Area, Length, Curvature[1]

Lecture 07

See Section 3.2 in
Reinhard Klette: Concise Computer Vision
Springer-Verlag, London, 2014

`ccv.wordpress.fos.auckland.ac.nz`

[1]See last slide for copyright information.

Agenda

Geometry in Digital Images

Images are given with a resolution of $N_{cols} \times N_{rows}$, i.e. the size of Ω

This resolution influences the accuracy when solving geometric tasks

Examples of geometric tasks

Area or perimeter of an object region

Length or curvature of a path in an image

An increase in image resolution should support an increase in accuracy for measured properties (known as *multigrid convergence* of measurements)

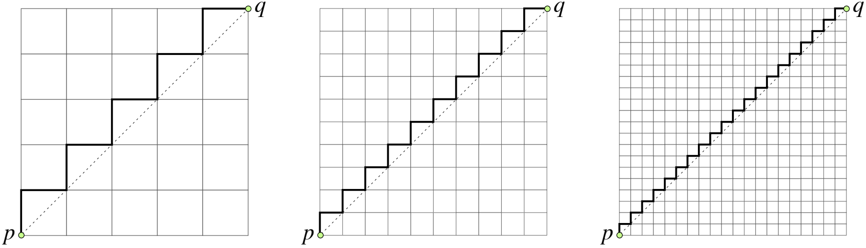
This needs to be ensured by the used measurement algorithms

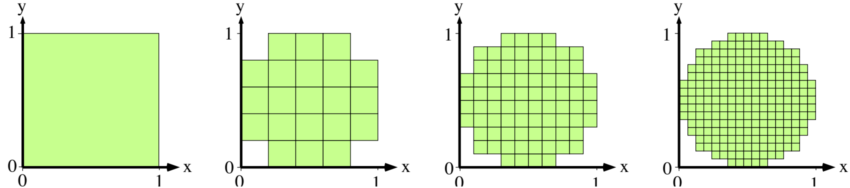
Agenda

The "Staircase Effect"

Length $a\sqrt{2}$ of a diagonal pq in a square with sides of length a

But: Length of diagonal 4-paths always equal to $2a$





Also: Perimeter of a digitised unit disks always equal to 4

How to Measure Length in the Euclidean Plane?

Length is measured for arcs

Consider the length of a polygonal approximation of an arc

- Defined by points $\phi(t_i)$ on the arc
- Let points $\phi(t_i)$ move closer and closer together (i.e. increase of n)
- We obtain more line segments on the polygonal approximation

Their length *converges* against the length of the given arc

Observation

The use of the length of a 4-path for estimating the length of a digital arc can lead to errors of 41.4% compared to arcs prior to digitisation

without any chance to reduce these errors in some cases by using higher grid resolution. This method is not recommended for length measurements in image analysis.

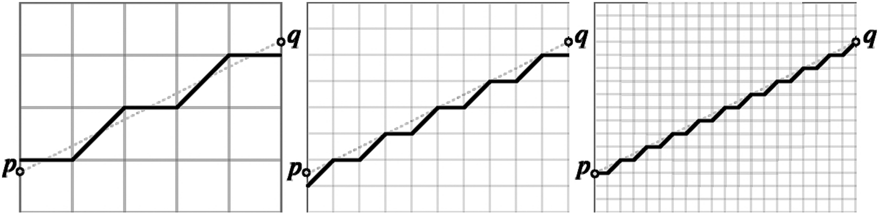
May be Weighted Edges Can Solve the Problem?

Another attempt: Use the length of an 8-path for length measurements

Use *weight* $\sqrt{2}$ for diagonal edges and weight 1 for *isothetic* edges (i.e. parallel to one of the coordinate axes)

Example

Consider digitised line segment pq with slope 22.5° and length $5\sqrt{5}/2$



What are the lengths of those 8-paths when using the proposed weights?

Answer

For a grid with edges of length 1 (shown on the left)

The length equals $3 + 2\sqrt{2}$

For *any* grid with edges of length $1/2^n$, for $n \geq 1$

The length equals $(5 + 5\sqrt{2})/2$

Result

Length of 8-paths not converging to $5\sqrt{5}/2$ as grid edge length goes to 0

Observation

The use of the length of an 8-path for estimating the length of a digital arc can lead to errors of 7.9% compared to arcs prior to digitisation

This upper bound for errors might be acceptable in some applications

A Solution with Convergence to True Length

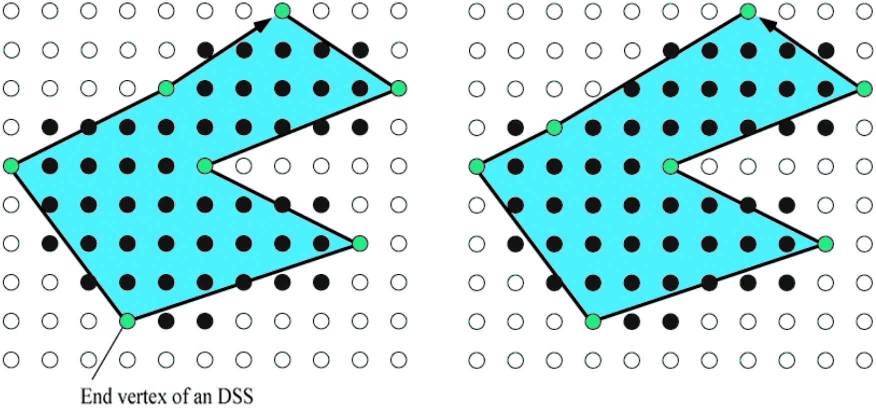
Polygonal Simplification of Borders

We recall the scheme illustrated on Page 6

1. Segment a given digital arc into maximum-length *digital straight segments* (DSS*s*) as on the next page
2. Take the sum of lengths of those straight segments

Measurement converges to the true length of a digitized arc when going for images with finer and finer grid resolution

Illustration of Arc Segmentation into DSSs



Clockwise (*left*) and counterclockwise (*right*) polygonal approximation of the border of a region by maximum-length DSS*s*

(For details see algorithms for DSS calculation)

Agenda

Area Estimation is Simpler

First: How is area defined in the Euclidean plane?

(1) *Area of triangle* $\langle p, q, r\rangle$, for $p = (x_1, y_1)$, $q = (x_2, y_2)$, $r = (x_3, y_3)$

$$A = \frac{1}{2} \cdot |x_1 y_2 + x_3 y_1 + x_2 y_3 - x_3 y_2 - x_2 y_1 - x_1 y_3|$$

(2) *Area of simple polygon* $\langle p_1, p_2, \ldots, p_n\rangle$, for $p_i = (x_i, y_i)$, $i = 1, 2, \ldots, n$

$$A = \frac{1}{2} \left| \sum_{i=1}^{n} x_i \, (y_{i+1} - y_{i-1}) \right|$$

with $y_0 = y_n$ and $y_{n+1} = y_1$

(3) *In general:* Area of a measurable set $R \subset \mathbb{R}^2$

$$A = \int_R \mathrm{d}x \, \mathrm{d}y$$

How to Measure the Area of a Region in an Image?

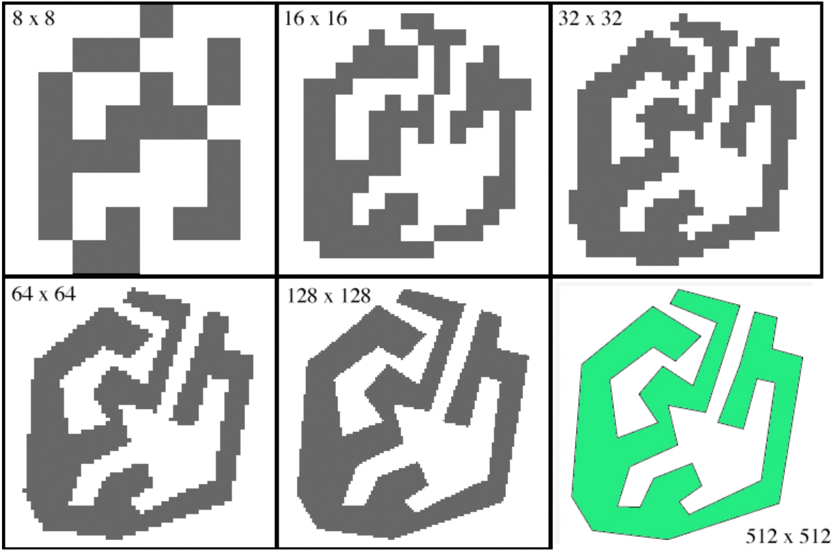
Answer by C. F. Gauss in the early 19th century:

Count all the grid points (i.e. pixel locations) in the digitised set and multiply by the size of a grid square (i.e. the pixel size)

Next page: Illustration of an experiment

1. Given: Simple polygon defined in a grid of size 512×512
   having area $A = 102,742.5$ and perimeter $P = 4,040.7966\ldots$
2. Subsample this polygon in images of reduced resolution
3. Estimate area and perimeter in images and compare with A and P

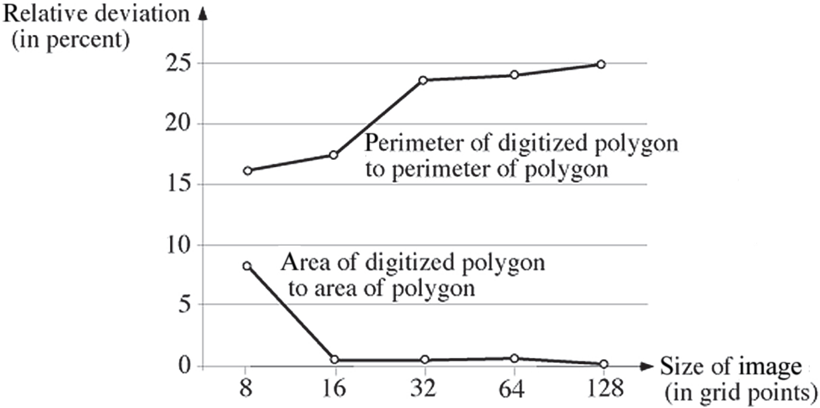
Different Digitizations



Results

Area: As proposed by Gauss, the number of pixels (i.e. grid cells) times the square of the edge length

Perimeter: Number of cell edges on the frontier of the polygon times the length of an edge (i.e. illustrating failure of 4-adjacency again)



Relative Deviation

Relative deviation

is the absolute difference between estimated property values P_{est} and P for subsampled polygon and original polygon, respectively, divided by P

$$\frac{|P_{est} - P|}{P}$$

Relative deviation in percent

$$\frac{|P_{est} - P|}{P} \cdot 100$$

Agenda

Curvature for Border Characterization

Shapes may also be characterised by
- high-curvature points or
- change in curvature along the border

A Third Property: Curvature

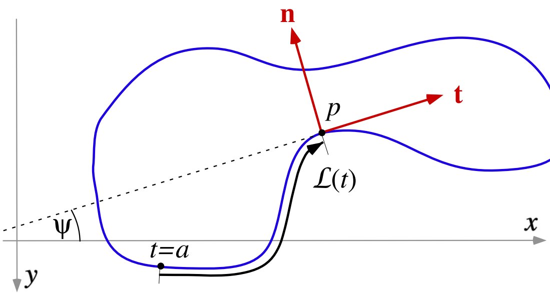
Curvature can be defined at non-singular points of a smooth arc $\gamma(t)$
in the Euclidean plane

Curvature at a point p on γ
can be defined in different ways

Option 1: Rate of change of angle of a tangential line at p

Option 2: Derivative at p (requires a parameterised representation of γ)

Option 1: Rate of Change



Starting at $t = a$, the arc to $p = \gamma(t)$ has the length $l = \mathcal{L}(t)$

n is the normal, and **t** the tangent defining angle ψ

While p is sliding along γ, angle ψ will change

Rate of Change in ψ

Defines *curvature* $\kappa_{tan}(p)$

$$\kappa_{tan}(t) = \frac{\mathrm{d}\psi(t)}{\mathrm{d}l}$$

(1) “Fast change” in ψ = “high curvature”

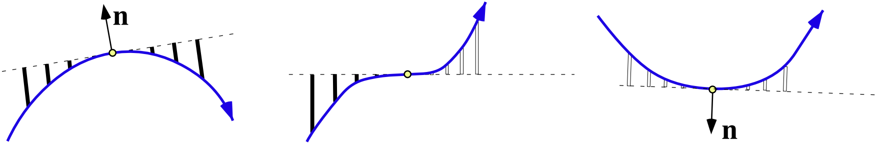
(2) “Slow change” in ψ = “low curvature”

(3) No change in $\psi = p$ on a straight segment of γ

Convex, Inflection or Straight, or Concave

$\kappa_{tan}(t)$ can be

1. negative: p is a *convex point*
2. zero: p is a point of inflection or on a straight segment
3. positive: p is a *concave point*



Option 2: Curvature of a Parametrized Arc

Assume a parametric representation $\gamma(t) = (x(t), y(t))$. Then:

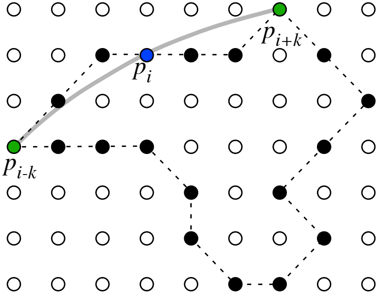
$$\kappa_{tan}(t) = \frac{\dot{x}(t) \cdot \ddot{y}(t) \; - \; \dot{y}(t) \cdot \ddot{x}(t)}{[\, \dot{x}(t)^2 + \dot{y}(t)^2 \,]^{1.5}}$$

with

$$\dot{x}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}, \; \dot{y}(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}, \; \ddot{x}(t) = \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}, \; \ddot{y}(t) = \frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}$$

Now: Curvature in an Image

How to define curvature of a border of an object region at pixel location p?



Example:

Go $k = 3$ steps forward and backward for points p_{i-k} and p_{i+k}
Use those three points now for estimating curvature

Example: Following Option 2

Given: Digital curve $\langle p_1, \ldots, p_m \rangle$, where $p_j = (x_j, y_j)$ for $1 \leq j \leq m$

Assume: Samples along parametrized curve $\gamma(t) = (x(t), y(t))$, $t \in [0, m]$
At p_i thus $\gamma(i) = p_i$

Functions $x(t)$ and $y(t)$ locally interpolated by second order polynomials

$$\begin{aligned} x(t) &= a_0 + a_1 t + a_2 t^2 \\ y(t) &= b_0 + b_1 t + b_2 t^2 \end{aligned}$$

Let $x(0) = x_i$, $x(1) = x_{i-k}$, $x(2) = x_{i+k}$ for $k \geq 1$; analogously for $y(t)$

Curvature at p_i then defined by

$$\kappa_i = \frac{2(a_1 b_2 - b_1 a_2)}{[\, a_1^2 + b_1^2 \,]^{1.5}}$$

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