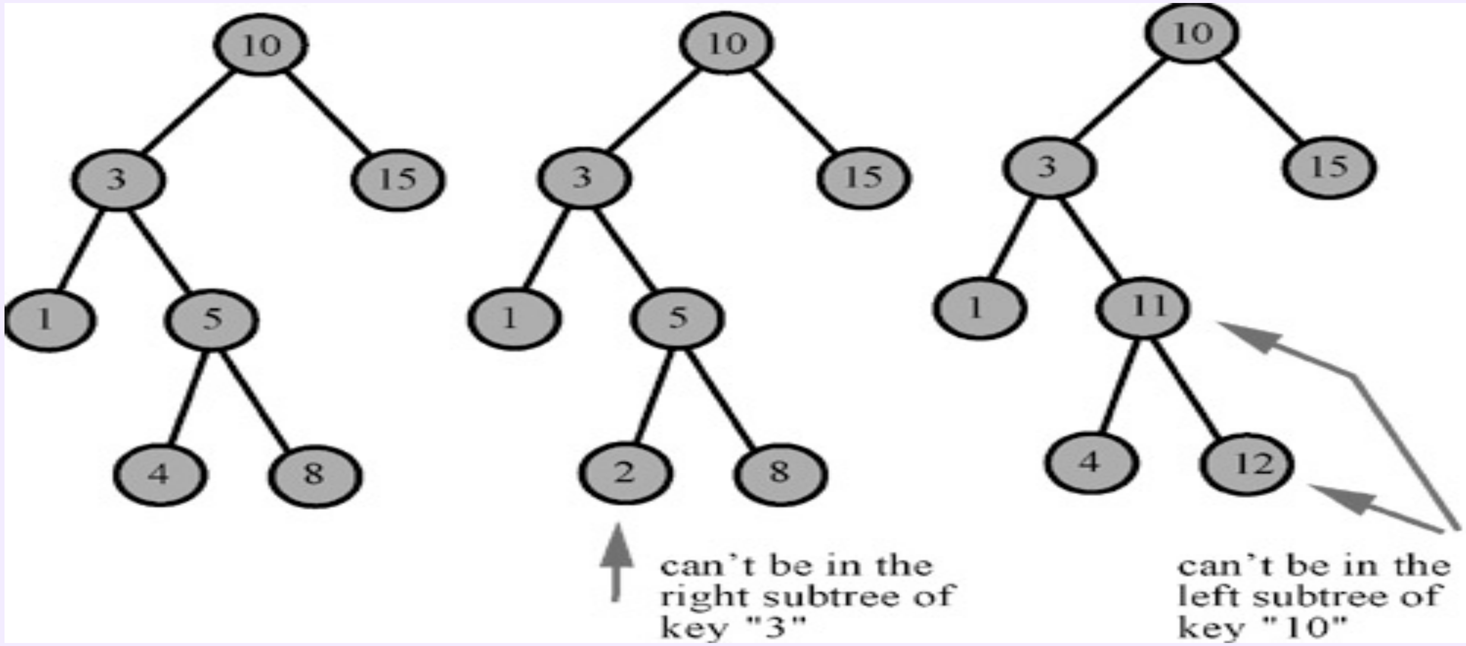
- Left-to-right ordering in a tree:
  - for every node x, the values of all the keys k_{left} in the left subtree are **smaller** than the key k_{parent} in x and
  - the values of all the keys k_{right} in the right subtree are larger than the key in x**:**

$$k_{\text{left}} < k_{\text{parent}} < k_{\text{right}}$$



Binary Search Tree

Compare the left–right ordering in a BST to the bottom–up ordering in a heap where the key of each parent node is greater than or equal to the key of any child node



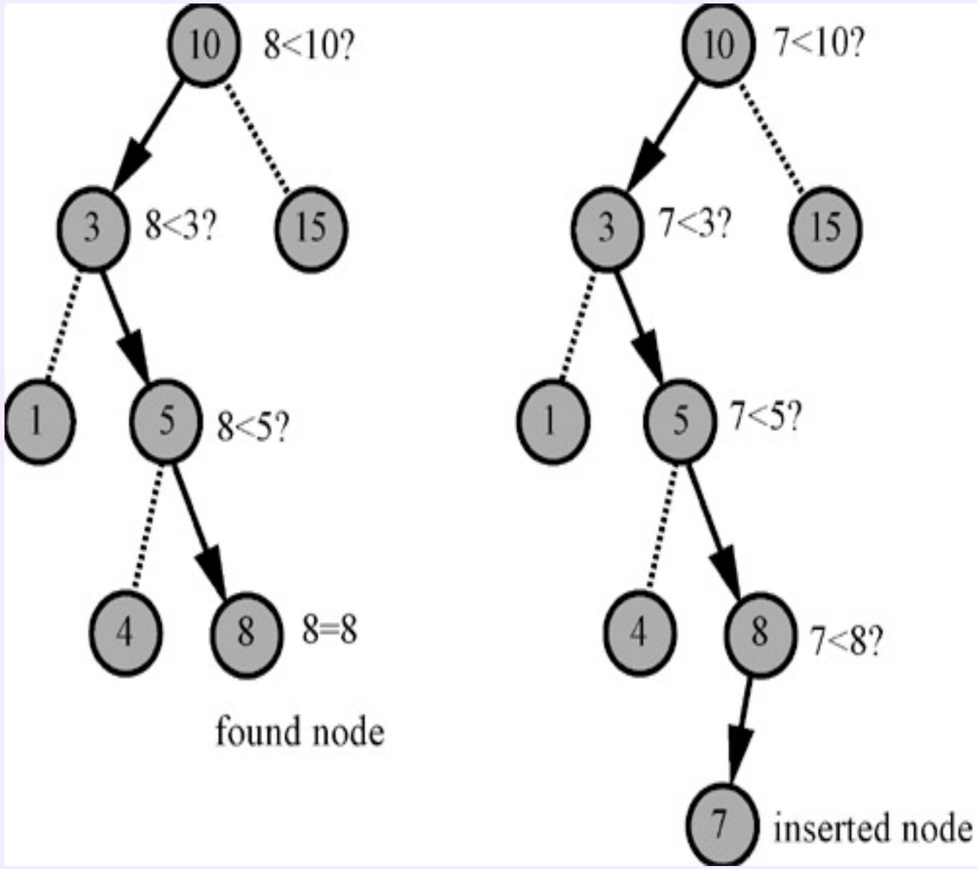
Binary Search Tree

- No duplicates! (attach them all to a single item)
- Basic operations:
  - **find**: find a given search **key** or detect that it is not present in the tree
  - **insert**: insert a node with a given **key** to the tree if it is not found
  - **findMin**: find the minimum **key**
  - **findMax**: find the maximum **key**
  - **remove**: remove a node with a given **key** and restore the tree if necessary

BST: find / insert operations

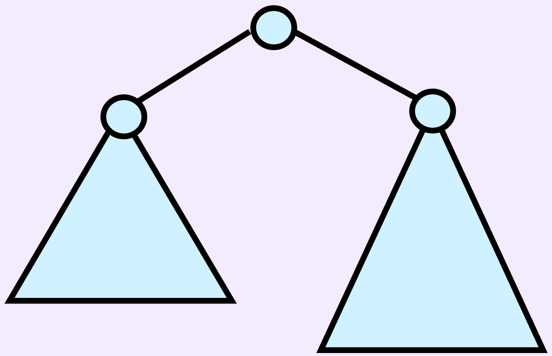
find is a successful binary search

insert creates a new node at the point at which an unsuccessful search stops



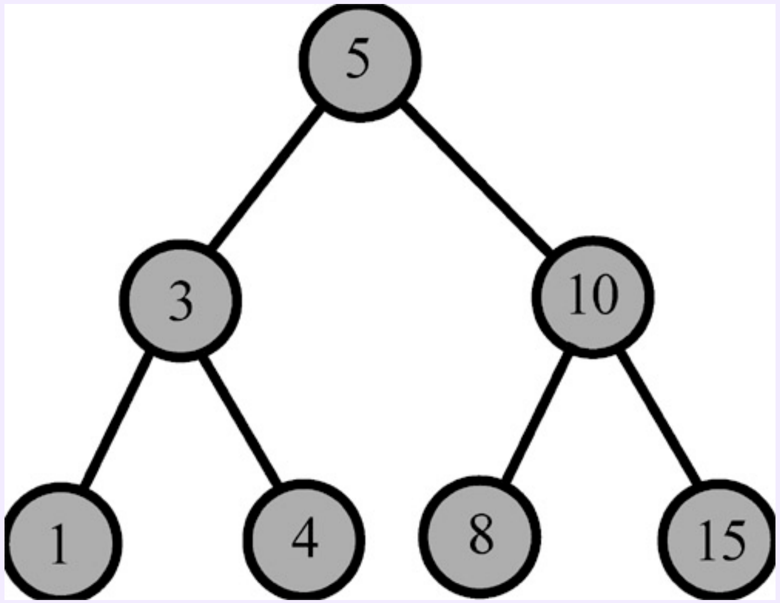
Binary Search Trees: findMin / findMax / sort

- **findMin/findMax** are extremely simple:
  - starting at the root, branch repeatedly left (**findMin**) or right (**findMax**) as long as a corresponding child exists
- The root of the tree plays a role of the pivot in QuickSort
- As in QuickSort, the recursive traversal of the tree can **sort** the items:
  - First visit the left subtree
  - Then visit the root
  - Then visit the right subtree



Binary Search Tree: running time

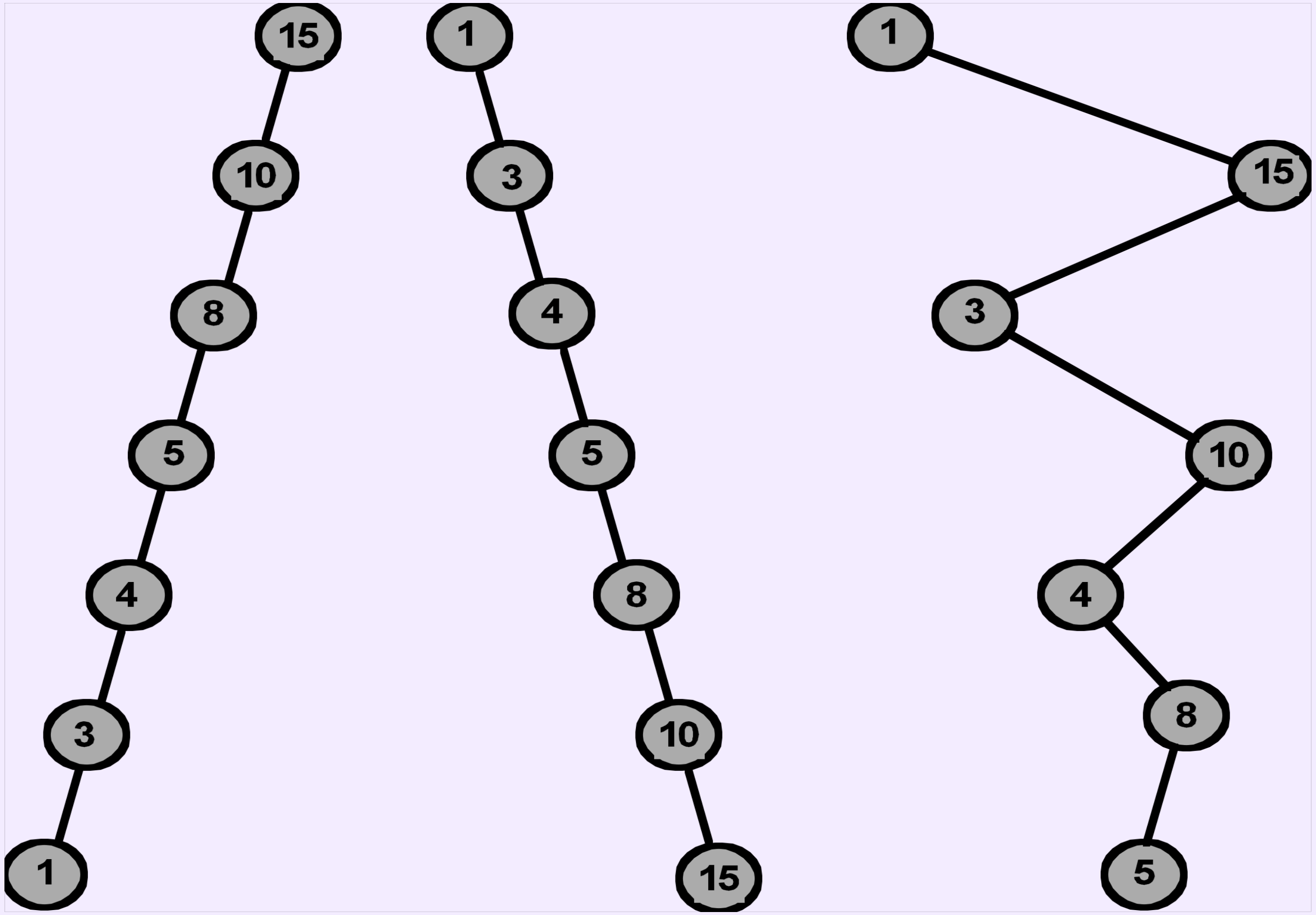
Time for **find**, **insert**, **findMin**, **findMax**, **sort** a single item: $O(\log n)$ average-case and $O(n)$ worst-case complexity



(just as in QuickSort)

BST of the depth about $\log n$

BST of the depth about n



Binary Search Tree: node removal

- **remove** is the most complex operation:
  - The removal may disconnect parts of the tree
  - The reattachment of the tree must maintain the **binary search tree property**
  - The reattachment **should not** make the tree unnecessarily deeper as the depth specifies the running time of the tree operations

BST: how to remove a node

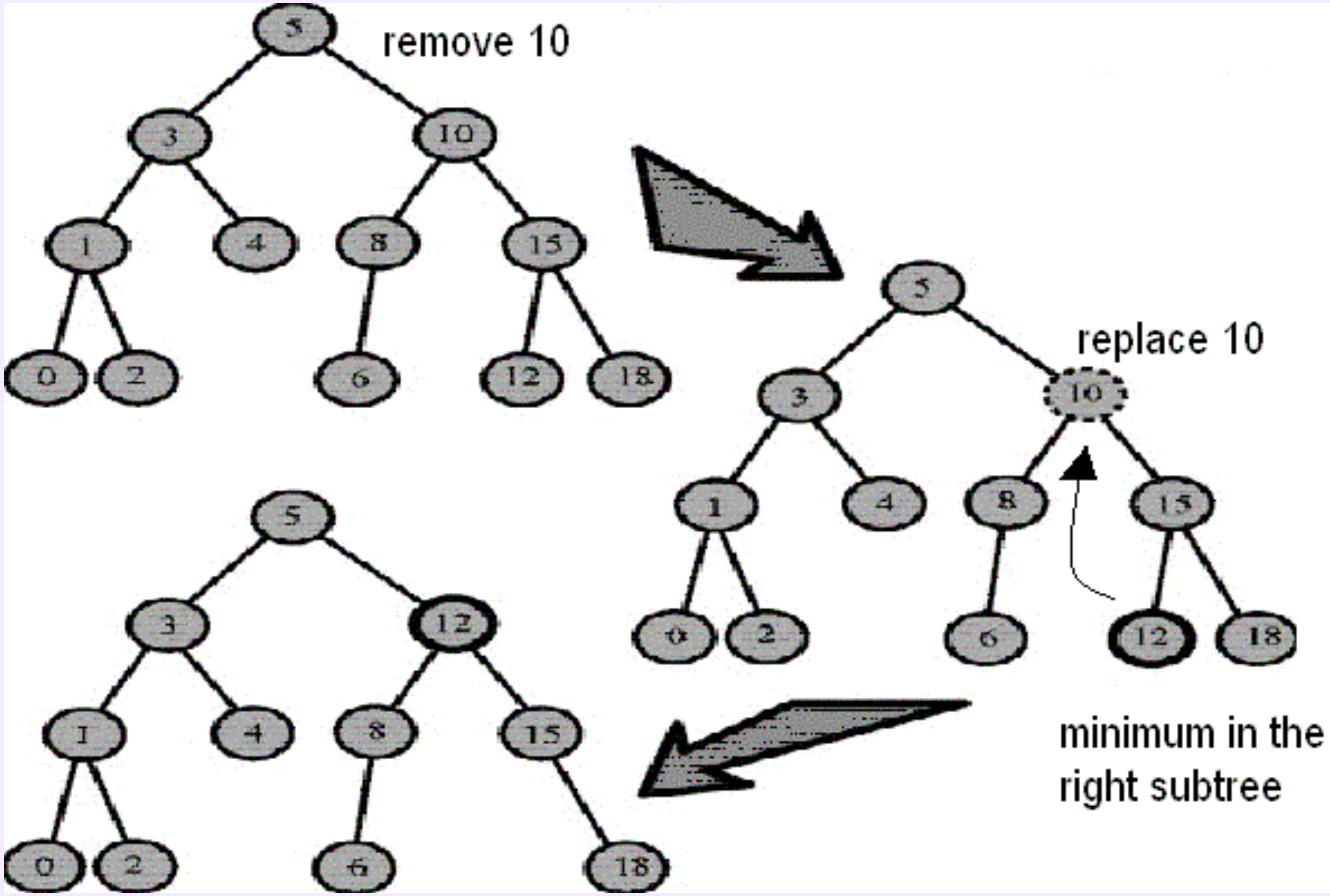
- If the node k to be removed is a leaf, delete it
- If the node k has only one child, remove it after linking its child to its parent node
- Thus, **removeMin** and **removeMax** are not complex because the affected nodes are either leaves or have only one child

BST: how to **remove** a node

If the node k to be removed has two children:

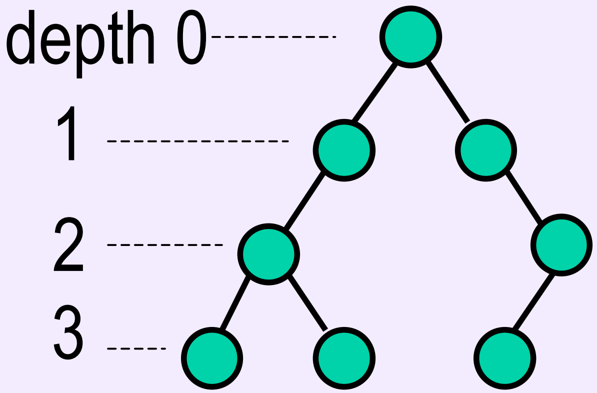
- Replace the item in this node with the item with the **smallest** key in the **right** subtree
  - The smallest node is easily found as in **findMin**
- Remove the latter node from the right subtree
  - This removal is very simple as the node with the smallest key does not have a left child

BST: an Example of Node Removal



Average-Case Performance of Binary Search Tree Operations

Internal path length of a binary tree is the sum of the **depths** of its nodes:



IPL = 0 + 1 + 1 + 2 + 2 + 3 + 3 + 3 = 15

Average internal path length $T(n)$ of the binary search trees with n nodes is $O(n \log n)$

Average-Case Performance of Binary Search Tree Operations

- If the n-node tree contains the root, the i-node left subtree, and the $(n-i-1)$-node right subtree:

$$T(n) = n - 1 + T(i) + T(n-i-1)$$

  - The root contributes 1 to the path length of each of the other $n - 1$ nodes

- Averaging over all i; $0 \le i < n \rightarrow$ the same recurrence as for QuickSort:

$$T(n) = (n-1) + \tfrac{2}{n}\big(T(0) + T(1) + \ldots + T(n-1)\big)$$

so that $T(n)$ is $O(n \log n)$

Average-Case Performance of Binary Search Tree Operations

- Therefore, the average complexity of **find** or **insert** operations is $T(n)/n = O(\log n)$
- For n^2 pairs of random **insert** / **remove** operations, an expected depth is $O(n^{0.5})$
- In practice, for random input, all operations are about $O(\log n)$ but the worst-case performance can be $O(n)$!

Balanced Trees

- **Balancing** ensures that the internal path lengths are close to the optimal $n \log n$
- The average-case and the worst-case complexity is about $O(\log n)$ due to their balanced structure
- But, **insert** and **remove** operations take more time on average than for the standard binary search trees
  - **AVL** tree (1962: Adelson-Velskii, Landis)
  - **Red-black** and **AA-tree**
  - **B-tree** (1972: Bayer, McCreight)

AVL Tree

- An AVL tree is a binary search tree with the following additional **balance property**:
  - for any node in the tree, the height of the left and right subtrees can differ by at most 1
  - the height of an empty subtree is -1
- The **AVL-balance** guarantees that the AVL tree of height h has at least c^h nodes, $c > 1$, and the maximum depth of an n-item tree is about $\log_c n$

AVL Tree

- Let S_h be the **size** of the smallest AVL tree of the height h (it is obvious that $S_0 = 1$, $S_1 = 2$)
- This tree has two subtrees of the height $h-1$ and $h-2$, respectively, by the AVL-balance condition
- It follows that $S_h = S_{h-1} + S_{h-2} + 1$, or $S_h = F_{h+3} - 1$ where F_i is the i-th Fibonacci number

AVL Tree

- Therefore, for each n-node AVL tree:

$$n \geq S_h \approx \left(\varphi^{h+3} / \sqrt{5}\right) - 1$$

where $\varphi = \left(1 + \sqrt{5}\right) / 2 \cong 1.618$, or

$$h \leq 1.44 \log_2(n+1) - 1.328$$

- The worst-case height is **at most 44%** more than the minimum height of the binary trees