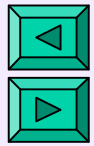
Data Sorting

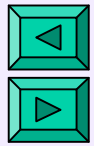
- **Ordering relation:** places each pair α, β of ***countable*** items in a fixed order denoted as (α,β) or $<\alpha,\beta>$
- **Order notation:** $\alpha \leq \beta$ (*less than or equal to*)
- **Countable item:** labelled by a specific ***integer key***
- **Comparable objects in Java:** if an object can be ***less than***, ***equal to***, or ***greater than*** another object:

  object1.compareTo(object2) **<0, =0, >0**



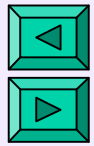
Order of Data Items

- **Numerical order** - by value:
  $5 \leq 5 \leq 6.45 \leq 22.79 \leq \ldots \leq 1056.32$
- **Alphabetical order** - by position in an alphabet:
  a ≤ b ≤ c ≤ d ≤ … ≤ z
  Such ordering depends on the alphabet used: look into any bilingual dictionary...
- **Lexicographic order** - by first differing element:
  5456 ≤ 5457 ≤ 5500 ≤ 6100 ≤ …
  pork ≤ ward ≤ word ≤ work ≤ …



Features of Ordering

- Relation on an array $\mathbf{A} = \{a, b, c, \ldots\}$ is:
  - ***reflexive:*** $a \leq a$
  - ***transitive:*** if $a \leq b$ and $b \leq c$, then $a \leq c$
  - ***symmetric:*** if $a \leq b$ then $b \leq a$
- **Linear order** if for any pair of elements a and b either $a \leq b$ or $b \leq a$: $a \leq b \leq c \leq \ldots$
- **Partial order** if there are incomparable elements



Sorting with Insertion Sort

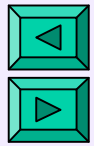
- Split an array into a unordered and ordered parts
- Sequentially contract the unordered part, one element per stage:

ordered part | unordered part

$a_0, \ \ldots, \ a_{i-1}$ | $a_i, \ \ldots, \ a_{n-1}$

- At each stage $i = 1, \ldots, n-1$:

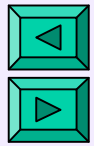
n – i unordered and ***i*** ordered elements



Insertion Sort: Step $i = 4$

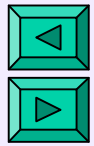
- N_c - number of comparisons per insertion
- N_m - number of moves per insertion

13	18	35	44	15	10	20	N_c	N_m
			15	44			<	→
		15	35				<	→
	15	18					<	→
15							≥	
13	15	18	35	44	*10*	*20*	4	3



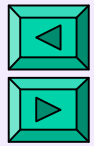
Insertion Sort : Step $i = 5$

13	15	18	35	44	*10*	*20*	N_c	N_m
				10	44		<	→
			10	35			<	→
		10	18				<	→
	10	15					<	→
10	13						<	→
10	13	15	18	35	44	*20*	5	5



Pseudocode of Insertion Sort

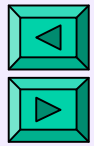
```
begin InsertionSort ( integer array a[] of size n )
1.      for i ← 1 while i < n  step i ← i + 1 do
2.          s_tmp ← a[ i ];      k ← i  – 1
3.          while k ≥ 0 AND s_tmp < a[k] do
4.              a[ k + 1 ] ← a[ k ];      k ← k – 1
5.          end while
6.          a[ k + 1 ] ← s_tmp
7.      end for
 end InsertionSort
```

Average Complexity at Stage i

- $i + 1$ positions to place a next item: $0\ 1\ 2\ \ldots\ i$ -1 i
- $i - j + 1$ comparisons and $i - j$ moves for each position $j = i, i-1, \ldots, 1$
- i comparisons and i moves for position $j = 0$
- Average number of comparisons:

$$E_i = \frac{1 + 2 + \ldots + i + i}{i + 1} = \frac{i}{2} + \frac{i}{i + 1}$$



Total Average Complexity

- $n-1$ stages for n input items: the total average number of comparisons:

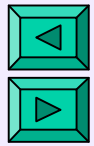
$$E = E_1 + E_2 + \ldots + E_{n-1} = \frac{n^2}{4} + \frac{3n}{4} - H_n$$

$$\left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{2}{2} + \frac{2}{3}\right) + \ldots + \left(\frac{n-1}{2} + \frac{n-1}{n}\right)$$

$$= \frac{1}{2}(1 + 2 + \ldots + (n-1)) + \left(\frac{1}{2} + \frac{2}{3} + \ldots + \frac{n-1}{n}\right)$$

$$= \frac{(n-1)n}{4} + n - \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

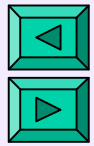
- $H_n \cong \ln n + 0.577$ when $n \to \infty$ is the n-th **harmonic** number



Analysis of Inversions

- An **inversion** in an array $\mathbf{A} = [a_1, a_2, \ldots, a_n]$ is any ordered pair of positions (i, j) such that $i < j$ but $a_i > a_j$: e.g., […, 2,…, 1] or [100, …, 35, …]

A	Number of inversions	$\mathbf{A}_{\text{reverse}}$	Number of inversions	Total number
3,2,5	1	5,2,3	2	3
3,2,5,1	4	1,5,2,3	2	6
1,2,3,4,7	0	7,4,3,2,1	10	10

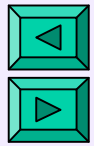


Analysis of Inversions

- Total number of inversions both in an arbitrary array $\mathbf{A}$ and its reverse $\mathbf{A}_{\text{reverse}}$ is equal to the **total number of the ordered pairs** ($i < j$):

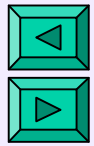
$$\binom{n}{2} = \frac{(n-1)\cdot n}{2}$$

- A sorted array has no inversions
- A reverse sorted array has $\frac{(n-1)n}{2}$ inversions



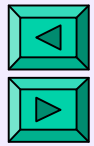
Analysis of Inversions

- Exactly **one inversion** is removed by swapping two neighbours $a_{i-1} > a_i$
- An array with k inversions results in $O(n + k)$ running time of insertionSort
- Worst-case time: $c\frac{n^2}{2}$, or $O(n^2)$
- Average-case time: $c\frac{n^2}{4}$, or $O(n^2)$



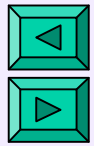
More Efficient Shell's Sort

- **Efficient sort** must eliminate more than just one inversion between the neighbours per exchange!
  - Insertion sort eliminates **one** inversion per exchange
- D.Shell (1959): compare first the keys at a distance of gap_T, then of $gap_{T-1} < gap_T$, and so on until of $gap_1 = 1$
- After a stage with gap_t, all elements spaced gap_t apart are sorted; it can be proven that any gap_t-sorted array remains gap_t-sorted after being then gap_{t-1}-sorted



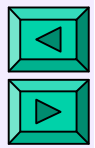
Pseudocode of ShellSort

```
begin  Shell Sort ( integer array a of size n )
1.    for gap ← ⌊n/2⌋ while gap > 0
          step gap ← ( if gap = 2 then 1 else ⌊gap/2.2⌋ ) do
2.        for i ← gap while i < n step i ← i + 1 do
3.            s_tmp ← a[i];  k ← i
4.            while( k ≥ gap AND s_tmp < a[ k – gap ]  do
5.                a[ k ] ← a[ k – gap ];   k ← k  –  gap
6.            end while
7.            a[ k ] ← s_tmp
8.        end for
9.    end for
end Shell Sort
```

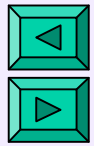
Example of ShellSort: step 1

gap	*i*:C:M	Data to be sorted									
		25	8	2	91	70	50	20	31	15	65
5	5:1:0	25					50				
	6:1:0		8					20			
	7:1:0			2					31		
	8:1:1				**15**					**91**	
	9:1:1					**65**					**70**
		25	8	2	15	65	50	20	31	91	70



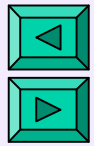
Example of ShellSort: step 2

ga	*i*:C:M	25	8	2	15	65	50	20	31	91	70
p	2:1:1	**2**		**25**							
	3:1:0		8		15						
	4:1:0			25		65					
	5:1:0				15		50				
	6:3:2	2		**20**		**25**		**65**			
	7:2:1				15		**31**		**50**		
	8:1:0							65		91	
	9:1:0								50		70



Example of ShellSort: step 3

gap	*i*:C:M	2	8	20	15	25	31	65	50	91	70
9	1:1:0	2	8								
	2:1:0		8	20							
	3:2:1		8	**15**	**20**						
	4:1:0				20	25					
	5:1:0					25	31				
	6:1:0						31	65			
	7:2:1						31	**50**	**65**		
	8:1:0								65	91	
	9:2:1								65	**70**	**91**



Time complexity of ShellSort

- Heavily depends on gap sequences
- Shell's sequence: $^{n}/_{2}, ^{n}/_{4}, \ldots, 1$:

  $O(n^2)$ worst; $O(n^{1.5})$ average
- "Odd gaps only" (if even: $^{gap}/_{2} + 1$):

  $O(n^{1.5})$ worst; $O(n^{1.25})$ average
- Heuristic sequence: $^{gap}/_{2.2}$: better than $O(n^{1.25})$
- A very simple algorithm with an extremely complex analysis!