Teaching Matters: Lecturer and Assessment

COMPSCI 220 S1 C: Lecturer and supervisor

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Assessment (separate pass of theoretic and practical sections!):

28% 4 assignments ($4 \times 7\%$).

12% Closed-book written test (April 14, 2016, the lecture time).

60% Closed-book written examination.

Textbook:
M. J. Dinneen, G. Gimel'farb, and M. C. Wilson,
INTRODUCTION TO ALGORITHMS AND DATA STRUCTURES.
4^{th} ed.: e-book; 2016 (on the class website).

Teaching Matters: Lectures and Assignments

ASSIGNMENTS:

#	Topic	Out date	Due date
1	Analysing complexity of algorithms	March 7	March 21
2	Data sorting efficiency	March 24	April 8
	Midterm test (lecture time):	April 14; 4–5 p.m.	
3	Data search efficiency	April 15	May 6
4	Graph algorithms	May 9	May 27

LECTURES:

Week	Topic	Days	Lect.#s
1–2	Complexity of algorithms	6	01 – 06
3–4	Sorting algorithms	5	07 – 11
5–6	Search algorithms	6	12 – 17
7–8	**Midterm test** / *break*		
9–13	Graphs and graph algorithms	16	18 – 33

Teaching Matters: Class Representative

Class representative - what are your responsibilities?

- Provide a link between staff and students to affect decision making
- Represent the collective views of your class
- Represent course at Student-Staff Consultative Committee meetings
- Work with the AUSA Educational Vice-President on larger issues within faculties and University as a whole
- Continually provide feedback on progress made on issues
- Help promote AUSA services and events to your classes
- Ensure AUSA surveys are completed by your classes

Benefits of being a class rep:

- Important and recognised addition to your resume
- Improve your leadership skills set
- Ability to make significant changes to your education
- Collaborate and work with AUSA

Key Learning Outcomes

After this course, you will know how to:

- Use basic asymptotic notation to express and predict algorithm performance on large input.
- Compare performance of various algorithms in a given situation and select the best one.
- Write a recurrence describing performance of a formally or informally described algorithm and solve that recurrence.
- Compare performance of basic data structures (lists, trees, tables, and graphs) for a given problem and select the best one.
- Use and program the fast algorithms for standard graph depth- or breadth-first search and optimisation, such as, e.g., the shortest paths or minimum spanning trees.

Note these are minimal expectations and excellence requires practice!

COMPSCI 220

Math Background: Sets, Logarithms, Series, ...

Lecturer: Georgy Gimel'farb

Important notes regarding slides and other course materials:

Drs. Nevil Brownlee, Michael J. Dinneen, Georgy Gimel'farb, Simone Linz, Ulrich Speidel, Mark C. Wilson, and others contributed to these teaching aids.

But all the shortcomings should be attributed to only the current lecturer...

Sets of Elements

A set X is an unordered collection of zero or more elements.

1. $X = \{3, 4, 5, 6, 7\}$ – the set of integers from 3 to 7.
2. $X = \{\text{Aragorn},\text{Boromir},\text{Frodo},\text{Gandalf}\}$ – "The Lord of the Rings" heroes.
3. $X = \{\mathrm{A}, \alpha, \mathrm{B}, \beta, \ldots, \Omega, \omega\}$ – the Greek alphabet.

Equivalent specifications:

- $X = \{3, 5, 6, 7, 4\}$, $\{6, 7, 4, 3, 5\}$, $X = \{7, 3, 5, 6, 4\}$ etc.
- $X = \{x \mid x \text{ is an integer and } x \geq 3 \text{ and } x \leq 7\}$
  - Reads "X is the set of all x such that x is an integer and x is greater than or equal to 3 and x is less than or equal to 7".
  - Common notation: "$\{x \mid \ldots\}$" or "$\{x : \ldots\}$ reads "the set of all x for which [some condition applies]".

$x \in X$ reads "x is an **element** of X: $5 \in \{3, 7, 6, 5, 4\}$, but $8 \notin \{3, 7, 6, 5, 4\}$.

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Cardinalities, Sets, and Subsets

$|X|$ – the number of elements, or the *cardinality* of a set X.

- $|\{3,4,5,6,7\}| = 5$
- $|\ \{\alpha,\beta,\gamma,\delta,\epsilon,\zeta,\eta,\theta,\iota,\kappa,\lambda,\mu,\nu,\xi,o,\pi,\rho,\sigma,\tau,\upsilon,\phi,\chi,\psi,\omega\}| = 24.$
- Zero cardinality of the empty set, $\emptyset$, with no elements: $|\emptyset| = 0$.
- Sometimes the cardinality of X is denoted $\#X$ or $\operatorname{card} X$.

A set Y whose elements are all also elements in X is a subset of X.

1. Y is a subset of X: $\begin{cases} Y \subset X \text{ if } |Y| < |X| \text{ (a "true" subset)} \\ Y \subseteq X \text{ if } |Y| \le |X| \end{cases}$
   - $\{3,4\} \subset \{3,5,6,7,4\}$, but $\{3,7,9\} \not\subset \{3,4,5,6,7\}$
2. X is a **superset** of Y: $\begin{cases} X \supset Y \text{ if } Y \subset X \text{ (a "true" superset)} \\ X \supseteq Y \text{ if } Y \subseteq X \end{cases}$

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Unions, Intersections, and Complements of Sets

The **union** of two sets, X and Y, is the set of the elements of X and elements of Y: $X \bigcup Y = \{x \mid x \in X \text{ OR } x \in Y\}$.

If $X = \{3, 5, 6, 7, 4\}$ and $Y = \{2, 5, 7\}$, then $X \bigcup Y = \{2, 3, 4, 5, 6, 7\}$.

The **intersection** of two sets, X and Y, is the set of the elements that are in both X and Y: $X \bigcap Y = \{x \mid x \in X \text{ AND } x \in Y\}$.

- If $X = \{3, 5, 6, 7, 4\}$ and $Y = \{2, 5, 7\}$, then $X \bigcap Y = \{5, 7\}$.

Two sets, X and Y, with no common elements are **disjoint**: $X \bigcap Y = \emptyset$.

- If $X = \{3, 5, 6, 7, 4\}$ and $Y = \{2, 8, 9\}$, then $X \bigcap Y = \emptyset$.

The **complement** of a subset Y of X is the set of all elements of X that are not in Y: $X \backslash Y = \{x \mid x \in X \text{ AND } x \notin Y\}$.

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Summing a Set of Values: $\sum\limits_{x \in X} f(x) = \sum_{x \in X} f(x)$

A sum is denoted by $\sum$, the uppercase Greek letter for "sigma".

- $f(x)$ – a scalar function of a numerical argument, $x \in X$.
- For the integer $i \in \mathbb{I} = \{1, 2, 3, 4, 5\}$, and $f(i) = i^2$:

$$\sum_{i=1}^{5} f(i) = \textstyle\sum_{i=1}^{5} f(i) = \displaystyle\sum_{i \in \mathbb{I}} f(i) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

  - Both a subscript, $_{i=1}$, and a superscript, ^{5}: the start and the end value for the variable, i, incrementing by 1 for each term of the sum.
  - **Only** a subscript, $_{i \in \mathbb{I}}$: summing up the term after $\sum$ for all $i \in \mathbb{I}$.
- For $X = \{x_1, x_2, x_3, x_4\}$, $\sum_{x \in X} f(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$:

$$\begin{cases} X = \{0, -1, 5, -2\} \quad \Longrightarrow \\ \\ \displaystyle\sum_{x \in X} f(x) = \sum_{i=1}^{4} f(x_i) = 0^2 + (-1)^2 + 5^2 + (-2)^2 = 30 \end{cases}$$

Sequences of Numbers, $a_0, a_1, a_2, \ldots$, and Series

A **sequence** of the length n: an **ordered list** of n terms (elements).

- **Arithmetic** sequences: a first term, a_0, and a constant difference, $\Delta = a_i - a_{i-1}$; $i = 1, 2, 3, \ldots$.
  - The i^{th} term of an arithmetic sequence: $a_i = a_0 + i\Delta$.
- **Geometric** sequences: a first term, a_0, and a constant ratio, $\rho = \frac{a_i}{a_{i-1}}$; $i = 1, 2, 3, \ldots$.
  - The i^{th} term of a geometric sequence: $a_i = a_0\rho^i$.
- Finite, $n < \infty$, and infinite, $n = \infty$, sequences.

A **series** sums up the terms of an infinite sequence: $\sum_{i=0}^{\infty} a_i$.

- A sequence, $\{s_0, s_1, s_2 \ldots\}$, of partial sums: $s_n = \sum\limits_{i=0}^{n} a_i$.
- The partial sum for an arithmetic sequence: $s_n = (n+1)a_0 + (0+1+\ldots+n))\Delta = (n+1)a_0 + \frac{n(n+1)}{2}\Delta$.
- **Convergent** series: the partial sums have a finite limit, $\left|\lim\limits_{n\to\infty} s_n\right| < \infty$.

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Telescoping Series for an Infinite Sequence $a_0, a_1, \ldots$

Telescoping series is defined as

$$\left\{ \begin{array}{ccccccccccc} a_0 & + & a_1 & + & a_2 & + & \ldots & + & a_n & & \\ & - & a_1 & - & a_2 & - & \ldots & - & a_n & - & a_{n+1} \end{array} \right\};\ n = 1, 2, \ldots, \infty:$$

$$\sum_{i=0}^{\infty} (a_i - a_{i+1}) = \lim_{n\to\infty} (s_n - s_{n+1}) = \lim_{n\to\infty} (a_0 - a_{n+1}) = a_0 - \lim_{n\to\infty} a_n$$

- The telescoping series converges if the sequence terms, a_i; $i = 0, 1, \ldots, \infty$, converge to a finite limit, $a_{\lim} = \lim_{i\to\infty} a_i$.
- The value of the convergent telescoping series:

$$\sum_{i=0}^{\infty} (a_i - a_{i+1}) = a_0 - a_{\lim}$$

Rounding: replacing a real number x with the closest integer.

1. **Ceil** notation: $\lceil x \rceil$ rounds up to the nearest integer larger than or equal to x, e.g., $\lceil 3.2 \rceil = 4$.
2. **Floor** notation: $\lfloor x \rfloor$ rounds down to the nearest integer smaller than or equal to x: e.g., $\lfloor 3.2 \rfloor = 3$.

Exponential functions a^x with the base a and the exponent x:

- For integers, i and any real number a, a^i is equal to a multiplied i times; however, i may be and, in fact, often is a real number.
- Simple rules for exponential functions:

$$a^{-x} = \frac{1}{a^x}; \quad a^{x+y} = a^x \cdot a^y; \quad a^{x-y} = \frac{a^x}{a^y}; \quad (a^x)^y = a^{x \cdot y}$$

- Special case: e^x where e is Euler's (pronounced "Oil-ah") constant ($e \approx 2.718\ldots$): the function e^x is its own derivative: $\frac{d}{dx}e^x = e^x$.
  - Using the natural logarithm (see Slide 14), $\ln$, all functions a^κ are mapped into e^x by setting $x = \kappa \ln a$, because $e^{\ln a} = a$.

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Logarithms: $\log y$

The inverse of exponential functions: if $a^x = y$, then $\log_a y = x$.

- $\log_a y = x$ reads "x is the logarithm to base a of y".
  - Logarithm to another base differs by a factor: $\log_a y = \log_a b \cdot \log_b y$ because $y = a^{\log_a y} = b^{\log_b y} = ((a^{\log_a b})^{\log_b y} = a^{\log_a b \cdot \log_b y}$.
  - Thus, the base is often neglected by writing $\log y$ instead of $\log_a y$.
- Commonly used logarithms:
  - "logarithm to base 2" (in computing)
  - "logarithm to base 10" (in engineering)
  - "logarithm to base $e \approx 2.718\ldots$" (the "natural logarithm"):

    $\ln x$ always means $\log_e x$.
- Simple rules for logarithms (from those for exponential functions):

$$\log(x \cdot y) = \log x + \log y; \quad \log\left(\frac{x}{y}\right) = \log x - \log y;$$

$$\log(x^y) = y \cdot \log x$$

Mathematical Induction

It proves that a math statement is true for all integers, such that $n \geq n_0$, where n_0 is usually a non-negative constant.

- If the proof should be for all non-negative integers, $n_0 = 0$.
- If the proof should be for all positive integers, $n_0 = 1$.

Proof by mathematical induction (or simply induction):

1. **Basis:** Prove that the statement is true for n_0.
2. **Induction hypothesis:** Assume that the statement is true for some n.
3. **Inductive step from n to $n+1$:** If the induction hypothesis holds, prove that the statement is also true for $n+1$.

The inductive step completes the proof.

Example 1: The Gauss Formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Prove by the math induction that for all $n \geq 1$,

$$S_n = 1 + 2 + \ldots + n \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1. *Basis*: For $n_0 = 1$, the statement is correct: $S_1 = 1 = \frac{1 \cdot 2}{2}$.
2. *Induction hypothesis*: $S_n = \frac{n(n+1)}{2}$ holds for n, i.e.,

$$1 + \ldots + n = \frac{n(n+1)}{2}$$

3. *Inductive step*:

$$S_{n+1} = S_n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

Hence, the same formula holds for $n+1$, so that the formula is valid for all $n \geq 1$.

Mathematical Induction: Example 2

Prove that $S_n = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$.

1. *Basis*: For $n_0 = 1$, the statement is correct: $S_1 = 1 = \frac{1 \cdot 2 \cdot 3}{6}$.
2. *Induction hypothesis*: $S_n = \frac{n(n+1)(2n+1)}{6}$ holds for n, i.e.,

$$1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. *Inductive step*: $S_{n+1} = S_n + (n+1)^2 =$

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)[n(2n+1)+6(n+1)]}{6} =$$

$$\frac{(n+1)[2n^2+7n+6]}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

i.e., the same formula holds for $n+1$.

Mathematical Induction: Example 3

Prove that $S_n = 1 + a + a^2 + \ldots + a^n = \frac{a^{n+1}-1}{a-1}$ for $n \geq 0$ and $a \neq 1$ (the geometric series with the ratio a).

1. *Basis*: For $n_0 = 0$, the statement is correct: $S_1 = 1 = \frac{a-1}{a-1}$.
2. *Induction hypothesis*: $S_n = \frac{a^{n+1}}{a-1}$ holds for n, i.e.,

$$1 + \ldots + a^n = \frac{a^{n+1} - 1}{a - 1}$$

3. *Inductive step*: $S_{n+1} = S_n + a^{n+1} = \frac{a^{n+1}-1}{a-1} + a^{n+1}$

$$= \frac{a^{n+1} - 1 + a^{n+2} - a^{n+1}}{a - 1} = \frac{a^{n+2} - 1}{a - 1}$$

i.e., the same formula holds for $n + 1$.

Systematic Analysis of Algorithms

Some COMPSCI 220 contexts use systematically, step-by-step, the established mathematical tools: definitions, lemmas, and theorems.

- A **definition** is used to make it clear what a certain term means, what we are going to call something, or how we will be using a certain notation.
- A **theorem** is a statement we claim to be true, and it always requires a **proof**.
- A **lemma** is like a small theorem that we prove to lead us up to the proof of a more extensive theorem.
- As a general rule, we do not claim that something is true unless we can also prove it.