"Big-Oh", "Big-Omega", and "Big-Theta": Properties and Rules

Lecturer: Georgy Gimel'farb

COMPSCI 220 Algorithms and Data Structures

1. Big-Oh rules
   - Scaling
   - Transitivity
   - Rule of sums
   - Rule of products
   - Limit rule
2. Examples

Big-Oh: Scaling

Scaling (Lemma 1.15)

For all constant factors $c > 0$, the function $cf(n)$ is $\mathrm{O}(f(n))$, or in shorthand notation cf is $\mathrm{O}(f)$.

The proof: $cf(n) < (c + \varepsilon)f(n)$ holds for all $n > 0$ and $\varepsilon > 0$.

- Constant factors are ignored.
- Only the powers and functions of n should be exploited

It is this ignoring of constant factors that motivates for such a notation! In particular, f is $\mathrm{O}(f)$.

$$\text{Examples: } \begin{cases} 50n \in \mathrm{O}(n) & 0.05n \in \mathrm{O}(n) \\ 50,000,000n \in \mathrm{O}(n) & 0.0000005n \in \mathrm{O}(n) \end{cases}$$

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Transitivity (Lemma 1.16)

If h is $\mathrm{O}(g)$ and g is $\mathrm{O}(f)$, then h is $\mathrm{O}(f)$.

The proof:

- ◦ $h(n) \leq c_1 g(n)$ for $n > n_1$; $c_1 > 0$, because $h \in \mathrm{O}(g)$.
- • $g(n) \leq c_2 f(n)$ for $n > n_2$; $c_2 > 0$, because $g \in \mathrm{O}(f)$.

→ Substituting the second inequality (•) into the first inequality (◦) leads to the inequality

$$h(n) \leq \underbrace{c_1 c_2}_{c;\, c>0} f(n) \text{ for } n > \underbrace{\max\{n_1, n_2\}}_{n_0}$$

proving the transitivity rule.

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Big-Oh: Transitivity

Informal meaning of the transitivity rule:

If function $h(n)$ grows at most as fast as $g(n)$,
which grows at most as fast as $f(n)$,
then $h(n)$ grows at most as fast as $f(n)$.

Examples:

- If $h \in O(g)$ and $g \in O(n^2)$, then $h \in O(n^2)$.
- If $\log_{10} n \in O(n^{0.01})$ and $n^{0.01} \in O(n)$, then $\log_{10} n \in O(n)$.
- If $n^{50} \in O(2^n)$ and $2^n \in O(3^n)$, then $n^{50} \in O(3^n)$.

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Big-Oh: Rule of Sums

Rule-of-sums (Lemma 1.17)

If $g_1 \in \mathrm{O}(f_1)$ and $g_2 \in \mathrm{O}(f_2)$, then $g_1 + g_2 \in \mathrm{O}(\max\{f_1, f_2\})$.

The proof:

- ∘ $g_1(n) \le c_1 f_1(n)$ for $n > n_1$, because $g_1 \in \mathrm{O}(f_1)$.
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→ Summing the inequalities (∘) and (•) leads to the inequality

$$\begin{aligned} g_1(n) + g_2(n) &\le c_1 f_1(n) + c_2 f_2(n) \\ &\le \max\{c_1, c_2\} \left(f_1(n) + f_2(n)\right) \\ &\le \underbrace{2 \cdot \max\{c_1, c_2\}}_{c;\, c>0} \cdot \max\left\{f_1(n), f_2(n)\right\} \end{aligned}$$

for $n > \underbrace{\max\{n_1, n_2\}}_{n_0}$, proving the rule of sums.

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Big-Oh: Rule of Sums

Informal meaning of the rule of sums:

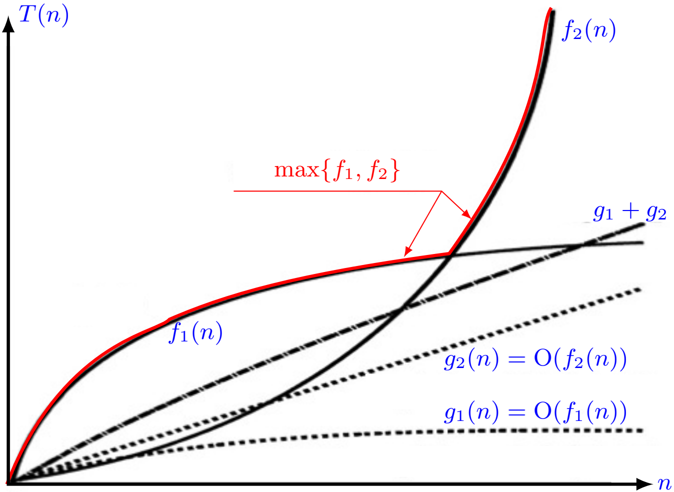
The sum of functions grows as its fastest-growing term. Therefore,

- If $g \in \mathrm{O}(f)$ and $h \in \mathrm{O}(f)$, then $g + h \in \mathrm{O}(f)$.
- If $g \in \mathrm{O}(f)$, then $g + f \in \mathrm{O}(f)$.
- If $g(n) = a_0 + a_1 n + \ldots + a_k n^k$ (a polynomial of degree k), then $g(n) \in \mathrm{O}(n^k)$.

Examples:

$$\begin{cases} \text{If} \quad h \in \mathrm{O}(n) & \text{and} \quad g \in \mathrm{O}(n^2), \quad \text{then} \quad g + h \in \mathrm{O}(n^2) \\ \text{If} \quad h \in \mathrm{O}(n \log n) & \text{and} \quad g \in \mathrm{O}(n), \quad \text{then} \quad g + h \in \mathrm{O}(n \log n) \end{cases}$$

Big-Oh: Rule of Sums



Big-Oh: Rule of Products

Rule-of-products (Lemma 1.18)

If $g_1 \in O(f_1)$ and $g_2 \in O(f_2)$, then $g_1 g_2 \in O(f_1 f_2)$.

The proof:

- ○ $g_1(n) \leq c_1 f_1(n)$ for $n > n_1$, because $g_1 \in O(f_1)$.
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proving the rule of products.

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proving the rule of products.

Big-Oh: Rule of Products

Informal meaning of the rule of products:

The product of upper bounds of functions gives an upper bound for the product of the functions.
Therefore,

- If $g \in O(f)$ and $h \in O(f)$, then $gh \in O(f^2)$.
- If $g \in O(f)$ and $h \in O(f^k)$, then $gh \in O(f^{k+1})$.
- If $g \in O(f)$ and $h(n)$ is a given function, then $gh \in O(fh)$.

Examples:

- If $h \in O(n)$ and $g \in O(n^2)$, then $gh \in O(n^3)$.
- If $h \in O(\log n)$ and $g \in O(n)$, then $gh \in O(n \log n)$.

Big-Oh: The Limit Rule

Suppose the ratio's limit $\lim_{n\to\infty}\frac{f(n)}{g(n)} = L$ exists (may be infinite, ∞).

$$\text{Then}\begin{cases}\text{if} & L = 0 & \text{then} & f \in \mathrm{O}(g)\\ \text{if} & 0 < L < \infty & \text{then} & f \in \Theta(g)\\ \text{if} & L = \infty & \text{then} & f \in \Omega(g)\end{cases}$$

When f and g are positive and differentiable functions for $x > 0$, but $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ or $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, the limit L can be computed using the standard **L'Hopital** rule of calculus:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

where $z'(x) \equiv \frac{dz(x)}{dx}$ denotes the first derivative of the function $z(x)$.

Example 1.22 (Textbook)

Prove that exponential functions grow faster than powers:

n^k is $\mathrm{O}(b^n)$ for all $b > 1$, $n > 1$, and $k \geq 0$.

The proof – either by induction, or what is simpler, by the limit rule using successive ($k + 1$ times) differentiation of $f(x) = x^k$ and $g(x) = b^x$ by x:

- Derivatives of $f(x) = x^k$ by x for $k \geq 0$:

$$\frac{dx^k}{dn} = kx^{k-1}; \qquad \frac{d^2x^k}{dx^2} = k(k-1)x^{k-2}; \quad \ldots$$

$$\frac{d^kx^k}{dx^k} = k(k-1)\cdots 2 \cdot 1 = k!; \quad \frac{d^{k+1}x^k}{dx^{k+1}} = 0$$

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$$\frac{d^kb^x}{dx^k} = b^x(\ln b)^k; \quad \frac{d^{k+1}b^x}{dx^{k+1}} = b^x(\ln b)^{k+1}$$

- Therefore, by the L'Hopital rule, the limit of the ratio

$$\lim_{n\to\infty} \frac{n^k}{b^n} = \lim_{n\to\infty} \frac{0}{b^n(\ln b)^{k+1}} = 0$$

for $b > 1$, proving that $n^k \in \mathrm{O}(b^n)$ for all $b > 1$, $n > 1$, and $k \geq 0$.

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$\log_b n$ is $\mathrm{O}(n^k)$ for all $b > 1$, $n > 1$, and $k > 0$.

The proof:

- The first derivative of $f(x) = x^k$ by x is $\frac{dx^k}{dx} = kx^{k-1}$.
- The first derivative of $g(x) = \log_b x$ by x is $\frac{d \log_b x}{dx} = \frac{1}{x \ln b}$.
- By the limit rule, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} (k \ln b) n^k = \infty$ for $n > 1$, $b > 1$, and $k \geq 0$, proving $n^k \in \Omega(\log_b n)$, i.e. $\log_b n \in \mathrm{O}(n^k)$.

As a result, $\log n \in \mathrm{O}(n)$; $n \log n \in \mathrm{O}(n^2)$, and $\log n \in \mathrm{O}(n^{0.0001})$.

$\log_b n$ is $\mathrm{O}(\log n)$ for all $b > 1$ because $\log_b n = \log_b a \times \log_a n$

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The proof:

- The first derivative of $f(x) = x^k$ by x is $\frac{dx^k}{dx} = kx^{k-1}$.
- The first derivative of $g(x) = \log_b x$ by x is $\frac{d \log_b x}{dx} = \frac{1}{x \ln b}$.
- By the limit rule, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} (k \ln b) n^k = \infty$ for $n > 1$, $b > 1$, and $k \geq 0$, proving $n^k \in \Omega(\log_b n)$, i.e. $\log_b n \in O(n^k)$.

As a result, $\log n \in O(n)$; $n \log n \in O(n^2)$, and $\log n \in O(n^{0.0001})$.

$\log_b n$ is $O(\log n)$ for all $b > 1$ because $\log_b n = \log_b a \times \log_a n$