Running Time Evaluation

Quadratic Vs. Linear Time

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COMPSCI 220 Algorithms and Data Structures

1. Running time

2. Examples

3. "Big-Oh", "Big-Omega", and "Big-Theta" Tools

4. Time complexity

Running Time $T(n)$: Estimation Rules

It is proportional to the **most significant term** in $T(n)$:

- n for a linear time, $T(n) = c_0 + c_1 n$; or
- n^k if $T(n) = c_0 + c_1 n + \ldots + c_k n^k$ for a polynomial time.

Once a problem size n becomes large, the most significant term is that which has the largest power of n.

- The most significant term increases faster than other terms which reduce in significance.

Constants of proportionality depend on a compiler, language, computer, programming, etc.

- It is useful to ignore the constants when analysing algorithms.
- Reducing constants of proportionality by using faster hardware or minimising time spent on the "inner loop" does not effect an algorithm's behaviour for a large problem!

Elementary Operations and Data Inputs

Basic elementary computing operations

- Arithmetic operations $(+; -; *; /; \%)$
- Relational operators ($==; != ; >; <; \geq; \leq$)
- Boolean operations (AND; OR; XOR; NOT)
- Branch operations
- Return

Input size for problem domains (meaning of n)

Sorting: n items
Graph / path: n vertices / edges
Image processing: n pixels (2D images) or voxels (3D images)
Text processing: n characters, i.e. the string length n

Estimating Running Time

Simplifying assumptions: all elementary statements / expressions take the same amount of time to execute, e.g. simple arithmetic assignments, `return`, etc.

- A single loop increases in time **linearly** as $\lambda \cdot T_{\text{body of a loop}}$ where λ is number of times the loop is executed.
- Nested loops result in **polynomial** running time $T(n) = cn^k$ if the number of elementary operations in the innermost loop is constant (k is the highest level of nesting and c is some constant).
- The first three values of k have special names:
  - **linear time** for $k = 1$ (a single loop);
  - **quadratic time** for $k = 2$ (two nested loops), and
  - **cubic time** for $k = 3$ (three nested loops).

Estimating Running Time

Conditional / switch statements like

if {condition} **then** {const time T_1} **else** {const time T_2}

are more complicated.

- One has to account for branching frequencies $f_{\text{condition=true}}$ and $f_{\text{condition=false}} = 1 - f_{\text{condition=true}}$:

$$T = f_{\mathrm{true}} T_1 + (1 - f_{\mathrm{true}})\, T_2 \leq \max\{T_1, T_2\}$$

Function calls:

$$T_{\mathrm{function}} = \sum T_{\mathrm{statements\ in\ function}}$$

Function composition:

$$T(f(g(n))) = T(g(n)) + T(f(n))$$

Estimating Running Time

Function calls in more detail: $T = \sum_i T_{\text{statement } i}$

```
...  x.myMethod( 5, ...  );
...
public void myMethod( int a, ...  )  {
    statements 1, 2, ..., M
}
```

Function composition in more detail: $T(f(g(n)))$:

- Computation of $x = g(n) \longrightarrow T(g(n))$
- Computation of $y = f(x) \longrightarrow T(f(n))$
- $T(f(g(n))) = T(g(n)) + T(f(n))$

Example 1.5: Textbook, p.19

Logarithmic time for a simple loop due to an exponential change

$$i = 1, k, k^2, k^3, \ldots, k^m$$

of the control variable in the range $1 \leq i \leq n$:

for $i \leftarrow 1$ **step** $i \leftarrow i * k$ **until** n **do**
    `...constant number of elementary operations`
end for

m iterations such that $k^{m-1} < n \leq k^m \longrightarrow T(n) = c\lceil \log_k n \rceil$

- The **ceil** $\lceil z \rceil$ of the real number z is the least integer not less than z.
- Additional conditions for executing inner loops only for special values of the outer variables also decrease running time.

Example 1.6: Textbook, p.19

Linearithmic $n \log n$ running time of the conditional nested loops:

$m \leftarrow 2$
for $j \leftarrow 1$ **to** n **do**
  if j == m **then**
    $m \leftarrow 2 * m$
    for $i \leftarrow 1$ **to** n **do**
      ...`constant number of elementary operations`
    end for
  end if
end for

The inner loop is executed k times for $j = m = 2, 4, \ldots, 2^k$

- $2^k \leq n < 2^{k+1}$ implies that $k \leq \log_2 n < k + 1$
- In total, $T(n)$ is proportional to kn, that is, $T(n) = n \lfloor \log_2 n \rfloor$.
- The **floor** $\lfloor z \rfloor$ is the greatest integer not greater than z.

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Exercise 1.2.1: Textbook

Is the running time quadratic or linear for the nested loops below?

$m \leftarrow 1$
for $j \leftarrow 1$ **to** n **do**
    if $j == m$ **then**
        $m \leftarrow (n-1) * m$
        for $i \leftarrow 1$ **to** n **do**
            ...`constant number of operations`
        end for
    end if
} **end for**

The inner loop is executed only twice, for $j = 1$ and $j = n - 1$; in total: $T(n) = 2n \rightarrow$ **linear** running time.

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"Big-Oh", "Big-Omega", and "Big-Theta" Tools

How does the relative running time change if the input size, n, increases from n_1 to n_2, all other things equal?

$$\text{By a factor of } \frac{T(n_2)}{T(n_1)} = \frac{cf(n_1}{cf(n_1)} = \frac{f(n_2)}{f(n_1)}$$

- "Big-Oh", "Big-Omega", and "Big-Theta" help to avoid imprecise statements like "*roughly proportional to...*"
- Can be applied to all non-negative-valued functions, $f(n)$ and $g(n)$, defined on non-negative integers, n.
- Running time is such a function, $T(n)$, of data size, n; $n > 0$.

Basic assumption:

Two algorithms have essentially the same complexity if their running times as functions of n differ only by a constant factor.

Definition of "Big-Oh", $g(n)$ is $\mathrm{O}(f(n))$

Let $f(n)$ and $g(n)$ be non-negative-valued functions, defined on non-negative integers, n.

Then $g(n)$ is $\mathrm{O}(f(n))$ (read "$g(n)$ is Big Oh of $f(n)$) **iff** there exists a positive real constant, c, and a positive integer, n_0, such that $g(n) \le cf(n)$ for all $n > n_0$.

- The notation "iff" is an abbreviation of "if and only if".
- Meaning: $g(n)$ is a member of the set $\mathrm{O}(f(n))$ of functions that increase **at most** as fast as $f(n)$, when $n \to \infty$.
- In other words, $g(n) \in \mathrm{O}(f(n))$ if $g(n)$ increases eventually at the same or lesser rate than $f(n)$, to within a constant factor.
- $g(n) \in \mathrm{O}(f(n))$ specifies a generalised "asymptotic upper bound", such that $g(n)$ for large n may approach closer and closer to $cf(n)$.

Definition of "Big-Omega", $g(n)$ is $\Omega(f(n))$

$g(n)$ is $\Omega(f(n))$ (read "$g(n)$ is Big Omega of $f(n)$) iff there exists a positive real constant, c, and a positive integer, n_0, such that $g(n) \geq cf(n)$ for all $n > n_0$.

- Meaning: $g(n)$ is a member of the set $\Omega(f(n))$ of functions that increase **at least** as fast as $f(n)$, when $n \to \infty$.
- In other words, $g(n) \in \Omega(f(n))$ if $g(n)$ increases eventually at the same or larger rate than $f(n)$, to within a constant factor.
- "Big Omega" is complementary to "Big Oh" and generalises the concept of "asymptotic lower bound" ($\geq_{n\to\infty}$) just as "Big Oh" generalises the asymptotic upper bound ($\leq_{n\to\infty}$).
- If $g(n)$ is $\mathrm{O}(f(n))$, then $f(n)$ is $\Omega(g(n))$.

Definition of "Big Theta", $g(n)$ is $\Theta(f(n))$

$g(n)$ is $\Theta(f(n))$ (read "$g(n)$ is Big Theta of $f(n)$) iff there exist two positive real constants, c_1 and c_2, and a positive integer, n_0, such that $c_1 f(n) \leq g(n) \leq c_2 f(n)$.

- Meaning: $g(n)$ is a member of the set $\Theta(f(n))$ of functions that increase as fast as $f(n)$, when $n \to \infty$
- Im other words, $g(n) \in \Theta(f(n))$ if $g(n)$ increases eventually at the same rate as $f(n)$, to within a constant factor.
- "Big Theta" generalises the concept of "asymptotic tight bound".
- If $g(n) \in \mathrm{O}(f(n))$ and $f(n) \in \mathrm{O}(g(n))$, then $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(f(n))$, i.e. both algorithms are of the same time complexity.

Proving $g(n)$ is $O(f(n))$, or $\Omega(f(n))$, or $\Theta(f(n))$

Proving the 'Big-X" property means finding constants, (c, n_0) or (c_1, c_2, n_0) specified in Definitions.

- It might be done by a chain of inequalities, starting from $f(n)$.
- Mathematical induction can be used in more intricate cases.

Proving $g(n)$ is **not** "Big-X" of $f(n)$ finds the required constants do not exist, i.e. lead to a contradiction.

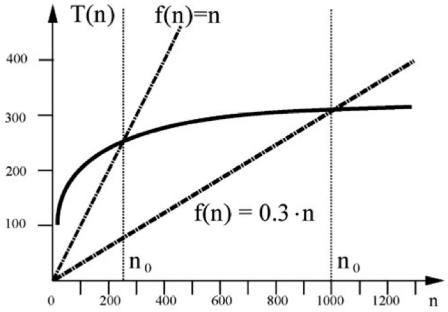
Example 1: Prove that $g(n) = 5n^2 + 3n$ is not $O(n)$.

If $g(n) = 5n^2 + 3n \leq c \cdot n$ for $n > n_0$, then for any n_0 the factor $c > 5n_0 + 3$, i.e. it cannot be constant. Therefore, $g(n) \notin O(n)$.

Example 2: Prove that $g(n) = 5n^2 + 3n$ is $\Omega(n)$.

If $g(n) = 5n^2 + 3n \geq c \cdot n$ for $n > n_0$, then for any n_0 there exist the required factor $c < 5n_0 + 3$. Therefore, $g(n) \in \Omega(n)$.

Time Complexity of Algorithms



$$T(n) = 100 \log_{10} n$$

$T(n) \leq n$ for all $n > 238$

$T(n) \leq 0.3n$ for all $n > 1000$

$$T(n) \in \mathrm{O}(n)$$

In analysing running time, $T(n) \in \mathrm{O}(f(n))$, functions $f(n)$ measure approximate time complexity like $\log n$, n, n^2 etc.

- Polynomial algorithms: $T(n)$ is $O(n^k)$; $k = \text{const}$.
- Exponential algorithms otherwise.

Intractable problems: if no polynomial algorithm is known for solution.

Time Complexity Growth

$f(n)$	Approximate number of data items processed per:			
	1 minute	1 day	1 year	1 century
n	10	$14,400$	5.3×10^6	5.3×10^8
$n \log_{10} n$	10	$3,997$	8.8×10^5	6.7×10^7
$n^{1.5}$	10	$1,275$	$65,128$	1.4×10^6
n^2	10	379	$7,252$	$72,522$
n^3	10	112	807	$3,746$
2^n	10	20	29	35

Beware Exponential Complexity!

- A linear, $O(n)$, algorithm processing $\mathbf{10}$ items per minute, can process $\mathbf{1.4 \times 10^4}$ items per day, $\mathbf{5.3 \times 10^6}$ items per year, and $\mathbf{5.3 \times 10^8}$ items per century.
- An exponential, $O(2^n)$, algorithm processing **10** items per minute, can process only **20** items per day and only **35** items per century...

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Big-Oh vs. Actual Running Time

Example 1:

Algorithms A and B with running times $T_A(n) = 20n$ time units and $T_B(n) = 0.1n\log_2 n$ time units, respectively.

- In the "Big-Oh" sense, the linear algorithm A is better than the linearithmic algorithm B...
- **But:** on which data volume can A outperform B, i.e. for which value n the running time for A is less than for B?

$$\begin{array}{lll} & T_A(n) < T_B(n) & \text{if} \quad 20n < 0.1n\log_2 n, \\ \text{or} & \log_2 n > 200, & \text{that is, when } \mathbf{n > 2^{200} \approx 10^{60}}! \end{array}$$

Thus, in all practical cases the algorithm B is better than A...

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Example 2:

Algorithms A and B with running times $T_A(n) = 20n$ time units and $T_B(n) = 0.1n^2$ time units, respectively.

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- **But:** on which data volume can A outperform B, i.e. for which value n the running time for A is less than for B?

$$T_A(n) < T_B(n) \text{ if } 20n < 0.1n^2, \text{ or } n > 200$$

Thus the algorithm A is better than B in most of practical cases except for $n < 200$ when B becomes faster...

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