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Abstract

The aim of this note is to introduce the notion of *random sequences of reals* and to prove that the answer to the question in the title is negative, as anticipated by the informal discussion of Longpré and Kreinovich [15].

Keywords: Riemann zeta-function, random real, random sequence of reals

1 Introduction

Riemann's Hypothesis, a famous open problem of mathematics, states that all complex roots (zeros) $s = \mathrm{Re}(s) + i\,\mathrm{Im}(s)$ of the Riemann's zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(i.e., the values for which $\zeta(s) = 0$) are located on the straight line $\mathrm{Re}(s) = 1/2$ in the complex plane (except for the known zeros, which are the negative even integers). This hypothesis has appeared as Problem No. 8 in Hilbert's famous 1900 list of 23 open problems (see Hilbert [11], Browder [1], and Karatsuba and Voronin [13]).

It has been proven that the real parts of the non-trivial zeros s of the Riemann's zeta-function are close to $1/2$, so they form a highly organized set. In fact a large proportion of them lies on the line $\mathrm{Re}(s) = 1/2$. The *imaginary* parts $\mathrm{Im}(s)$ of the same zeros s are far from displaying any order and here is the argument.

Let us note that the zeta-function has infinitely many, but only countably many zeros. The non-trivial zeros lie in the stripe $0 < \mathrm{Re}(s) < 1$ and are symmetric with respect to the real axis. Hence it is sufficient to consider the zeros with positive imaginary part. Since they do not have an accumulation point in the complex plane, we can order them to a sequence $s_k = \mathrm{Re}(s_k) + i\,\mathrm{Im}(s_k)$, $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, by the size of the imaginary

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part. (Zeros with the same imaginary part–if any–are ordered by their real parts and multiple zeros are listed according to their multiplicities.)

Rademacher [9] has proven — using Riemann's Hypothesis — that the imaginary parts of this sequence form a *uniformly distributed sequence.* Hlawka [12] has shown how to prove Rademacher's result without using Riemann's Hypothesis. More exactly, the Rademacher-Hlawka theorem states that for every real number $t \neq 0$, the fractional parts $f_k(t)$ of the sequence

$$\frac{t}{2\pi} \cdot \mathrm{Im}(s_k)$$

are uniformly distributed in the sense that for every subinterval $[a,b] \subseteq [0,1]$, the portion of points $f_0(t), \ldots, f_N(t)$ that belong to this subinterval tends to $b-a$ as $N \to \infty$. In the language of probabilities, the uniform distribution of the sequence $f_k(t)$ means that the probability of $f_k(t)$ lying within an interval coincides with the measure of the interval.

A real x is (Borel) *normal* to the base $b \geq 2$ in case for every non-negative integer $n \geq 1$, each block of digits $0, 1, \ldots, b-1$ of size n will occur in the limit exactly b^{-n} of the time. There is a close relation between the properties of uniform distribution and normality (see Kuipers and Niederreiter [14]): *The number x is normal to the base b iff the sequence $(b^n x)_{n \geq 0}$ is uniformly distributed.* Normality is a weaker form of Chaitin/Martin-Löf randomness (for various equivalent definitions see Chaitin [7], Martin-Löf [16], Calude [4]): *a random sequence is normal in every base, but the converse implication fails to be true* (cf. Calude [3, 4]).

2 The Problem

The above facts led the first author ([4], 219; see also [5]) to ask the question:

> *Do zeros of Riemann's zeta-function form a random sequence?*

Longpré and Kreinovich [15] have argued that the answer is *negative*:

> For a computable real number t, the sequence formed by merging the binary digits of the fractional parts $f_0(t), f_1(t), \ldots$ is not random because the zeros of Riemann's zeta-function can be computed algorithmically with any given accuracy. Thus, the sequence of zeros is algorithmic and hence, it cannot be random (cf. Calude [4], 138-139).

We agree with the above intuition and we will offer a complete solution along the strategy suggested by Longpré and Kreinovich. Before going to technicalities we would like to make several comments.

First, one of the goals of the question was to invite researchers to define the notion of a *random (infinite) sequence of reals.* This part of the question is methodological since approaches can be quite different and inequivalent. It is desirable that any definition implies that any random sequence of reals is uniformly distributed.

Secondly, suppose that "we have" a definition of random sequence of reals. Consider the set

$$Z = \{\mathrm{Im}(s_k) | k \geq 0\}$$

of imaginary parts of all non-trivial zeros (with positive imaginary part) of the Riemann's zeta-function. An important point about this notation is that it does *not* specify how the sequence $\mathrm{Im}(s_k)$ was being constructed. Therefore, in order to answer the original question one has to specify how the imaginary parts of the zeros are being (effectively) enumerated. As we have seen, the Rademacher-Hlawka theorem tells us that with respect to a specific, natural enumeration, the sequence of imaginary parts of the zeros of the Riemann's zeta-function is uniformly distributed modulo 1.

Two questions arise: a) is this sequence random?, b) is the sequence defined by another enumeration of the zeros random?

Note that neither randomness nor uniform distribution are invariant with respect to arbitrary enumerations. Thus, the true nature of the original question was *to develop an appropriate theory of randomness for sequences of reals*, and, according to it, *to decide whether a sequence of imaginary parts of the zeros of Riemann's zeta-function is or not random.*

3 Random Sequences of Real Numbers

In this section we give several definitions of random sequences of real numbers which lead to a possible way to make formal and precise the ideas of Longpré and Kreinovich [15]. We prove the useful fact that a sequence of real numbers which contains a non-random real is non-random itself. Furthermore, we show that each random sequence of real numbers is uniformly distributed.

We start by recalling the definition of a Chaitin/Martin-Löf random sequence [7, 16, 4]. Let Σ be a finite alphabet and $\Sigma^\omega = \{p : \mathbb{N} \to \Sigma\}$ be the set of all infinite sequences over Σ. For a finite word $w \in \Sigma^*$ we denote by $w\Sigma^\omega = \{p \in \Sigma^\omega \mid p(0) \ldots p(|w|-1) = w\}$ the open subset of all infinite sequences with prefix w. Let μ be the usual measure on Σ^ω, defined by $\mu(w\Sigma^\omega) = 2^{-|w|}$, for all $w \in \Sigma^*$. Let $\nu : \mathbb{N} \to \Sigma^*$ be the quasi-lexicographical ordering. Finally, let $\langle\ .,.\ \rangle : \mathbb{N}^2 \to \mathbb{N}$ be the bijective pairing function defined by $\langle i, j\rangle = \frac{1}{2}(i+j)(i+j+1)+j$, and $\pi^{-1} : \mathbb{N} \to \mathbb{N}^2$ be its inverse.

A *randomness test on* Σ^ω is a sequence $(U_n)_{n\geq 0}$ of open sets $U_n \subseteq \Sigma^\omega$ such that

1. $\mu(U_n) \leq 2^{-n}$, for all n,
2. there is an r.e. set A with $U_n = \bigcup_{\langle i,n\rangle\in A} \nu(i)\Sigma^\omega$, for all n.

A sequence $p \in \Sigma^\omega$ is called *non-random* if there is a randomness test $(U_n)_{n\geq 0}$ with $p \in \bigcap_{n\geq 0} U_n$. A sequence $p \in \Sigma^\omega$ is called *random* if it is not non-random.

Next we introduce randomness for real numbers via representations of real numbers with respect to some natural base. Fix a base $b \geq 2$, set $\Sigma_b = \{0, 1, \ldots, b-1\}$, and consider the representation of reals in the unit interval

$$\rho_b : \Sigma_b^\omega \to [0,1], \qquad \rho_b(\sigma_0\sigma_1 \ldots \sigma_n \ldots) = \sum_{i=0}^{\infty} \sigma_i 2^{-(i+1)}\,.$$

The expansion to base b is unique for all reals $x \in [0, 1]$ except for those rationals corresponding to sequences ending in an infinite sequence of 0s, respectively, of $(b-1)$s. A real x is called *random to base* b if its fractional part has a random b-adic expansion. This definition is base invariant; see Calude and Jürgensen [6] or Calude [4]. For a different proof and a base invariant characterization of random reals see Weihrauch [19], Weihrauch, Hertling [20].

For sequences of real numbers we can proceed in the same way by "merging" the digits of the expansions of the fractional parts of the real numbers in a computable way.

Definition 3.1. Let $f : \mathbb{N}^2 \to \mathbb{N}$ be a bijection and let $b \geq 2$ be an integer. A sequence $(a_n)_{n\geq 0}$ of real numbers a_n is called *f-random to base b* if there exists a random sequence $q = q_0q_1q_2\ldots \in \Sigma_b^\omega$ such that $\rho_b(q_{f(0,n)}q_{f(1,n)}q_{f(2,n)}\ldots)$ is the fractional part of a_n, for all $n \in \mathbb{N}$.

Note that this definition leads to the same randomness notion for all computable bijections f.

Lemma 3.2. *Let $f : \mathbb{N}^2 \to \mathbb{N}$ be an arbitrary computable bijection and $b \geq 2$ be an arbitrary base. Then any sequence $(a_n)_{n\geq 0}$ of real numbers is f-random to base b iff it is $\langle .,. \rangle$-random to base b.*

Proof. We can assume that all numbers a_n lie in the interval $[0, 1)$. The bijection $f \circ \pi^{-1}$ is computable. Fix a sequence $q \in \Sigma_b^\omega$. The sequence $q = q_0q_1q_2\ldots$ is random iff the sequence $p = q_{f\circ\pi^{-1}(0)}q_{f\circ\pi^{-1}(1)}q_{f\circ\pi^{-1}(2)}\cdots$ is random (see Lemma 3.4 below). Furthermore, $q_{f(i,n)} = p_{\langle i,n\rangle}$, for all i, n, hence $a_n = \rho_b(q_{f(0,n)}q_{f(1,n)}q_{f(2,n)}\ldots)$ iff $a_n = \rho_b(p_{\langle 0,n\rangle}p_{\langle 1,n\rangle}p_{\langle 2,n\rangle}\ldots)$, for all n. This proves the assertion. □

Lemma 3.2 justifies the following definition.

Definition 3.3. Let $b \geq 2$ be an integer. A sequence $(a_n)_{n\geq 0}$ of real numbers a_n is called *random to base b* iff there exists a random sequence $q = q_0q_1q_2\ldots \in \Sigma_b^\omega$ such that $a_n = \rho_b(q_{\langle 0,n\rangle}q_{\langle 1,n\rangle}q_{\langle 2,n\rangle}\cdots)$.

This notion of randomness has natural properties, as Proposition 3.5 and Theorem 3.6 show. Our proofs will use the following well-known fact, see e.g. Lemma 3.4 in Book, Lutz, Martin [2].

Lemma 3.4. *Let $f : \mathbb{N} \to \mathbb{N}$ be a computable one-to-one function. If $\sigma_0\sigma_1\sigma_2\ldots \in \Sigma^\omega$ is a random sequence, then the sequence $\sigma_{f(0)}\sigma_{f(1)}\sigma_{f(2)}\cdots$ is random as well.*

Proposition 3.5. *If a sequence of reals contains a non-random real, then the sequence itself is non-random to any base b.*

Proof. Let $(a_n)_{n\geq 0}$ be a sequence of reals which is random to some base $b \geq 2$. We can assume that $a_n \in [0, 1)$, for all n. There is a random sequence $q \in \Sigma_b^\omega$ such that $a_n = \rho_b(q_{\langle 0,n\rangle}q_{\langle 1,n\rangle}q_{\langle 2,n\rangle}\ldots)$, for all n. But, by Lemma 3.4, the sequence $q_{\langle 0,n\rangle}q_{\langle 1,n\rangle}q_{\langle 2,n\rangle}\cdots$ is also random, for each $n \in \mathbb{N}$. Thus, all real numbers a_n are random to base b, hence, random. □

Theorem 3.6. *If a sequence of real numbers is random to some base b, then it is uniformly distributed modulo 1.*

Proof. Let the sequence $(a_n)_{n\geq 0}$ of reals be random to some base $b \geq 2$. We can assume that all reals a_n lie in $[0, 1)$. For each integer $N \geq 1$ and $0 \leq r < s \leq 1$ put

$$A([r, s), N) = \#\{i \leq N - 1 \,|\, a_i \in [r, s)\}/N .$$

We have to show $\lim_{N\to\infty} A([r, s), N) = s - r$, for all $0 \leq r < s \leq 1$.

For each n, let $p(n) = p(n)_0 p(n)_1 p(n)_2 \dots$ be the expansion of a_n in base b. We know that the sequence q with $q_{\langle j,n\rangle} = p(n)_j$, for all n, j, is random. Fix a number $k \geq 1$. By Lemma 3.4 the sequence

$$q^{(k)} = p(0)_0 p(0)_1 \dots p(0)_{k-1} p(1)_0 p(1)_1 \dots p(1)_{k-1} \dots$$

is random. Let us consider each block $p(n)_0 \dots p(n)_{k-1}$ in this sequence as one digit in the alphabet Σ_{b^k}. In other words, consider the sequence $r^{(k)} \in \Sigma_{b^k}$ with $\rho_{b^k}(r^{(k)}) = \rho_b(q^{(k)})$. This sequence is random as well and therefore also normal, see Calude [7]. Its normality implies that for each interval $[l/b^k, (l+1)/b^k)$ with $0 \leq l < b^k$ the asymptotic portion of numbers a_n in this interval is $\lim_{N\to\infty} A([l/b^k, (l+1)/b^k), N) = 1/b^k$. This immediately implies

$$\lim_{N\to\infty} A([l/b^k, m/b^k), N) = (m-l)/b^k,$$

for $l, m \in \mathbb{N}$, $0 \leq l \leq m \leq b^k$. Let $[r, s)$ be now an arbitrary interval with $0 \leq r < s \leq 1$. It contains an interval $[l/b^k, m/b^k)$ with $l, m \in \mathbb{N}$, $0 \leq l \leq m \leq 2^k$ and with length at least $r - s - 2 \cdot b^k$. Hence

$$\liminf_{N\to\infty} A([r,s), N) \geq r - s - 2 \cdot b^k.$$

In the same way one proves $\limsup_{N\to\infty} A([r,s), N) \leq r - s + 2 \cdot b^k$. Note that we have proved this for arbitrary $k \geq 1$. Thus, the desired assertion $\lim_{N\to\infty} A([r,s), N) = r - s$ follows. □

Finally, we note that the randomness notion for sequences of real numbers introduced in Definition 3.3 is also base independent (the base independence of the randomness notion for real numbers has been proved by Calude and Jürgensen [6]; see also Calude [4]). Even more, the randomness of real numbers as well as of sequences of real numbers can be characterized directly over the real numbers without using representations with respect to some base. These results — which we will not prove here — are contained in the forthcoming paper by Weihrauch and Hertling [20] (for real numbers see also Weihrauch [19]), in which randomness is introduced not just for spaces of sequences but more generally for spaces which are endowed with a numbering of a base of the topology and with a measure. Here, we formulate the result for the case of sequences of real numbers.

Consider the space $[0,1]^\omega$ of infinite sequences of real numbers in the unit interval. This space is endowed with the product topology of the usual topology of $[0,1]$ and with the product measure μ^∞ of the Lebesgue measure μ on $[0,1]$, see Halmos [10]. Let $\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q} \cap [0,1)$ be the total numbering of the rational numbers in the unit interval (except the number 1) defined by $\nu_{\mathbb{Q}}(\langle i,j\rangle) = \frac{i}{j+1}$. We define a numbering of a base of the topology of the unit interval by

$$B_{\langle i,j\rangle} = \{x \in [0,1] \mid |x - \nu_{\mathbb{Q}}(i)| \leq 2^{-j}\},$$

and we define a numbering B^∞ of a base of the topology of $[0,1]^\omega$ by means of a standard numbering of all finite words of natural numbers. Define $\lambda : \mathbb{N}^* \to \mathbb{N}$ by

$$\begin{aligned} \lambda(\text{empty word}) &= 0, \\ \lambda(n_1 n_2 \dots n_l) &= p_1^{n_1} p_2^{n_2} \cdots p_l^{n_l+1} - 1, \qquad \text{for } l \geq 1, \end{aligned}$$

where $p_1 = 2$, $p_2 = 3$, $p_3 = 5\ldots$ is the sequence of prime numbers. The function λ is a bijection. Let $\kappa : \mathbb{N} \to \mathbb{N}^*$ be the inverse function to λ. We set

$$B_i^\infty = B_{n_1} \times B_{n_2} \times \cdots \times B_{n_l} \times [0,1]^\omega ,$$

for $\kappa(i) = n_1 n_2 \cdots n_l$. Using this numbering of a base of the product topology on $[0,1]^\infty$ we can define randomness for infinite sequences directly.

Definition 3.7 (Weihrauch and Hertling [20]). A *randomness test on* $[0,1]^\omega$ is a sequence $(U_n)_{n\geq 0}$ of open sets $U_n \subseteq [0,1]^\omega$ such that

1. $\mu^\infty(U_n) \leq 2^{-n}$,
2. there is an r.e. set $A \subseteq \mathbb{N}$ with $U_n = \bigcup_{\langle i,n\rangle \in A} B_i^\infty$, for all n.

A sequence $(a_n)_{n\geq 0}$ of real numbers in the unit interval is called *non-random* if there is a randomness test $(U_n)_{n\geq 0}$ on $[0,1]^\omega$ such that $(a_n)_{n\geq 0} \in \bigcap_{m\geq 0} U_m$. It is called *random* if it is not non-random.

Theorem 3.8 (Weihrauch and Hertling [20]). *Let $b \geq 2$. A sequence $(a_n)_{n\geq 0}$ of reals in the unit interval is random iff it is random to base b.*

Corollary 3.9. *The randomness notion introduced in Definition 3.3 is base independent.*

"Most" sequences of real numbers are random: in a perfect analogy with the case of reals (see [4]), with probability one every sequence of real numbers is random. Examples of random sequences of real numbers can be easily constructed from random reals; however, *we do not have a natural example*, for instance, as natural as Chaitin's Omega number [7, 8].

4 The Solution

Finally, we return to the zeros of the Riemann's zeta-function. As we have seen, by the Rademacher-Hlawka theorem, the sequence $(\mathrm{Im}(s_k))_{k\geq 0}$ of the (positive) imaginary parts of the non-trivial zeros of the Riemann's zeta-function is uniformly distributed modulo 1. By Theorem 3.6, this is a property shared by all random sequences. But neither the sequence $(\mathrm{Im}(s_k))_{k\geq 0}$ nor any other sequence containing imaginary parts of zeros of Riemann's zeta-function is random. We formulate a slightly more general result.

Lemma 4.1. *Let $U \subseteq \mathbb{C}$ be a connected open subset of the complex plane, $f : U \to \mathbb{C}$ be an analytic function which is computable at least on some open subset of U. If a sequence of real numbers contains a real or imaginary part of a zero of f, then this sequence is not random.*

Proof. By Pour-El and Richards [18] (Chapter 1.2, Proposition 1), the function f is computable on any compact subset of its domain U. By a result of Orevkov [17] each zero of f is a computable complex number, that is, its real and imaginary parts are computable real numbers. Any computable real number is non-random, hence, every sequence $(y_n)_{n\geq 0}$ which contains at least one real or imaginary part of a zero of f is not random by Lemma 3.5. □

Theorem 4.2. *No sequence $(y_k)_{k \geq 0}$ of reals which contains at least one imaginary part of a zero of the Riemann's zeta-function is random.*

Proof. For complex numbers s with $\mathrm{Im}(s) > 1$ the value $\zeta(s)$ of the zeta-function is given by the absolutely convergent sum $\sum_{n=1}^{\infty} n^{-s}$. Hence the zeta-function is computable in the half plane $\{s \,|\, \mathrm{Im}(s) > 1\}$. The assertion follows from Lemma 4.1 since the domain of definition of the zeta-function is the connected open set $\mathbb{C} \setminus \{1\}$. $\square$

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