COMPSCI 210 S1T
Computer Systems

Floating Point Numbers

Agenda & Reading

- Agenda:
  - Introduction
  - Normalization
  - IEEE 754 Floating Point Representation
    - Single Precision (32-bit)
    - Double Precision (64-bit)
  - Examples
  - Special Numbers
  - Conversion
- Recommended Reading:
  - IEEE Floating Point Numbers
    - http://en.wikipedia.org/wiki/IEEE_floating-point_standard

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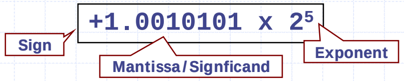
Introduction

- Given the following numbers:
  - $37.25_{10} = 100101.01_2$
  - $7.625_{10} = 111.101_2$
  - $0.3125_{10} = 0.0101_2$
  - $+300000000$ ms^{-1}
  - $+299, 792,458$ ms^{-1}
- How can we store the above number in 4 bytes?
- People usually represent very large and small numbers in “scientific notation”, as a fixed point number times a power of 10.
  - Scientific Notation:
    - $3 * 10^8$
    - $2.9979 * 10^8$
    - $3.125 * 10^{-1}$
    - $3.725 * 10^1$
  - Floating point numbers are represented in the computer in a similar manner, but using base 2 rather than base 10.
    - 1.0010101×2^5
    - 1.11101×2^2
    - 1.01×2^{-2}

Normalization

- Given the following number
  - $12.34 = 0.1234 \times 10^2 = 1.234 \times 10^1 = 123.4 \times 10^{-1}$
- Scientific numbers are always written with one digit before the point, giving a “normalized” representation.
  - After Normalization = 1.234×10^1
- Normalization in Base 2
  - One digit to the left of the binary point. It must be 1.
  - Examples:
    - $100101.01_2 = 1.0010101 \times 2^5$ **Radix point move to left by 5 places**
    - $111.101_2 = 1.11101 \times 2^2$ **Radix point move to left by 2 places**
    - $0.0101_2 = 1.01 \times 2^{-2}$ **Radix point move to right by 2 places**
- After normalization, the numbers now have a standard format

Floating Point



- Consists of three parts
  - Sign bit
  - Exponent
  - Mantissa/Signficand
  - Note: Normalized binary numbers always start with a 1 (the leftmost bit of the signficand value). We don't need to store the 1. IEEE 754 representation uses this idea. All numbers must be normalized.

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- IEEE 754 Floating Point Representation
  - Single Precision (32 bits)
    - a 1-bit sign, an 8-bit exponent with a bias of 127, and a 23-bit significand
    - (–1)*sign* *(1.0+*significand*) *2 $^{(exponent–127)}$
    - Example:

```
0  10000100 00101010000000000000000
```

  - Double Precision (64 bits)
    - a 1-bit sign, an 11-bit exponent with a bias of 1023, and a 52-bit significand
    - (-1)*sign* * (1.0 + *significand*) * 2 $^{(exponent–1023)}$
    - Example:

```
0  10000000100
0010101000000000000000000000000000000000000000000000
```

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IEEE 754 Floating Point Representation

- Single Precision (float)
  - Sign Bit: (1 bit)
    - 0 +ve, 1 –ve
  - Exponent Bits: (8 bits)
    - Excess/Biased representation
      - To store positive and negative exponents
        - Subtracting a bias 127 (2^8 -1) from the value
        - 0 < e < 255; Actual exponent is: E = e - 127
      - Example:
        - 00000001 is the representation of -126;
        - 10000000 is the representation of +1
        - An exponent of 5 is therefore stored as 127+5 = 132 (10000100);
        - An exponent of -2 is stored as = 127-2 = 125 (1111101)
  - Mantissa Bits: (23 bits)
    - Mantissa is the set of 0's and 1's to the right of the radix point of the normalized binary number
      - 1.0010101 x 2^5 , Mantissa = 00101010...0
      - Numbers stored in normalized form; i.e. 1.xxx..., therefore it is assumed (and therefore "1" is not stored) in the format

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IEEE 754 Floating Point Representation

- Double Precision (double)
  - Sign Bit: (1 bit)
    - 0 +ve, 1 –ve
  - Exponent Bits: (11 bits)
    - Excess/Biased representation
      - Subtracting a bias 1023 from the value
        - Note: The exponent is 11 bits, so the bias = 2^{11} -1
      - Example:
        - 00000000001 is the representation of -1022
        - 10000000000 is the representation of +1
  - Mantissa Bits: (55 bits)
    - Mantissa is the set of 0's and 1's to the right of the radix point of the normalized binary number
      - Numbers stored in normalized form; i.e. 1.xxx..., therefore it is assumed (and therefore "1" is not stored) in the format

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Examples

- 32-bit IEEE 754 Floating point representation
  - 0 01111101 101 0000 0000 0000 0000 0000
    - Sign = 0
    - Exponent = 01111101 = 125 =>
      - 125 – 127 = -2
    - Mantissa = 1010...0 = 0.5 + 0.125 = 0.625
    - Answer = $(-1)^0$ * (1.0 + 0.625) * 2 $^{(-2)}$
    - = (1.625) * 2 $^{(-2)}$ = 0.40625
- 64-bit IEEE 754 Floating point representation
  - 0 01111111101 101 0000 0000 0000 0000 0000 … 0
    - Sign = 0
    - Exponent = 01111111101 = 1021 =>
      - 1021 – 1023 = -2
    - Mantissa = 1010...0 = 0.5 + 0.125 = 0.625
    - Answer = $(-1)^0$ * (1.0 + 0.625) * 2 $^{(-2)}$
    - = (1.625) * 2 $^{(-2)}$ = 0.40625

Special Numbers

- Special Exponents (Single)
  - 00000000
    - Case 1: Represent number in Denormalized format!
      - Exponent is all zeros, the floating-point number is Denormalized
      - = (0.0+*significand*) *2 $^{(exponent–bias)}$
    - Case 2: Represent ZERO with all zeros in exponent and mantissa bits
      - +0, -0
  - 11111111
    - NaN
      - An exponent of all ones with any other mantissa is interpreted to mean "not a number". Example: 0 11111111 00000000000000000000001
    - Infinity:
      - An exponent of all ones with a mantissa whose bits are all zero indicates an infinity. The sign of the infinity is indicated by the sign bit. Example: 0 11111111 00000000000000000000000

Table

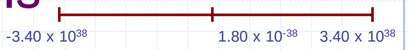
Range Name		
-NaN	1 11...11 00...01	FF800001
-Infinity	1 11...11 00...00	FF800000
Negative Normalized	1 11..10 11..11	FF7FFFFF
	...	
Negative Normalized	1 00...01 00..00	80800000
Negative Denormalized	1 00..00 11...11	807FFFF
	...	
Negative Denormalized	1 00..00 00...01	80000001
-0	1 00..00 00..00	80000000
+0	0 00..00 00..00	00000000
Positive Denormalized	0 00..00 00...01	00000001
	...	
Positive Denormalized	0 00..00 11...11	007FFFFF
Positive Normalized	0 00..01 00...00	00800000
	...	
Positive Normalized	0 11..10 11...11	7F7FFFFF
+Infinity	0 11..11 00..00	7F800000
+NaN	0 11..11 00..01	7F800001

Conversion:

- Example 1: Decimal -> IEEE Floating Point Representation
  - -1.25_{10}
  - Binary = -1.01
  - Normalization => -1.01
    - = -1.01 * 2^0
  - Bits:
    - Exponent = 127 = 01111111
    - Sign = 1
    - Mantissa = 010...0
  - Answer = 1 01111111 010...0 = BFA00000
- Example 2: IEEE Floating Point Representation -> Decimal
  - 40900000 = 0100 0000 1001 0000 ... 0000
    - Sign = 0
    - Exponent = 100 0000 1
      - = 129 = 129-127
    - Mantissa = 001 0000 ... 0000
      - = 0.125
    - Answer = $(-1)^0$ * (1.0 + 0.125) * 2 $^{(2)}$ = 4.5 q

```
float num = -1.25;
int temp;
temp = *((int*) &num); //convert the float to the Hex bit patterns
printf("\nThe number is %f = %x", num, temp);
```

Range & Problems



- Range:
  - Magnitude of numbers that can be represented is in the range:
  - Single Precision (Normalized)
    - 2^{-126} * (1.0) to 2^{127} * (2 -2^{-23})
      - 2^{-126} * (1.0): Exponent: 00000001 Mantissa: 000...000
      - 2^{127} * (2 -2^{-23}): Exponent: 11111110, Mantissa: 111...111 = 3.40 x 10^{38}
    - = 1.18 x 10^{-38} to 3.40 x 10^{38} (approximately)
  - Double Precision
    - 2^{-1022}*(1.0) to 2^{1023}*(2 – 2^{-52})
    - = 2.23*10^{-308} to 1.8*10^{308} (double)
- Problems
  - Overflow occurs when the exponent is larger than the allocated space
  - Underflow occurs when a negative exponent is too large in absolute value to fit within the bits allocated to store it
  - Truncation occurs when there are not enough digits in the mantissa to represent the number:
    - Examples: 1/3, 1/10 etc