Data selection. Lower complexity bound for sorting

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COMPSCI 220 Algorithms and Data Structures

1. Data selection: Quickselect
2. Lower complexity bound for sorting
3. The worst-case complexity bound
4. The average-case complexity bound
5. Lower sorting complexity under additional constraints

Data Selection vs. Data Sorting

- **Selection**: finding only the k^{th} smallest element, called the element of **rank** k, or the k^{th} **order statistic** in a list of n items.
- Main question: can selection be done faster without sorting?

Quickselect: the average $\Theta(n)$ and worst-case $\Theta(n^2)$ complexity

1. If $n = 0$ or 1, return "*not found*" or the list item, respectively.
2. Otherwise, choose one of the list items as a pivot, p, and partition the list into disjoint "head" and "tail" sublists with j and $n - j - 1$ items, respectively, separated by p at position with index j[a].
3. Return the result of quickselect on the head if $k < j$; the element p if $k = j$, or the result of quickselect on the tail otherwise.

[a] All head (tail) items are less (greater) than the pivot p and precede (follow) it.

Analysis of Quickselect: Average-case Complexity

Theorem 2.33: The average-case time complexity of quickselect is linear, or $\Theta(n)$.

Proof. Up to cn operations to partition the list into the head and tail sublists of size j and $n-1-j$, respectively, where $0 \leq j \leq n-1$.

- As in quicksort, each final pivot index j with equal probability $\frac{1}{n}$.
- Average time $T(n)$ to select the k^{th} smallest item out of n items:

$$T(n) = \frac{1}{n}\sum_{j=0}^{n-1}\frac{T(j)+T(n-j-1)}{2} + cn = \frac{1}{n}\sum_{j=0}^{n-1} T(j) + cn.$$

- Therefore, $nT(n) = \sum_{j=0}^{n-1} T(j) + cn^2$.
  - $nT(n) - (n-1)T(n-1) = T(n-1) + c(2n-1)$, or
  - $\boxed{T(n) \approx T(n-1) + c', \text{ so that } T(n) \in \Theta(n).}$

Implementation of Quickselect

algorithm `quickSelect`

Array-based quickselect

finds k^{th} smallest element in the subarray $a[l..r]$

Input: array $a[0..n-1]$; array indices l, r; integer k

begin

  if $l \leq r$ **then**

    $i \leftarrow \texttt{pivot}(a, l, r)$ return initial position of pivot

    $j \leftarrow \texttt{partition}(a, l, r, i)$ return final position of pivot

    $q \leftarrow j - l + 1$ the pivot's rank in $a[l..r]$

    if $k = q$ **then return** $a[j]$

    else if $k < q$ **then return** $\texttt{quickSelect}(a, l, j-1, k)$

    else return $\texttt{quickSelect}(a, j+1, r, k-q)$

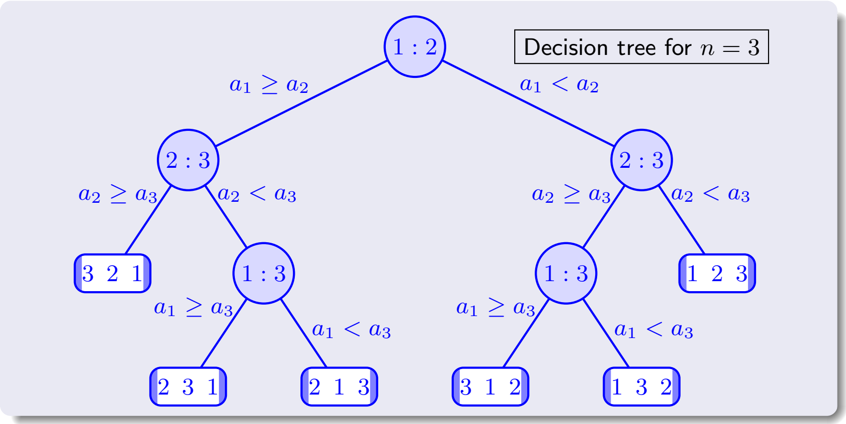
  end if

  else return *"not found"*

end

Sorting by Pairwise Comparisons: Decision Tree

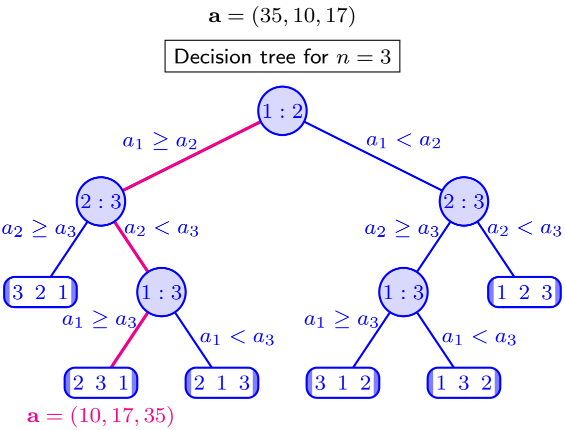
Representing any sorting of n items by pairwise comparisons with a binary **decision tree** having $n!$ leaves (internal nodes: comparisons).



Sorting by Pairwise Comparisons: Decision Tree

- Each leaf ijk:
  a sorted array a_i, a_j, a_k obtained from the initial list a_1, a_2, a_3.
- Each internal node:
  a pairwise comparison $i : j$ between the elements a_i and a_j.
  - Two downward arcs: two possible results: $a_i \geq a_j$ or $a_i < a_j$.
- Any of $n!$ permutations of n arbitrary items $a_1, \ldots, a_n$ may be met after sorting: so the decision tree must have at least $n!$ leaves.
- The path length from the root to a leaf is equal to the total number of comparisons for getting the sorted list at the leaf.
  - The longest path (the tree height) is equal to the worst-case number of comparisons.
  - Example: 3 items are sorted with no more than 3 comparisons, because the height of the tree for $n = 3$ is equal to 3.

Sorting by Pairwise Comparisons: Decision Tree



Example:

$a_1 = 35, a_2 = 10, a_3 = 17$

1. Comparison $1 : 2$
   $(35 > 10) \rightarrow$
   left branch $a_1 > a_2$
2. Comparison $2 : 3$
   $(10 < 17) \rightarrow$
   right branch $a2 < a3$
3. Comparison $1 : 3$:
   $(35 > 17) \rightarrow$
   left branch $a1 > a3$
4. Sorted array 231 $\rightarrow$
   $a_2 = 10, a_3 = 17, a_1 = 35$

The Worst-case Complexity Bound

Lemma: A decision tree of height h has at most 2^h leaves.

Proof: by mathematical induction.

- **Base cases**: A tree of height 0 has at most 2^0 leaves (i.e. one leaf).
- **Hypothesis**: Let any tree of height $h-1$ have at most 2^{h-1} leaves.
- **Induction**:
  - Any tree of height h consists of a root and two subtrees of height at most $h-1$ each.
  - The number of leaves in the whole decision tree of height h is equal to the total number of leaves in its subtrees, that is, at most $2^{h-1} + 2^{h-1} = 2^h$. $\square$

The Worst-case Complexity Bound

Theorem 2.35 (Textbook): Every pairwise-comparison-based sorting algorithm takes $\Omega(n \log n)$ time in the worst case.

Proof:

- Each binary tree, as shown in Slide 9, has at most 2^h leaves.
- The least height h such that $2^h \geq n!$ has the lower bound $h \geq \lg(n!)$.
- By the Stirling's approximation, $n! \approx n^n e^{-n}\sqrt{2\pi n}$ as $n \to \infty$.
- Therefore, asymptotically, $\lg(n!) \approx n \lg n - 1.44n$, or $\lg(n!) \in \Omega(n \log n)$. $\square$

Therefore, heapsort and mergesort have the asymptotically optimal worst-case time complexity for comparison-based sorting.

The Average-case Complexity Bound

Theorem 2.36 (Textbook): Every pairwise-comparison-based sorting algorithm takes $\Omega(n \log n)$ time in the average case.

Proof: Let $H(k)$ be the sum of all heights of k leaves in a balanced decision tree with equal numbers, $\frac{k}{2}$, of leaves on the left and right subtrees.

- Such a tree has the smallest height, i.e., in any other decision tree, the sum of heights cannot be smaller than $H(k)$.
- $H(k) = 2H\left(\frac{k}{2}\right) + k$ as the link to the root adds one to each height, so that $H(k) = k \lg k$.
- When $k = n!$ (the number of permutations of an array of n keys), $H(n!) = n! \lg(n!)$.
- Given equiprobable permutations, the average height of a leaf is $H_{\text{avg}}(n!) = \frac{1}{n!} H(n!) = \lg(n!) \approx n \lg n - 1.44n$.
- Thus, the lower bound of the average-case complexity of sorting n items by pairwise comparisons is $\Omega(n \log n)$. $\square$

Counting Sort – Exercise 2.7.2 (Textbook)

Input: an integer array $\mathbf{a}_n = (a_1, \ldots, a_n)$; each $a_i \in \mathbb{Q} = \{0, \ldots, Q-1\}$.

- Make a counting array $\mathbf{t}_Q$ and set $t_q \leftarrow 0$ for $q \in \mathbb{Q}$.
- Scan through $\mathbf{a}_n$ to accumulate in the counters t_q; $q \in \mathbb{Q}$, how many times each item q is found: if $a_i = q$, then $t[q] \leftarrow t[q] + 1$.
- Loop through $0 \leq q \leq Q-1$ and output t_q copies of q at each step.

Linear worst- and average-case time complexity, $\Theta(n)$ when Q is fixed.

- $Q + n$ elementary operations to first set $\mathbf{t}_Q$ to zero; count then how many times t_q each item q is found in $\mathbf{a}_n$, and successively output the sorted array $\mathbf{a}_n$ by repeating t_q times each entry q.

Theorems 2.35 and 2.36 do not hold under additional data constraints!